THE DERIVED CATEGORY
OF A GRADED COMMUTATIVE NOETHERIAN RING

BASED ON THE TALKS
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1. Introduction

The following theorem is a motivation for our main result.

Theorem (Neeman). *If R is a commutative noetherian ring, then there exists an inclusion preserving bijection between the subsets of the spectrum* $\operatorname{Spec} R$ *of R and the localizing subcategories of the derived category $\mathcal{D}(R)$ of R, which restricts to a bijection between the specialization closed subsets of* $\operatorname{Spec} R$ *and the smashing subcategories of $\mathcal{D}(R)$.*

The above theorem implies the following corollaries.

Corollary. *If R is a commutative noetherian ring, then the telescope conjecture for R holds, i.e. each smashing subcategory $\mathcal{L}$ of $\mathcal{D}(R)$ is the smallest localizing subcategory of $\mathcal{D}(R)$ containing the intersection of $\mathcal{L}$ with the subcategory $\mathcal{D}(R)^c$ of the compact objects in $\mathcal{D}(R)$.*

Corollary (Hopkins Theorem). *If R is a commutative noetherian ring, then there exists a bijection between the specialization closed subsets of* $\operatorname{Spec} R$ *and the thick subcategories of $\mathcal{D}(R)^c$.*

Thomason has proved that the latter corollary holds for arbitrary commutative rings and schemes. On the other hand, Keller has given an example showing that the former corollary is not true for arbitrary commutative rings.

2. Graded rings

In this section we fix a finitely generated abelian group G and a G-graded ring R. If $\varepsilon : G \times G \to \mathbb{Z}/2$ is bilinear and symmetric, then we say that R is ε-commutative if

$$r \cdot s = (-1)^{\varepsilon(g,h)} \cdot s \cdot r$$

for any homogeneous elements r and s of R of degrees g and h, respectively. For example, if $\varepsilon = 0$, then R is ε-commutative if and only if R is commutative. On the other, hand, if $G = \mathbb{Z}$ and ε is the multiplication map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ composed with the canonical map $\mathbb{Z} \to \mathbb{Z}/2$, then R is ε-commutative if and only if R is graded commutative. Finally, if

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$G = \mathbb{Z}/2$ and ε is the multiplication map $\mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2$, then R is ε-commutative if and only if R is a commutative superalgebra.

If R is ε-commutative, then we have the following important properties:

(1) Localization of modules work fine if we consider multiplicative systems which consist of the homogenous elements of even degrees.
(2) There exists the inner tensor product $\otimes$, which induces the derived tensor product $\otimes_R^L$ in $\mathcal{D}(R)$.
(3) The compact objects in $\mathcal{D}(R)$ coincide with the strongly dualizable objects in $\mathcal{D}(R)$, hence $\mathcal{D}(R)^c$ is a tensor triangulated category.
(4) $\mathcal{D}(R)$ is the smallest localzing subcategory of $\mathcal{D}(R)$ containing all the twists of R by the elements of G.

3. Result

Let G be a finitely generated abelian group and let R be a G-graded ε-commutative (graded) noetherian ring. Under these assumptions the homogeneous spectrum $\operatorname{Spec}^h(R)$ of R is noetherian and Matlis theory of injectives applies. Since $\mathcal{D}(R)$ is not generated by R in general, we can only classify localizing tensor ideals in $\mathcal{D}(R)$. However, the localizing tensor ideals in $\mathcal{D}(R)$ coincide with the twist closed localizing subcategories of $\mathcal{D}(R)$.

For an object X of $\mathcal{D}(R)$ we denote by $\operatorname{ssupp} X$ the subsets of $\operatorname{Spec}^h(R)$ consisting of those $\mathfrak{p} \in \operatorname{Spec}^h(R)$ that $X \otimes_R^L (R/\mathfrak{p})_{(\mathfrak{p})} \neq 0$.

Theorem. *The assignment*

$$\mathcal{L} \mapsto \bigcup_{X \in \mathcal{L}} \operatorname{ssupp} X$$

induces an inclusion preserving bijection between the twist closed localizing subcategories of $\mathcal{D}(R)$ and the subsets of $\operatorname{Spec}^h(R)$ – the inverse map is given by the assignment

$$S \mapsto \{X \in \mathcal{D}(R) : \operatorname{ssupp} X \subseteq S\}.$$

4. Application

Neeman's result implies that for every noetherian quasiaffine scheme U the localizing subcategories of $\mathcal{D}(\operatorname{QCoh}(U))$ are classified by the subsets of U. We give an analogous interpretation of our result.

Let G be a finitely generated abelian group and let R be a G-graded noetherian commutative ring. We fix a closed subset Z of $\operatorname{Spec}^h R$ and denote its complement by U. Let $\mathcal{A}'$ be the subcategory of the category of graded R-modules consisting of the modules M such that $M_{(\mathfrak{p})} = 0$ for all $\mathfrak{p} \in U$. Let $\mathcal{A}$ be the quotient of the category of the graded R-modules by $\mathcal{A}'$, i.e. the objects of $\mathcal{A}$ are the graded R-modules and for two graded R-modules M and N the homomorphism space $\operatorname{Hom}_{\mathcal{A}}(M, N)$ equals

$$\operatorname{colim}_{\substack{M' \subseteq M,\, N' N \\ M/M', N' \in \mathcal{A}'}} \operatorname{Hom}(M', N/N').$$

We have the following consequence of our main result.

Corollary. *In the above situation there exists a bijection between the subsets of U and the localizng tensor ideals of $\mathcal{D}(\mathcal{A})$, which restricts to a bijection between the specialization closed subsets of U and the thick tensor ideals of $\mathcal{D}(\mathcal{A})$.*

We present two more concrete instances of the above result.

First, assume that $G = \mathbb{Z}$ and R is non-negatively graded and generated by R_1 over R_0. If Z is the set of the homogeneous prime ideals of R containing $\bigoplus_{i>0} R_i$, then by a result of Serre we get that $\mathcal{A}$ is equivalent to the category $\operatorname{QCoh}(\operatorname{Proj} R)$. Consequently, it follows that the subsets of $\operatorname{Proj} R$ parameterize the tensor ideals of $\mathcal{D}(\operatorname{Proj} R)$.

Now assume that $G = \mathbb{Z}$ and R is non-negatively graded and finitely generated over R_0, which is a field k. The grading on R induces an action of $k^\times$ on $\operatorname{Spec} R$ and $\bigoplus_{i>0} R_i$ corresponds to a fixed point x of this action. If $Z := \{x\}$, then $\mathcal{A}$ is equivalent to the category of the quasi-coherent sheaves on the stack $(\operatorname{Spec} R \setminus \{x\})/k^\times$.