THE COMBINATORICS AROUND AN INFINITE IDEMPOTENT MATRIX

BASED ON THE TALK BY DOLORS HERBERA

First we recall the following theorem.

Theorem (Prihoda)**.** *Let R be a ring and P and Q be projective right R-modules. Then P and Q are isomorphic if and only if $P/P \cdot J(R) \simeq Q/Q \cdot J(R)$, where $J(R)$ is the Jacobson radical of R.*

Let E be a countable column-finite idempotent matrix with coefficients in a ring R. If $P := E \cdot R^{(\mathbb{N})}$, then P is a projective right R-module. Moreover, every countably generated projective right R-module is of this form. Let

$$I := \sum_{i,j \in \mathbb{N}} R \cdot E(i,j) \cdot R.$$

We call the ideals of this form trace ideals. For $k \in \mathbb{N}$ we put

$$I'_k := \sum_{\substack{i,j \in \mathbb{N} \\ j \leq k}} R \cdot E(i,j).$$

Since $E^2 = E$, for each $k \in \mathbb{N}$ there exists $l > k$ such that $I'_l \cdot I'_k = I'_k$. Consequently, there exists a chain

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$$

of finitely generated left ideals such that

$$I_{n+1} \cdot I_n = I_n$$

for each $n \in \mathbb{N}$ and

$$I = \bigcup_{n \in \mathbb{N}} I_n.$$

We have the following theorem.

Theorem (Whitehead)**.** *A two-sided ideal I of a ring R is a trace ideal if and only if there exists a chain*

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$$

of finitely generated left ideals such that

$$I_{n+1} \cdot I_n = I_n$$

for each $n \in \mathbb{N}$ and

$$I = \bigcup_{n \in \mathbb{N}} I_n.$$

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As a consequence of the above theorem we obtain the following facts:

(1) If I and I' are trace ideals, then $I = I'$ if and only if $I + J(R) = I' + J(R)$.
(2) If I is a trace ideal and $R/J(R)$ is noetherian, then $I = L \cdot R$ for a finitely generated left ideal L such that $L^2 = L$.
(3) If I is a trace ideal and M is a countably generated right projective R/I-module, then there exists a countable column-finite matrix E such that $M \simeq P/P \cdot I$ and
$$I = \sum_{i,j \in \mathbb{N}} R \cdot E(i,j) \cdot R,$$
where $P := E \cdot R^{(\mathbb{N})}$.

Now assume again that E is a countable column-finite idempotent matrix with coefficients in a ring R and $P := E \cdot R^{(\mathbb{N})}$. For $k \in \mathbb{N}$ we put
$$L'_k := \sum_{\substack{i,j \in \mathbb{N} \\ i \geq k}} E(i,j) \cdot R.$$
Using again the assumption $E^2 = E$ we obtain that for each $k \in \mathbb{N}$ there exists $l > k$ such that
$$L'_l \cdot L'_k = L'_l.$$
Consequently, there exists a chain
$$L_0 \supseteq L_1 \supseteq L_2 \supseteq \cdots$$
of right ideals such that
$$L_{n+1} \cdot L_n = L_{n+1}$$
for each $n \in \mathbb{N}$. Moreover, $P/P \cdot L_n$ is finitely generated for every $n \in \mathbb{N}$ and for every ideal K such that $P/P \cdot K$ is finitely generated there exists $n \in \mathbb{N}$ such that $K \supseteq L_n$. There exists unique minimal element in the set $\mathcal{I}(P)$ of ideals K of R such that $P/P \cdot K$ is finitely generated if and only if the sequence
$$L_0 \supseteq L_1 \supseteq L_2 \supseteq \cdots$$
becomes stationary.

Assume now that for every chain
$$L_0 \supseteq L_1 \supseteq L_2 \supseteq \cdots$$
of right ideals such that
$$L_{n+1} \cdot L_n = L_{n+1}$$
for each $n \in \mathbb{N}$, becomes stationary. This condition is satisfied for example for integral group rings of finite groups. For each countably generated projective R-module P we denote by L_P the minimal element in $\mathcal{I}(P)$.

Theorem (Prihoda)**.** *Assume in addition that R is noetherian. If P and Q are countably generated, then P and Q are isomorphic if and only if $L_P = L_Q$ and $P/P \cdot L_P \simeq Q/Q \cdot L_P$.*

For a ring R we denote by $V(R)$ the set of the isomorphism classes of a countably generated projective right R-modules. It is a commutative monoid with an addition given by direct sums.

Now assume that R is semilocal, i.e.

$$R/J(R) \simeq \mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_k}(D_k)$$

for some division rings $D_1, \ldots, D_k$ and positive integers $n_1, \ldots, n_k$. For each $i \in [1, k]$ we fix a simple $\mathbb{M}_{n_i}(D_i)$-module V_i. Then for each projective right R-module P there exist sets $I_1, \ldots, I_k$ such that

$$P/P \cdot J(R) \simeq V_1^{(I_1)} \oplus \cdots \oplus V_k^{(I_k)}.$$

Thus we may associate to P the element $(|I_1|, \ldots, |I_k|)$ the monoid $(\mathbb{N}^*)^k$, where $\mathbb{N}^* := \mathbb{N} \cup \{\infty\}$. This assignment induces a monomorphism $\dim : V(R) \to (\mathbb{N}^*)^k$ of monoids. The following theorem describes the submonoids of $(\mathbb{N}^*)^k$ which can be obtained as the images of the maps of the above form.

Theorem (Herbera/Prihoda)**.** *Let $n_1, \ldots, n_k$ be positive integers. If M is a submonoid of $(\mathbb{N}^*)^k$ such that $(n_1, \ldots, n_k) \in M$, then there exists a semilocal noetherian ring R such that*

$$R/J(R) \simeq M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$$

for some division rings $D_1, \ldots, D_k$ and $\dim(V(R)) = M$ if and only if there exists matrices D, E_1 and E_2 with nonnegative integral coefficients and a vector m with nonnegative integral coefficients such that

$$M = \{x \in (\mathbb{N}^*)^k : E_1 \cdot x = E_2 \cdot x \text{ and } D \cdot x \in \mathbb{N}^* \cdot m\}.$$

As an application of the above theorem we obtain that there exists a semilocal noetherian ring R such that

$$R/J(R) \simeq M_2(D_1) \times D_2$$

for some division rings D_1 and D_2 and

$$\dim(V(R)) = \langle (2, 1), (\infty, \infty), (0, \infty), (1, \infty) \rangle.$$

Indeed,

$$\langle (2, 1), (\infty, \infty), (0, \infty), (1, \infty) \rangle = \{(x, y) \in (\mathbb{N}^*)^2 : x + 2 \cdot y = 3 \cdot y\}.$$

Similarly,

$$\langle (2, 1), (\infty, \infty), (0, \infty) \rangle = \{(x, y) \in (\mathbb{N}^*)^2 : x + 2 \cdot y = 3 \cdot y \text{ and } 2 \mid x\}.$$