FINITE INJECTIVE DIMENSION OVER RINGS WITH NOETHERIAN COHOMOLOGY

BASED ON THE TALK BY JESSE BURKE

For a finite group G and a field K we denote by $H^*(G,K)$ the ring $\operatorname{Ext}^*_{KG}(K,K)$. The ring $H^*(G,K)$ is graded-commutative, i.e., if x and y are homogeneous elements of the ring $H^*(G,K)$ of degrees m and n, respectively, then

$$x \cdot y = (-1)^{mn} \cdot y \cdot x.$$

We put

$$H^\circ(G,K) := \begin{cases} H^*(G,K) & \text{if } \operatorname{char} K = 2, \\ \bigoplus_{i\in\mathbb{N}} H^{2i}(G,K) & \text{otherwise.} \end{cases}$$

For each KG-module M we denote by ξ_M the function

$$H^\circ(G,K) \to \operatorname{Ext}^*_{KG}(M,M)$$

given by tensoring the elements of $H^\circ(G,K)$ with the module M. These functions induce a structure of an $H^\circ(G,K)$-module in $\operatorname{Ext}^*_{KG}(M,N)$ for all KG-modules M and N, since

$$\beta \circ \xi_M(\alpha) = \xi_N(\alpha) \circ \beta$$

for any $\alpha \in \operatorname{Ext}^*_{KG}(M,N)$ and $\beta \in \operatorname{Ext}^*_{KG}(M,N)$. Venkov and Evans have proved that $H^\circ(G,K)$ is a finitely generated K-algebra and, for any finite dimensional modules M and N, $\operatorname{Ext}^\circ_{KG}(M,N)$ is a finitely generated $H^\circ(G,K)$-modules. Using this result one may prove the following well-known fact.

Theorem. *If M is a finite dimensional KG-module such that*

$$\operatorname{Ext}^n_{KG}(M,M) = 0$$

for any $n \gg 0$, then the module M is projective.

Proof. Let S be be a simple KG-module. The above mentioned result of Venkov and Evand implies that $\operatorname{Ext}^*_{KG}(M,S)$ is a finitely generated $\operatorname{Ext}^*_{KG}(M,M)$-module. Since $\operatorname{Ext}^n_{KG}(M,M) = 0$ for any $n \gg 0$, this implies that $\operatorname{Ext}^n_{KG}(M,S) = 0$ for any $n \gg 0$. This holds for each simple KG-module S, hence $\operatorname{pdim}_{KG} M < \infty$, what implies that the module M is projective. □

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The aim of this talk is to generalize this observation.

We start with the following definition.

Let $\mathcal{T}$ be a triangulated category and S a commutative graded ring. We say that S is a ring of cohomology operators for the category $\mathcal{T}$ if for any object X of the category $\mathcal{T}$ there is given a graded ring homomorphism $\xi_X : S \to \operatorname{Hom}^*_{\mathcal{T}}(X, X)$ such that

$$\beta \cdot \xi_X(\alpha) = \xi_Y(\alpha) \cdot \beta$$

for any objects X and Y of the category $\mathcal{T}$, an element α of the ring S and $\beta \in \operatorname{Hom}^*_{\mathcal{T}}(X, Y)$. In the above, for objects X and Y of the category $\mathcal{T}$ we put

$$\operatorname{Hom}^*_{\mathcal{T}}(X, Y) := \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{T}}(\Sigma^n X, Y).$$

Moreover, if X is an object of the category $\mathcal{T}$, then we define a structure of a ring in the group $\operatorname{Hom}^*_{\mathcal{T}}(X, X)$ by the formula:

$$\alpha \cdot \beta := \Sigma^n \alpha \circ \beta$$

for homogeneous elements α and β of the group $\operatorname{Hom}^*_{\mathcal{T}}(X, Y)$ of degrees m and n, respectively. The analogous formulas also define, for any objects X and Y of the category $\mathcal{T}$, the structures of the left $\operatorname{Hom}^*_{\mathcal{T}}(Y, Y)$- and the right $\operatorname{Hom}^*_{\mathcal{T}}(X, X)$-module in the group $\operatorname{Hom}^*_{\mathcal{T}}(X, Y)$.

For a ring R we denote we denote by $\mathcal{K}(\operatorname{Inj} R)$ the homotopy category of the injective R-modules. Krause proved that if R is left noetherian, then the natural map

$$\mathcal{K}(\operatorname{Inj} R) \to \mathcal{D}(\operatorname{Mod} R)$$

restricts to an equivalence

$$\mathcal{K}(\operatorname{Inj} R)^c \to \mathcal{D}^b(\mathcal{R}),$$

where for a triangulated category $\mathcal{T}$ we denote by $\mathcal{T}^c$ the full subcategory of compact objects. We denote the inverse equivalence $\mathcal{D}^b(\mathcal{R}) \to \mathcal{K}(\operatorname{Inj} R)^c$ by i_R.

We have the following generalization of the fact mentioned at the beginning.

Theorem (Burke)**.** *Let R be a ring and S a non-negatively graded ring of cohomology operators for the category $\mathcal{K} := \mathcal{K}(\operatorname{Inj} R)$ such that $\operatorname{Hom}^*_{\mathcal{K}}(C, C)$ is a finitely generated S-module for any compact object C in the category $\mathcal{K}$. If M is a complex of R-modules such that $H^n(M) = 0$ for any $n \gg 0$ and $\operatorname{Hom}^*_{\mathcal{K}}(i_R M, i_R M)$ is $S_+ := \bigoplus_{i \in \mathbb{N}_+}$-torsion, then $\operatorname{idim}_R M < \infty$.*

Recall that a graded module E over a graded ring S is called S_+-torsion if for any homogeneous element x of the module E there exists $n \in \mathbb{N}$ such that $(S_+)^n \cdot x = 0$. In particular, this condition is satisfied if $E_n = 0$ for any $n \gg 0$. On the other hand, the two conditions are equivalent if the module E if finitely generated.