THE GRADED LIE ALGEBRA ON THE HOCHSCHILD COHOMOLOGY OF A MODULAR GROUP ALGEBRA

BASED ON THE TALK BY SELENE SANCHEZ

Throughout the presentation k is a field.

Let A be a k-algebra. For $n \in \mathbb{N}$ we define $C^n(A)$ by

$$C^n(A) := \begin{cases} A & n = 0, \\ \operatorname{Hom}_k(A^{\otimes n}, A) & n \in \mathbb{N}_+. \end{cases}$$

If $n, m \in \mathbb{N}$, $f \in C^n(A)$, $g \in C^m(A)$, and $i \in [1, n]$, then we define $f \circ_i g \in C^{n+m-1}(A)$ by

$$(f \circ_i g)(a_1 \otimes \cdots \otimes a_{n+m-1}) := \\ f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_{n+m-1})$$

$(a_1, \ldots, a_{n+m-1} \in A)$. Finally, if $n, m \in \mathbb{N}$, $f \in C^n(A)$ and $g \in C^m(A)$, then we put

$$f \circ g := \sum_{i \in [1,n]} (-1)^{(i-1)\cdot(m-1)} f \circ_i g$$

and

$$[f, g] := f \circ g - (-1)^{(n-1)\cdot(m-1)} g \circ f.$$

The above gives the Gerstenhaber bracket

$$[-, -] : \mathrm{HH}^n(A) \times \mathrm{HH}^m(A) \to \mathrm{HH}^{n+m-1}(A),$$

where, for $n \in \mathbb{N}$, $\mathrm{HH}^n(A)$ denotes the n-th Hochschild cohomology group of A, i.e. $\mathrm{HH}^n(A) := \operatorname{Ext}^n_{A \otimes A^{\mathrm{op}}}(A, A)$. Recall that

$$\mathrm{HH}^1(A) = \operatorname{Der}(A) / \operatorname{Der}^0(A),$$

where $\operatorname{Der}(A)$ is the set of the derivations of A and $\operatorname{Der}^0(A)$ is the set of the inner derivations of A, i.e. $\operatorname{Der}^0(A) := \{D_a : a \in A\}$, where

$$D_a(x) := a \cdot x - x \cdot a \qquad (a, x \in A).$$

Then the Gerstenhaber bracket $\mathrm{HH}^1(A) \times \mathrm{HH}^1(A) \to \mathrm{HH}^1(A)$ is induced by the commutator bracket given by

$$[D, D'] := D \circ D' - D' \circ D \qquad (D, D' \in \operatorname{Der}(A)).$$

For the rest of the presentations we assume that $p := \operatorname{char} k > 2$. Let G be a cyclic group of order p with generator g and $A := kG$. Then

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$\mathrm{HH}^1(A) = \mathrm{Der}(A)$ is the Witt algebra $\mathcal{W}$, i.e. the Lie algebra with a basis $(D_i : i \in [-1, p-2])$ such that

$$[D_i, D_j] = (j - i) \cdot D_j$$

for all $i, j \in [-1, p-2]$. Moreover, $\mathrm{HH}^n(A) = A \otimes k\beta_n$ for some β_n for each $n \in \mathbb{N}_+$. Finally, if $i, j \in [0, p-1]$ and $n, m \in \mathbb{N}_+$, then

$$[g^i \otimes \beta_n, g^j \otimes \beta_m] = \begin{cases} (j-i) \cdot g^{i+j} \otimes \beta_{n+m-1} & 2 \nmid n, m, \\ j \cdot g^{i+j} \otimes \beta_{n+m-1} & 2 \nmid n \text{ and } 2 \mid m, \\ 0 & 2 \mid n, m. \end{cases}$$

In particular, $\mathrm{HH}^{\mathrm{odd}}(A) \simeq \mathcal{W}(t)$. Moreover, if $n \in \mathbb{N}_+$, then $\mathrm{HH}^n(A)$ is the adjoint $\mathcal{W}$-module provided $2 \nmid n$, and the standard $\mathcal{W}$-module provided $2 \mid n$.