PARAMETRIZATIONS OF CANONICAL BASES AND IRREDUCIBLE COMPONENTS OF NILPOTENT VARIETIES

BASED ON THE TALK BY YONG JIANG

Throughout the talk Q is a fixed quiver with the set of vertices I.

Let $\mathfrak{g}$ be the Kac–Moody Lie algebra associated with Q. By the $\mathcal{U}_q^-(\mathfrak{g})$ we denote the subalgebra of the quantized enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ of $\mathfrak{g}$ generated by the elements f_i, $i \in I$. By the crystal basis of $\mathcal{U}_q^-(\mathfrak{g})$ we mean a pair $(\mathcal{L}, \mathcal{B})$ such that $\mathcal{L}$ is an $\mathcal{A}_0$-lattice of $\mathcal{U}_q^-(\mathfrak{g})$, i.e. $\mathcal{L} \otimes \mathbb{Q}(q) \simeq \mathcal{U}_q^-(\mathfrak{g})$, where

$$\mathcal{A}_0 := \left\{ \frac{f}{g} \in \mathbb{Q}(q) : g(0) \neq 0 \right\},$$

and $\mathcal{B}$ is a $\mathbb{Q}$-basis of $\mathcal{L}/q \cdot \mathcal{L}$ (observe that $\mathcal{A}_0/q \cdot \mathcal{A}_0 \simeq \mathbb{Q}$).

Now let w be an element of the Weyl group W associated with Q. We also fix a sequence $\mathbf{i} = (i_1, \dots, i_r)$ of vertices of Q inducing a reduced expression of w. For each $k \in [1, r]$ we put

$$F_{\mathbf{i},k} := (T_{i_1} \circ \dots \circ T_{i_{k-1}})(f_{i_k}),$$

where, for $i \in I$, T_i is the Lusztig's braid automorphism associated with i. Finally, if $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{N}^r$, then we put

$$F_{\mathbf{i}}(\mathbf{a}) := F_{\mathbf{i},1}^{(a_1)} \cdot \ldots \cdot F_{\mathbf{i},r}^{(a_r)},$$

where

$$x^{(a)} := \frac{x^a}{[a]!}$$

for an element x of $\mathcal{U}_q^-(\mathfrak{g})$ and $a \in \mathbb{N}$,

$$[a]! = [1] \cdot \ldots \cdot [a]$$

for $a \in \mathbb{N}$, and

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}}$$

for $a \in \mathbb{N}$. Lusztig has proved that for each $\mathbf{a} \in \mathbb{N}^r$ there exists unique element b of $\mathcal{B}$ such that the elements $F_{\mathbf{i}}(\mathbf{a})$ and b are congruent modulo $q \cdot \mathcal{L}$. We denote the map induced in this way by $\Phi_{\mathbf{i}}$.

Let Λ be the preprojective algebra associated with Q. For a dimension vector $\mathbf{d}$ we denote by $\Lambda_{\mathbf{d}}$ the variety of the nilpotent Λ-modules with dimension vector $\mathbf{d}$. By $\operatorname{Irr} \Lambda$ we denote the set of the irreducible components of the varieties $\Lambda_{\mathbf{d}}$, $\mathbf{d} \in \mathbb{N}^I$. Kashiwara and Saito have

Date: 04.11.2011.

proved that there is a natural bijection Ψ between $\mathcal{B}$ and $\operatorname{Irr}\Lambda$. Consequently, we have the function $\Psi_{\mathbf{i}} := \Psi \circ \Phi_{\mathbf{i}} : \mathbb{N}^r \to \operatorname{Irr}\Lambda$. We describe this map explicitly.

For $i \in I$ we denote by I_i the injective envelope of the simple Λ-module at i. For $k \in [1, r]$ we put

$$V_{\mathbf{i},k} := S_{i_1,\dots,i_k} I_{i_k}.$$

Here, for $i \in I$ and a Λ-module V, we denote by $S_i V$ the maximal submodule of V whose composition factors are isomorphic to S_i. Moreover, if $j_1, \dots, j_t \in I$ and $t > 1$, then $S_{j_1,\dots,j_t} V := S_{j_1}(S_{j_2,\dots,j_t} V)$. Next, for $k \in [1, r]$ we denote by k^- the maximal $s \in [1, k-1]$ such that $i_s = i_k$ (we put $k^- := 0$ if the is no such s). There is a natural embedding $V_{\mathbf{i},k^-} \hookrightarrow V_{\mathbf{i},k}$ and we put $M_{\mathbf{i},k} := V_{\mathbf{i},k}/V_{\mathbf{i},k^-}$. If $\mathbf{a} \in \mathbb{N}^r$, then we denote by $Z_{\mathbf{i}}(\mathbf{a})$ the closure of the set consisting of the Λ-modules X such that there exists a filtration

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X$$

such that $X_k / X_{k-1} \simeq M_{\mathbf{i},k}^{a_k}$ for each $k \in [1, r]$. Then $Z_{\mathbf{i}}(\mathbf{a}) \in \operatorname{Irr}\Lambda$ and $\Psi_{\mathbf{i}}(\mathbf{a}) = Z_{\mathbf{i}}(\mathbf{a})$ for each $\mathbf{a} \in \mathbb{N}^r$.