# TRANSITIVITY OF THE BRAID GROUP ACTION FOR COXETER GROUPS (AFTER IGUSA-SCHIFFLER)

BASED ON THE TALK BY DIRK KUSSIN

Fix a positive integer $n$ and a symmetric $n \times n$ matrix $m$, whose coefficients are non-negative integers such that $m(i,i) = 1$ for each $i \in \{1, \ldots, n\}$ and $m(i,j) \neq 1$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$. By the Coxeter group $W$ associated with these data we mean the group generated by the elements $s_1, \ldots, s_n$ subject to the conditions $(s_i \cdot s_j)^{m_{i,j}} = 1$ for all $i, j \in \{1, \ldots, n\}$.

We present a geometric interpretation of this group. Let $\alpha_1, \ldots, \alpha_n$ be the standard basis vectors of $V := \mathbb{R}^n$. We define the symmetric bilinear form $B$ on $V$ by

$$B(\alpha_i, \alpha_j) := \begin{cases} -1 & \text{if } m_{i,j} = 0, \\ -\cos(\frac{\pi}{m_{i,j}}) & \text{if } m_{i,j} \neq 0, \end{cases}$$

for $i, j \in \{1, \ldots, n\}$. For $i \in \{1, \ldots, n\}$ we define $\sigma_i : V \to V$ by

$$\sigma_i(x) := x - 2 \cdot B(\alpha_i, x) \cdot \alpha_i$$

for $x \in V$. One shows that the assignment

$$s_i \mapsto \sigma_i, \ i \in \{1, \ldots, n\},$$

induces an injective group homomorphism $W \to \mathrm{GL}(V)$, which we treat as identification.

Let

$$\Phi := \{w(\alpha_i) : w \in W \text{ and } i \in \{1, \ldots, m\}\}.$$

We call the elements of $\Phi$ roots. Let

$$\Phi^+ := \{x \in \Phi : x \geq 0\} \qquad \text{and} \qquad \Phi^- := \{x \in \Phi : x \leq 0\}.$$

Then $\Phi = \Phi^+ \cup \Phi^-$. For $\alpha \in \Phi$ we define $s_\alpha : V \to V$ by

$$s_\alpha(x) := x - 2 \cdot B(\alpha, x) \cdot \alpha.$$

One easily checks that $s_{w(\alpha)} = w \cdot s_\alpha \cdot w^{-1}$ for all $\alpha \in \Phi$ and $w \in W$. Consequently, the assignment

$$\alpha \mapsto s_\alpha, \alpha \in \Phi^+,$$

is a bijection between $\Phi^+$ and

$$R := \{w \cdot s_i \cdot w^{-1} : w \in W \text{ and } i \in \{1, \ldots, n\}\}$$

(we call the elements of $R$ reflections).

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For a positive integer $m$ we denote by $B_m$ be the braid group on $m$ strands, i.e. the group generated by the elements $\sigma_1, \ldots, \sigma_{m-1}$ such that $\sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i$ for all $i, j \in \{1, \ldots, m-1\}$ with $|i-j| > 1$ and $\sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1}$ for each $i \in \{1, \ldots, m-2\}$. For each group $G$ we have the action of $B_m$ on $G^m$ defined by the condition:

$$\sigma_i \cdot (g_1, \ldots, g_m) = (g_1, \ldots, g_{i-1}, g_i \cdot g_{i+1} \cdot g_i^{-1}, g_i, g_{i+2}, \ldots, g_m)$$

for all $i \in \{1, \ldots, m-1\}$ and $g_1, \ldots, g_m \in G$. Note that

$$g_1' \cdot \ldots \cdot g_m' = g_1 \cdot \ldots \cdot g_m$$

for all $\sigma \in B_m$ and $g_1, \ldots, g_m \in G$, where

$$(g_1', \ldots, g_m') := \sigma \cdot (g_1, \ldots, g_m).$$

Next, we also have the action of $B_m$ on $(\Phi^+)^m$ defined by the condition:

$$\sigma_i \cdot (\beta_1, \ldots, \beta_m) = (\beta_1, \ldots, \beta_{i-1}, |s_{\beta_i}(\beta_{i+1})|, \beta_i, \beta_{i+2}, \ldots, \beta_m)$$

for all $i \in \{1, \ldots, m-1\}$ and $g_1, \ldots, g_m \in G$. Observe that the above action is compatible with the action of $B_m$ on $W^m$ under the map which sends $(\beta_1, \ldots, \beta_m) \in B^m$ to $(s_{\beta_1}, \ldots, s_{\beta_m})$.

The aim of this talk is to sketch the proof of the following theorem.

**Theorem** (Igusa/Schiffler)**.** *If $t_1, \ldots, t_m \in R$ and $t_1 \cdot \ldots \cdot t_m = s_1 \cdot \ldots \cdot s_n$, then $m \geq n$. Moreover, if $m = n$, then there exists $\sigma \in B_n$ such that*

$$(t_1, \ldots, t_m) = \sigma \cdot (s_1, \ldots, s_n).$$

Without loss of generality we may assume that $m \leq n$. We also fix $\beta_1, \ldots, \beta_m \in \Phi^+$ such that $t_i = s_{\beta_i}$ for all $i \in \{1, \ldots, m\}$. Let $c := s_1 \cdot \ldots \cdot s_n$. A root $p \in \Phi^+$ is called projective if $c(p) < 0$. One shows that we have exactly $n$ projective roots, namely $p_1, \ldots, p_n$, where $p_i := (s_n \cdot \ldots \cdot s_{i+1})(\alpha_i)$ for $i \in \{1, \ldots, n\}$. The crucial point in the proof, whose proof we omit, is to observe that there exists $\sigma \in B_m$ such that

$$\sigma \cdot (\beta_1, \ldots, \beta_m) = (p_{i_1}, \ldots, p_{i_m})$$

for some $i_1, \ldots, i_m \in \{1, \ldots, n\}$ such that $i_1 > \ldots > i_m$. Since

$$s_{p_i} = s_n \cdot \ldots \cdot s_i \cdot \ldots s_n$$

for each $i \in \{1, \ldots, n\}$, by exploiting the equality

$$s_{p_{i_1}} \cdot \ldots \cdot s_{p_{i_m}} = s_1 \cdot \ldots \cdot s_n$$

we obtain the equality

$$s_1 \cdot \ldots \cdot \hat{s_{i_m}} \cdot \ldots \cdot \hat{s_{i_1}} \cdot \ldots \cdot s_n = 1,$$

which implies that $\{i_1, \ldots, i_m\} = \{1, \ldots, n\}$, hence finishes the proof.