CLUSTER PRESENTATIONS OF REFLECTION GROUPS

BASED ON THE TALK BY ROBERT MARSH

Throughout the talk n is a fixed positive integer.

Let M an $n \times n$ matrix with non-negative integer coefficients such that the following conditions are satisfied:

(1) if $i, j \in [1, n]$, then $M(i, j) = 1$ if and only if $i = j$.
(2) if $i, j \in [1, n]$, then $M(i, j) = M(j, i)$.

With a matrix M we associate the group W, which is the group generated by the elements $s_1, \ldots, s_n$ such that $(s_i \cdot s_j)^{M(i,j)} = e$ for any $i, j \in [1, n]$. The groups of this form are called the Coxeter groups.

The matrices of the above form can be constructed from the Coxeter matrices of finite type. Namely, given an $n \times n$ Coxeter matrix C we put

$$M(i,j) := \begin{cases} 1 & \text{if } i = j, \\ 2 & \text{if } i \neq j \text{ and } C(i,j) \cdot C(j,i) = 0, \\ 3 & \text{if } i \neq j \text{ and } C(i,j) \cdot C(j,i) = 1, \\ 4 & \text{if } i \neq j \text{ and } C(i,j) \cdot C(j,i) = 2, \\ 6 & \text{if } i \neq j \text{ and } C(i,j) \cdot C(j,i) = 3, \end{cases}$$

for $i, j \in [1, n]$. Consequently, we may associate a Coxeter group with each Coxeter matrix.

We put

$$\mathbb{F} := \mathbb{Q}(u_1, \ldots, u_n).$$

By a seed we mean every pair $(\mathbf{x}, B)$ consisting of a sequence $\mathbf{x}$ of algebraically independent generators of the field $\mathbb{F}$ and an $n \times n$ skew-symmetizable matrix B with integer coefficients. Recall that an $n \times n$ matrix B with integer coefficients is called skew-symmetizable if there exists a diagonal matrix D with positive coefficients on the main diagonal such that the matrix $D \cdot B$ is skew-symmetric. If $(\mathbf{x}, B)$ is a seed and $k \in [1, \ldots, n]$, then we define a new seed $(\mathbf{x}', B')$, called the mutation of the seed $(\mathbf{x}, B)$ at k and denoted by $\mu_k(\mathbf{x}, B)$, by the following formulas:

$$x_i' := \begin{cases} \frac{1}{x_k} \cdot \left(\prod_{\substack{j \in [1,n] \\ B(j,k) > 0}} x_j^{B(j,k)} + \prod_{\substack{j \in [1,n] \\ B(j,k) < 0}} x_j^{-B(j,k)} \right) & \text{if } i = k, \\ x_i & \text{otherwise,} \end{cases}$$

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for $i \in [1, n]$, and

$$B(i,j) := \begin{cases} -B(i,j) & \text{if } i = k \text{ and } j = k, \\ B(i,j) + \frac{|B(i,k)| \cdot B(k,j) + B(i,k) \cdot |B(k,j)|}{2} & \text{otherwise,} \end{cases}$$

for $i, j \in [1, n]$. If we fix a seed $(\mathbf{x}, B)$, called the initial seed, then we denote by $\mathcal{S}_0$ the smallest subset of the set of seeds containing $(\mathbf{x}, B)$ and closed under mutations. We call the elements of $\mathcal{S}_0$ the admissible seeds (for the seed $(\mathbf{x}, B)$). If $(\mathbf{x}, B)$ is an admissible seed, then we call the sequence $\mathbf{x}$ a cluster and the matrix B an exchange matrix. Finally, if $(x_1, \ldots, x_n)$ is a cluster, then we call the elements $x_1, \ldots, x_n$ cluster variables. By $\mathcal{A}(\mathbf{x}, B)$ we denote the $\mathbb{Q}$-subalgebra of the field $\mathbb{F}$ generated by the set of cluster variables. The algebras of this form are called the cluster algebras. If the set $\mathcal{S}_0$ is finite, then we say that the algebra $\mathcal{A}(\mathbf{x}, B)$ is of finite type.

Given a skew-symmetrizable $n \times n$ matrix B we define its Cartan counterpart A by the following formula:

$$A(i,j) := \begin{cases} 2 & \text{if } i = j, \\ -|B_{i,j}| & \text{otherwise,} \end{cases}$$

for $i, j \in [1, n]$. Fomin and Zelevinsky have proved that a cluster algebra is of finite type if and only if there exists an admissible seed with the exchange matrix, whose Cartan counterpart is a Cartan matrix of finite type. Consequently, with each cluster algebra of finite type we may associate a Coxeter group. Barot and Marsch have explained how to describe this group starting from an arbitrary admissible seed.