SUPERDECOMPOSABLE PURE-INJECTIVES OVER TUBULAR ALGEBRAS

BASED ON THE TALK BY MARK PREST

Throughout the talk R is a finite-dimensional algebra.

A monomorphism f of right R-modules is called *pure* if $f \otimes_R L$ is a monomorphism for each left R-module L. A right R-module M is called *pure-injective* if for every pure monomorphism $f : N \to M$ splits. It is known that a module is pure-injective if and only if it is a direct summand of a product of finite dimensional modules. Moreover, every pure injective module is the direct sum of the pure-injective hull of the direct sum of its indecomposable direct summands and an superdecomposable pure-injective module (a non-zero module is called *superdecomposable* if it has no indecomposable direct summands). It is an interesting problem to study for which algebras there exist superdecomposable pure-injective modules. It is known that they do not exist for the tame hereditary algebras, but they do exist for the strictly wild algebras and the non-domestic string algebras (in the latter case under the assumption that R is countable).

The aim of this talk is to present the following theorem.

Theorem. *If R is tubular, then there exists a superdecomposable pure-injective R-module.*

An imporant role in the proof is played by the *Ziegler spectrum* Zg_R of R which we define now. By definition Zg_R consists of the isomorphism classes of the indecomposable pure-injective R-modules. In Zg_R we introduce a topology, whose basis of open sets consists of the following sets: for each homomorphism $f : A \to B$ of finite dimensional R-modules we have the open set (f) consisting of the isomomorphism classes of the indecomposable pure-injective modules M such that

$$\mathrm{Hom}_R(A, M)/\operatorname{Im}\mathrm{Hom}_A(f, M) \neq 0.$$

Alternatively, we may describe the closed sets in Zg_R in the following way. A full subcategory of the category of right R-modules is called *definable* if it is closed under isomorphism, direct products, direct limits and pure submodules (a submodule N of a module M is called *pure* if the inclusion map $N \hookrightarrow M$ is a pure monomorphism). The assignement $\mathcal{D} \mapsto \mathcal{D} \cap \mathrm{Zg}_R$ induces a bijection between the definable subcategories of the category of right R-modules and the closed subsets of Zg_R. We

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also mention that the isomorphism class of a pure-injective module N is an isolated point of Zg_R if and only if N is finite dimensional.

Recall that if R tubular, then the finite dimensional indecomposable R-modules can be divided into the following classes:

- the class $\mathcal{P}_0$ of the preprojective modules,
- the class $\mathcal{T}_q$ of the finite dimensional indecomposable module of slope q, $q \in \{0\} \cup \mathbb{Q}_+ \cup \{\infty\}$,
- the class $\mathcal{Q}_\infty$ of the preinjective modules.

For each $r \in \{0\} \cup \mathbb{R}_+ \cup \{\infty\}$ we put

$$\mathcal{P}_r := \mathcal{P}_0 \cup \bigcup_{\gamma<r} \mathcal{T}_\gamma \qquad \text{and} \qquad \mathcal{Q}_r := \bigcup_{\gamma>r} \mathcal{T}_\gamma \cup \mathcal{Q}_\infty.$$

Then by $\mathcal{D}_r$ we denote the full subcategory of the category of right R-modules consisting of the modules M such that

$$\mathrm{Hom}_R(M, \mathcal{P}_r) = 0 = \mathrm{Hom}_R(\mathcal{Q}_r, M).$$

Then $\mathcal{D}_r$ is a definable subcategory of the category of right R-modules and we call it the subcategory of the modules of slope r.

We have the following theorem which implies the main theorem.

Theorem (Harland)**.** *For each irrational $r \in \mathbb{R}_+$ there exists a superdecomposable pure-injective module in $\mathcal{D}_r$.*

In order to show existence of superdecomposable pure-injective module we use the following criterion, which uses a notion of weight introduced by Ziegler.

Theorem (Ziegler)**.** *Let $\mathcal{D}$ be a definable subcategory of the category of right R-modules. If there exists a superdecomposable pure-injective module in $\mathcal{D}$, then* $\operatorname{width} \mathcal{D} = \infty$*. Moreover, if R is countable and* $\operatorname{width} \mathcal{D} = \infty$*, then there exists a superdecomposable pure-injective module in $\mathcal{D}$.*