ON QUIVER FLAG VARIETIES

BASED ON THE TALK BY JULIA SAUTER

Throughout the presentation K is an algebraically closed field.

Let Q be a quiver and $\mathbf{d} \in \mathbb{N}^{Q_0}$. By $R(\mathbf{d})$ we denote the space of the representations of Q of dimension vector $\mathbf{d}$. By a $\mathbf{d}$-admissible sequence of length $\nu \in \mathbb{N}_+$ we mean every sequence $\mathbb{d} = (\mathbf{d}^1, \ldots, \mathbf{d}^\nu) \in (\mathbb{N}^{Q_0})^\nu$ such that $\mathbf{d}^\nu = \mathbf{d}$ and $\mathbf{d}^k \leq \mathbf{d}^{k+1}$ for each $k \in [1, \nu - 1]$. For a $\mathbf{d}$-admissible sequence $\mathbb{d}$ of length ν we put

$$F(\mathbb{d}) := \{U = (U^k)_{k \in [1,\nu]} : U^\nu = K^{\mathbf{d}}, \text{ and } U^k \subseteq U^{k+1} \text{ and } \mathbf{dim}\, U^k = \mathbf{d}^k \text{ for each } k \in [1, \nu - 1]\}$$

and

$$\mathrm{RF}(\mathbb{d}) := \{(M, U) \in R(\mathbf{d}) \times F(\mathbb{d}) : U^k \text{ is a subrepresentation of } M \text{ for each } k \in [1, \nu]\}.$$

Moreover, we put

$$\langle \mathbb{d}, \mathbb{d} \rangle := \sum_{k \in [1,\nu]} \langle \mathbf{d}^k - \mathbf{d}^{k-1}, \mathbf{d}^k \rangle.$$

In the above situation we have the canonical projections

$$\pi_{\mathbb{d}} : \mathrm{RF}(\mathbb{d}) \to R(\mathbf{d}) \qquad \text{and} \qquad \mu_{\mathbb{d}} : \mathrm{RF}(\mathbb{d}) \to F(\mathbb{d}).$$

For $M \in R(\mathbf{d})$ we put $\mathrm{Fl}\binom{M}{\mathbb{d}} := \pi_{\mathbb{d}}^{-1}(M)$ (these schemes are called quiver flag varieties).

Now we present basic properties of the above construction.

Lemma 1. *Let Q be a quiver, $\mathbf{d} \in \mathbb{N}^{Q_0}$, and $\mathbb{d}$ be a $\mathbf{d}$-admissible sequence.*

1. *$\pi_{\mathbb{d}}$ is projective, hence $\mathrm{Fl}\binom{M}{\mathbb{d}}$ is a projective scheme for each $M \in R(\mathbf{d})$.*
2. *μ is a vector bundle, hence $\mathrm{RF}(\mathbb{d})$ is smooth of dimension*

$$\langle \mathbb{d}, \mathbb{d} \rangle + \sum_{i \in Q_0} d_i^2.$$

Lemma 2. *Let Q be a quiver, $\mathbf{d} \in \mathbb{N}^{Q_0}$, and $\mathbb{d}$ be a $\mathbf{d}$-admissible sequence. If $\mathrm{Im}\, \pi_{\mathbb{d}} = \overline{\mathcal{O}}_M$ for some $M \in R(d)$, then $\mathrm{Fl}\binom{M}{\mathbb{d}}$ is smooth and irreducible of dimension*

$$\dim \mathrm{RF}(\mathbb{d}) - \dim R(\mathbf{d}) + \dim_K \mathrm{Ext}^1_Q(M, M).$$

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Let Q be a quiver and $\mathbf{d} \in \mathbb{N}^{Q_0}$. If $M \in R(\mathbf{d})$ and $\mathbb{d}$ is a $\mathbf{d}$-admissible sequence, then we call $(M, \mathbb{d})$ a resolution pair if $\pi_{\mathbb{d}}$ is a resolution of singularities of $\overline{\mathcal{O}}_M$ (in particular, $\operatorname{Im} \pi_{\mathbb{d}} = \overline{\mathcal{O}}_M$).

Lemma 3. *Let Q be a quiver and $\mathbf{d} \in \mathbb{N}^{Q_0}$. If $M \in R(\mathbf{d})$ and $\mathbb{d}$ is a $\mathbf{d}$-admissible sequence, then $(M, \mathbb{d})$ is a resolution pair if and only if $\operatorname{Fl}\binom{M}{\mathbb{d}} \neq \varnothing$ and $\langle \mathbb{d}, \mathbb{d} \rangle = \dim_K \operatorname{End}_Q(M)$.*

Corollary 4. *Let Q be a quiver and $M_1, \ldots, M_\nu$, $\nu \in \mathbb{N}_+$, representations of Q. If*

- *$\operatorname{Hom}_Q(M_l, M_k) = 0$ for all $l, k \in [1, \nu]$ such that $l < k$, and*
- *$\operatorname{Ext}^1_Q(M_l, M_k) = 0$ for all $l, k \in [1, \nu]$ such that $l \geq k$,*

then $(M, \mathbb{d})$ is a resolution pair, where $M := \bigoplus_{k=1}^{\nu} M_k$ and

$$\mathbf{d}^k := \sum_{l \in [1,k]} \mathbf{dim}\, M_l \qquad (k \in [1, \nu]).$$

In particular, the above corollary implies that if Q is a Dynkin quiver, then for each representation M of Q there exists an admissible sequence $\mathbb{d}$ such that $(M, \mathbb{d})$ is a resolution pair. On the other hand, if

$$Q = \bullet \leftleftarrows \bullet \qquad \text{and} \qquad M = K \underset{1}{\overset{1}{\leftleftarrows}} K\ ,$$

then $\langle \mathbb{d}, \mathbb{d} \rangle = 0$ for each $\mathbf{dim}\, M$-admissible sequence $\mathbb{d}$ such that $\operatorname{Fl}\binom{M}{\mathbb{d}} \neq \varnothing$, hence there is no resolution pair of the form $(M, \mathbb{d})$. Next, if

$$Q = \underset{1}{\bullet} \leftleftarrows \underset{2}{\bullet} \qquad \text{and} \qquad M = P_2 \oplus I_1,$$

then $(M, \mathbb{d})$, where $\mathbb{d} := (\mathbf{dim}\, I_1, \mathbf{dim}\, M)$, is a resolution pair. However, Zwara has showed that $\overline{\mathcal{O}}_M$ is not normal, so $\pi_{\mathbb{d}}$ is not a crepant resolution.

Let Q be a quiver, $\mathbf{d} \in \mathbb{N}^{Q_0}$, and $\mathbb{d}$ is a $\mathbf{d}$-admissible sequence of length ν. If $M \in R(\mathbf{d})$, then $\operatorname{Fl}\binom{M}{\mathbb{d}}$ is in general neither irreducible nor reduced (not even generically). Wolf has showed that if $M \in R(\mathbf{d})$ and $U \in \operatorname{Fl}\binom{M}{\mathbb{d}}$, then

$$T_U \operatorname{Fl}\binom{M}{\mathbb{d}} = \operatorname{Hom}_{Q \otimes A_\nu}(U, M/U)$$

and we have a short exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}_{Q \otimes A_\nu}(U, M/U) &\to T_U \operatorname{RF}(\mathbb{d}) \\ &\to T_U(d) \to \operatorname{Ext}^1_{Q \otimes A_\nu}(U, M/U) \to 0, \end{aligned}$$

where

$$A_\nu := \underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{\nu-1}} \underset{\nu}{\bullet}$$

and we identify M with the Q-representation

$$M \xrightarrow{\mathrm{Id}_M} M \xrightarrow{\mathrm{Id}_M} \cdots \xrightarrow{\mathrm{Id}_M} M$$

of A_ν.

Theorem 5. *Let Q be a connected quiver and $\nu \in \mathbb{N}_+$.*

(1) *$Q \otimes A_\nu$ is of finite type if and only if one of the following conditions is satisfied:*
   - *Q is of Dynkin type and $\nu = 1$,*
   - *Q is of one of the types $\mathbb{A}_3$ and $\mathbb{A}_4$, and $\nu \leq 2$,*
   - *Q is of type $\mathbb{A}_2$ and $\nu \leq 4$,*
   - *Q is of type $\mathbb{A}_1$.*

(2) *$Q \otimes A_\nu$ is of infinite tame type if and only if one of the following conditions is satisfied:*
   - *Q is of Euclidean type and $\nu = 1$,*
   - *Q is of one of the types $\mathbb{A}_5$ and $\mathbb{D}_4$, and $\nu = 2$,*
   - *Q is of type $\mathbb{A}_3$ and $\nu = 3$.*

Let Q be a a quiver and $\nu \in \mathbb{N}_+$. By $\mathbb{X}(Q, \nu)$ we denote the full subcategory of the category of the Q-representations of A_ν consisting of the representations U such that U^{α_i} is a monomorphism for each $i \in [1, \nu - 1]$. Note that if $\mathbf{d} \in \mathbb{N}^{Q_0}$ and $\mathbb{d}$ is a $\mathbf{d}$-admissible sequence of length ν, then the $\mathrm{GL}(\mathbf{d})$-orbits in $\mathrm{RF}(\mathbb{d})$ correspond to the isomorphism classes of the objects in $\mathbb{X}(Q, \nu)$ of dimension vector $\mathbb{d}$. Moreover, if $M \in R(\mathbf{d})$, then the $\mathrm{Aut}_K(M)$-orbits in $\mathrm{Fl}\binom{M}{\mathbb{d}}$ correspond to the isomorphism classes of the objects U in $\mathbb{X}(Q, \nu)$ such that $U^\nu \simeq M$.

Theorem 6. *Let Q be a connected quiver and $\nu \in \mathbb{N}_+$. Then $\mathbb{X}(Q, \nu)$ is of finite type if and only if one the following conditions is satisfied:*

- *Q is of Dynkin type and $\nu = 1$,*
- *Q is of type $\mathbb{A}_4$, or $Q = A_5$, or Q or Q^{op} is the quiver*

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  and $\nu \leq 2$,
- *Q is of type $\mathbb{A}_3$ and $\nu \leq 3$,*
- *$Q = A_3$ and $\nu \leq 4$,*
- *Q is of one of the types $\mathbb{A}_1$ and $\mathbb{A}_2$.*