ON AUSLANDER–REITEN COMPONENTS FOR SELFINJECTIVE ALGEBRAS

BASED ON THE TALK BY DAN ZACHARIA

Throughout the talk R is a finite dimensional selfinjective algebra over an algebraically closed field K.

1. Motivation

Let M be an R-module. For $i \in \mathbb{N}$ we denote by $\beta_i(M)$ the i-th Betti number of M defined by $\beta_i(M) := \dim_K P_i$, where

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

is the minimal projective resolution of M. By the complexity $\operatorname{cx} M$ of M we mean

$$\min\{n \in \mathbb{N} : \text{there exists } c > 0 \text{ such that}$$
$$\beta_i(M) \leq c \cdot i^{n-1} \text{for each } i \in \mathbb{N}_+\},$$

where $\min \varnothing := \infty$. One shows that

$$\operatorname{cx} M = \min\{n \in \mathbb{N} : \text{there exists } c > 0 \text{ such that}$$
$$\dim_K \tau^i M \leq c \cdot i^{n-1} \text{ for each } i \in \mathbb{N}_+\},$$

Observe that $\operatorname{cx} M = 0$ if and only if $\operatorname{pd}_R M < \infty$ (thus M is projective). Similarly, $\operatorname{cx} M = 1$ if and only if the Betti numbers of M are bounded. For instance, if M is periodic, i.e. $\Omega^n M \simeq M$ for some $n \in \mathbb{N}_+$, then $\operatorname{cx} M = 1$. Eisenbud proved that if R is a group algebra, then $\operatorname{cx} M = 1$ if and only if M is periodic. Moreover, if R is a group algebra, then $\operatorname{cx} M < \infty$. Inspired by a result of Webb, who described the Auslander–Reiten components for the group algebras, we prove the following theorem.

Theorem 1.1 (Kerner/Zacharia)**.** *Let $\mathcal{C}$ be an Auslander–Reiten component of $\operatorname{mod} R$ such that $\operatorname{cx}(\mathcal{C}) < \infty$. If $\mathcal{C}$ is not τ-periodic, then $\mathcal{C}_s$ is of type $\mathbb{Z}\Delta$, where Δ is either Euclidean or infinite Dynkin. Moreover, if $\mathcal{C}$ is stable, then Δ is infinite Dynkin.*

If, in the situation of the above theorem, Δ is Euclidean, then $\operatorname{cx}(\mathcal{C}) = 2$.

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2. Ω-perfect modules

By an irreducible pair in $\operatorname{mod} R$ we mean every pair (B, C) of non-projective indecomposable R-modules such that $\operatorname{Irr}_R(B, C) \neq 0$. Such a pair (B, C) is called Ω-perfect if either $\dim_K \Omega^n B > \dim_K \Omega^n C$ for each $n \in \mathbb{N}$ or $\dim_K \Omega^n B < \dim_K \Omega^n C$ for each $n \in \mathbb{N}$. Similarly, a non-projective indecomposable R-module M is called Ω-perfect if each irreducible pair (B, C) is Ω-perfect. Finally, we say that a non-projective indecomposable R-module M is eventually Ω-perfect if there exists $n \in \mathbb{N}$ such that $\Omega^n M$ is Ω-perfect, and we define eventually Ω-perfect irreducible pairs similarly.

If (B, C) is an irreducible pair such that $\dim_K B > \dim_K C$, then $\dim_K \Omega B < \dim_K \Omega C$ if and only if $\dim_K B - \dim_K C = 1$. Moreover, if this is the case, $g \in \operatorname{Irr}_R(B, C)$, and $h \in \operatorname{Irr}_B(\Omega B, \Omega C)$, then $\operatorname{Ker} g \simeq \operatorname{Coker} h$. Obviously, $\operatorname{Ker} g$ is a simple R-module in the above situation. Green and Zacharia proved that if there are no periodic simple R-modules, then every R-module is eventually Ω-perfect. Moreover, they also showed that if $M \in \operatorname{mod} R$, $\operatorname{cx} M = 1$, and M is not τ-periodic, then M is eventually Ω-perfect. Next, if $\mathcal{C}$ is an Auslander–Reiten component of $\operatorname{mod} R$ of type $\mathbb{Z}\mathbb{A}_\infty$ and $M \in \partial\mathcal{C}$, then M is eventually Ω-perfect. Finally, if $\mathcal{C}$ is an Auslander–Reiten component of $\operatorname{mod} R$ of type $\mathbb{Z}\mathbb{A}_\infty^\infty$, $B, C \in \mathcal{C}_s$, and (B, C) is an Ω-perfect irreducible pair, then $\dim_K B > \dim_K C$.

3. The proof of theorem

Let $\mathcal{C}$ be an Auslander–Reiten component of $\operatorname{mod} R$ with $\operatorname{cx}(\mathcal{C}) < \infty$. If every non-projective module in $\mathcal{C}$ is eventually Ω-perfect, then we obtain our claim by studying the Auslander–Reiten sequences ending at Ω-perfect modules of finite complexity. In the other case, the claim follows from the following theorem.

Theorem 3.1. *If $\mathcal{C}$ is an Auslander–Reiten component of* $\operatorname{mod} R$ *containing a non-projective module which is not Ω-perfect, then $\mathcal{C}$ is of type $\mathbb{Z}\Delta$, where Δ is either $\mathbb{A}_\infty^\infty$ or $\mathbb{D}^\infty$, or of one of the types $\tilde{\mathbb{A}}$ or $\tilde{\mathbb{D}}$.*

Proof. Our assumptions imply that there exist a periodic simple R-module S and an irreducible pair (B, C) such that $B, C \in \mathcal{C}$ and we have an exact sequence of the form

$$0 \to S \to B \xrightarrow{g} C \to 0$$

with $g \in \operatorname{Irr}_R(B, C)$. Fix $k, m \in \mathbb{N}_+$ such that

$$\Omega^k S = S \qquad \text{and} \qquad \nu^m S = S,$$

and put

$$W := \bigoplus_{i \in [1,k]} \bigoplus_{j \in [1,m]} \Omega^i \nu^j S.$$

Then $\tau W \simeq W$. If we put $d_W(M) := \dim_K \underline{\mathrm{Hom}}_R(M, W)$ $(M \in \mathcal{C}_s)$, then by a result of Erdmann and Skowroński d_W is an additive function on $\mathcal{C}_s$, which is constant on the τ-orbits. Consequently, it follows by a result of Happel, Preiser and Ringel, that $\mathcal{C}$ is of type $\mathbb{Z}\Delta$, where Δ is either Euclidean or infinite Dynkin. Now we finish the proof by eliminating the cases $\mathbb{A}_\infty$ and $\tilde{\mathbb{E}}$. □

We finish with the following.

Proposition 3.2. *Let M be a non-projective indecomposable R-module. If there exists a non-projective indecomposable R-module B such that (B, M) and $(\tau M, B)$ are irreducible pairs which are not Ω-perfect, then* $\operatorname{cx} M = 2$.

Proof. Without loss of generality we may assume that

$$\dim_K B = \dim_K M + 1.$$

Fix non-zero $g \in \mathrm{Irr}_R(B, M)$ and put $S := \operatorname{Ker} g$. Then S is Ω-periodic, hence

$$\alpha := \max\{\dim_K \Omega^i S : i \in \mathbb{N}\} < \infty.$$

Moreover,

$$\dim_K \Omega^i M - \dim_K \Omega^i B \leq \alpha \qquad \text{and} \qquad \dim_K \Omega^i B - \dim_K \Omega^i \tau M \leq \alpha$$

for each $i \in \mathbb{N}$. Recall that $\dim_K \Omega^i \tau M = \dim_K \Omega^{i+2} M$ for each $i \in \mathbb{N}$, since $\tau = \nu\Omega^2$ and ν preserves dimensions. Consequently,

$$\dim_K \Omega^{i+2} M - \dim_K \Omega^i M \leq 2 \cdot \alpha$$

for each $i \in \mathbb{N}$. This implies that

$$\dim_K \tau^n M = \dim_K \Omega^{2 \cdot n} M \leq 2 \cdot n \cdot \alpha + \dim_K M$$

for each $n \in \mathbb{N}$, thus $\operatorname{cx} M \leq 2$. Moreover, $\operatorname{cx} M \neq 0$, since M is not projective, and $\operatorname{cx} M \neq 1$, since M is not eventually Ω-perfect. □