THE SIMPLE TRANSITIVITY OF THE BRAID GROUP ACTION AND NONCROSSING LOOPS (AFTER BESSIS)

BASED ON THE TALK BY PHILIPP LAMPE

For a positive integer n we denote by B_n be the braid group on n strands, i.e. the group generated by the elements $\sigma_1, \ldots, \sigma_{n-1}$ such that $\sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i$ for all $i, j \in \{1, \ldots, n-1\}$ with $|i - j| > 1$ and $\sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1}$ for each $i \in \{1, \ldots, n-2\}$. For each group G we have the action of B_n on G^n defined by the condition:

$$\sigma_i \cdot (g_1, \ldots, g_n) = (g_1, \ldots, g_{i-1}, g_i \cdot g_{i+1} \cdot g_i^{-1}, g_i, g_{i+2}, \ldots, g_n)$$

for all $i \in \{1, \ldots, n-1\}$ and $g_1, \ldots, g_n \in G$. Note that

$$g_1' \cdot \ldots \cdot g_n' = g_1 \cdot \ldots \cdot g_n$$

for all $\sigma \in B_n$ and $g_1, \ldots, g_n \in G$, where

$$(g_1', \ldots, g_n') := \sigma \cdot (g_1, \ldots, g_n).$$

Now we fix points $x_0, \ldots, x_n$ of $\mathbb{C}$. By F_n we denote the fundamental group of the space $\mathbb{C} \setminus \{x_1, \ldots, x_n\}$ at x_0. By a non-crossing loop we mean every element of F_n induced by a positively oriented continuous embedding of S^1 into $\mathbb{C} \setminus \{x_1, \ldots, x_n\}$ which maps 1 to x_0. By R we denote the set of the non-crossing loops whose interior contains exactly one of the points $x_1, \ldots, x_n$. One easily checks that $f \cdot g \cdot f^{-1} \in R$ for all $f, g \in R$. Consequently, the action of B_n on F_n^n induces the action of B_n on R^n.

Let W_n be the universal Coxeter group, i.e. the group generated by the elements $s_1, \ldots, s_n$ such that $s_i^2 = 1$ for each $i \in \{1, \ldots, n\}$. If we fix $f_1, \ldots, f_n \in R$ such that the interior of f_i contains x_i for each $i \in \{1, \ldots, n\}$, then there exists the group epimorphism $\pi : F_n \to S_n$ induced by the assignment $f_i \mapsto s_i$ for $i \in \{1, \ldots, n\}$.

For an element g of a group G and subset $A \subseteq G$ of by an A-decomposition of g we mean every sequence $(a_1, \ldots, a_k)$ of elements of A such that $g = a_1 \cdot \ldots \cdot a_k$. We denote by $\ell_A(g)$ the minimal k such that there exists an A-decomposition $(a_1, \ldots, a_k)$ of g. An A-decomposition $(a_1, \ldots, a_k)$ of g is called reduced if $k = \ell_A(g)$. We denote by $\mathrm{Red}_A(g)$ the set of the reduced A-decompositions of g.

Let $\pi : W_n \to S_n$ be as above and put $T := \pi(R)$, $g := f_1 \cdot \ldots \cdot f_n$ and $c := \pi(g) = s_1 \cdot \ldots \cdot s_n$. Then $\ell_R(g) = n = \ell_T(c)$. In particular, the action of B_n on R^n induces the actions of B_n on $\mathrm{Red}_R(g)$ and

Date: 22.07.2011.

$\mathrm{Red}_T(n)$. The main theorem of the talk states that these actions are simply transitive.