RAY CATEGORIES, IV

BASED ON THE TALK BY DIETER VOSSIECK

Assumption.

Throughout the talk k is a fixed algebraically closed field.

Definition.

A locally bounded category A is called distributive if its lattice of twosided ideals is distributive.

Remark.

A locally bounded category A is distributive if and only if the following equivalent conditions hold:

(1) for all objects x and y of the category A the morphism space $A(x, y)$ is a uniserial $A(y)$-$A(x)$-bimodule,

(2) for each object x of the category A the endomorphism ring $A(x)$ is uniserial and for for all objects x and y of the category A the morphism space $A(x, y)$ is a cyclic module over the rings $A(x)$ and $A(y)$.

Remark.

A locally bounded category A is not distributive if and only if the category A has a full subcategory with a residue category of one of the following forms:

$$\begin{tikzcd} \bullet \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd}, \qquad \alpha^2, \alpha\beta, \beta\alpha, \beta^2,$$

$$\begin{tikzcd} \bullet & \bullet \arrow[l, bend right, "\alpha"'] \arrow[l, bend left, "\beta"] \end{tikzcd},$$

$$\begin{tikzcd} \bullet \arrow[loop left, "\alpha"] & \bullet \arrow[l, "\beta"'] \arrow[loop right, "\gamma"] \end{tikzcd}, \qquad \alpha^2, \alpha\beta\gamma, , \gamma^2.$$

Definition (Kupisch).

For a distributive category A we denote by A^s the associated stem category defined as follows. The objects of the category A^s coincide with the objects of the category A. If x and y are objects of the category A, then the morphism space $A^s(x, y)$ consists of the sub-$A(y)$-$A(x)$-bimodules of $A(x, y)$. Finally, if x, y and z are objects of the category A, and U and V are subbimodules of $A(x, y)$ and $A(y, z)$, respectively,

Date: 11.09.2009.

then we define the composition $V \circ U$ by

$$V \circ U := VU := \Big\{ \sum_{i \in [1,n]} v_i u_i \mid n \in \mathbb{N} \ \wedge \ \forall i \in [1,n] : v_i \in V \wedge u_i \in U \Big\}.$$

EXAMPLE.

If A is the following category

$$\begin{tikzcd} \bullet & \bullet \arrow[l, "\alpha"'] \arrow[loop above, "\sigma"] & \bullet \arrow[l, "\beta"'] \end{tikzcd}, \quad \alpha\beta, \sigma^2,$$

then the stem category A^s is not a ray category.

DEFINITION.

Let x and y be objects of a distributive category A. By a ray in $A(x,y)$ we mean an orbit in $A(x,y)$ under the action of the group $A(x)^\times \times A(y)^\times$ defined as follows:

$$(r,s) * u := sur^{-1}$$

for $r \in A(x)^\times$, $s \in A(y)^\times$, and $u \in A(x,y)$.

LEMMA.

Let x, y and z be objects of a distributive category A, and $\vec{u}$ and $\vec{v}$ rays in $A(x,y)$ and $A(y,z)$, respectively. Then $\vec{v}\vec{u}$ is either a ray or a subbimodule of $A(x,z)$.

DEFINITION (BAUTISTA/GABRIEL/ROITER/SALMERON).

For a distributive category A we denote by $\vec{A}$ the associated ray category defined as follows. The objects of the category $\vec{A}$ coincide with the objects of the category A. If x and y are objects of the category A, then the morphism space $\vec{A}(x,y)$ consists of the rays in $A(x,y)$. Finally, if x, y and z are objects of the category A, and $\vec{u}$ and $\vec{v}$ are rays in $A(x,y)$ and $A(y,z)$, respectively, then we define the composition $\vec{v} \circ \vec{u}$ by

$$\vec{v} \circ \vec{u} := \begin{cases} \vec{v}\vec{u} & \text{if } \vec{v}\vec{u} \text{ is a ray,} \\ 0 & \text{otherwise.} \end{cases}$$

REMARK.

If X is a ray category, then the categories X and $k(\vec{X})$ are isomorphic.

DEFINITION.

For a ray category X and a 2-cocycle $f \in Z^2(X, k^\times)$ we define the twisted linearization $k^f(X)$ as follows. The category $k^f(X)$ has the same objects and morphisms as the category $k(X)$. Moreover, if ν and μ are composable morphisms in the category X, then we define composition $\nu \circ_f \mu$ by

$$\nu \circ_f \mu := f(\nu, \mu)\nu\mu.$$

REMARK.

If X is a ray category and f is a 2-coboundary, then the categories $k^f(X)$ and $k(X)$ are isomorphic.

LEMMA.

Let A be a distributive category such that $\vec{A}$ is either locally finite or minimal representation infinite. If $\operatorname{char} k \neq 2$, then there exists a 2-cocycle f such that the categories A and $k^f(\vec{A})$ are isomorphic.

PROPOSITION.

Let A be a distributive category such that $\vec{A}$ is either locally finite or minimal representation infinite. If $\operatorname{char} k \neq 2$, then A and $k^f(\vec{A})$ are isomorphic.

PROOF.

It is enough to use the fact that the group $H^2(\vec{A}, k^\times)$ is trivial in the above situation.

THEOREM.

Let A be a distributive category. Assume that $\operatorname{char} k \neq 2$.

(1) The category A is locally representation finite if and only if the category $\vec{A}$ is locally representation finite.

(2) The category A is minimal representation infinite if and only if the category $\vec{A}$ is minimal representation infinite.

In particular, in the above situations the categories A and $k(\vec{A})$ are isomorphic.

PROOF.

The crucial step of the proof is to show that the category $\vec{A}$ is representation finite provided the category A is representation finite. However, it follows by induction on $\dim A$ that the category $\vec{A}$ can be at most minimal representation infinite in the above situation, and the claim follows form the previous proposition.

REMARK.

The statements (1) and (2) of the above theorem are also valid if $\operatorname{char} k = 2$. Moreover, if $\operatorname{char} k = 2$, then a distributive category A is isomorphic to the linearization of its ray category, provided the category A is minimal representation infinite.