A BRIEF INTRODUCTION TO DERIVED CATEGORIES OVER DIFFERENTIAL GRADED ALGEBRAS

BASED ON THE TALK BY DONG YANG

§1. Differential graded algebras and categories

Definition.

By a differential graded algebra we mean a graded algebra $A = \bigoplus_{i \in \mathbb{Z}} A^i$ together with a differential $d : A \to A$ such that $d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} \cdot a \cdot d(b)$, i.e. d induces a chain map $A \otimes A \to A$.

Definition.

By a differential graded category we mean a category $\mathscr{A}$ such that for all objects X and Y of the category $\mathscr{A}$ the space $\operatorname{Hom}_{\mathscr{A}}(X, Y)$ is a complex and for all objects X, Y and Z of the category $\mathscr{A}$, the composition map $\operatorname{Hom}_{\mathscr{A}}(Y, Z) \otimes \operatorname{Hom}_{\mathscr{A}}(X, Y) \to \operatorname{Hom}_{\mathscr{A}}(X, Z)$ is a chain map.

Notation.

If $\mathscr{A}$ is a differential graded category, then by $Z^0\mathscr{A}$ we denote the category with the same objects as the category $\mathscr{A}$ and such that $\operatorname{Hom}_{Z^0\mathscr{A}} := Z^0 \operatorname{Hom}_{\mathscr{A}}$.

Notation.

If $\mathscr{A}$ is a differential graded category, then by $H^0\mathscr{A}$ we denote the category with the same objects as the category $\mathscr{A}$ and such that $\operatorname{Hom}_{H^0\mathscr{A}} := H^0 \operatorname{Hom}_{\mathscr{A}}$.

Definition.

By a differential graded functor between differential graded categories $\mathscr{A}$ and $\mathscr{B}$ we mean a functor $F : \mathscr{A} \to \mathscr{B}$ such that $F(X, Y) : \operatorname{Hom}_{\mathscr{A}}(X, Y) \to \operatorname{Hom}_{\mathscr{B}}(FX, FY)$ is a chain map for all objects X and Y of the category $\mathscr{A}$.

Notation.

If $F : \mathscr{A} \to \mathscr{B}$ is a differential graded functor, then we define $Z^0F : Z^0\mathscr{A} \to Z^0\mathscr{B}$ and $H^0F : H^0\mathscr{A} \to H^0\mathscr{B}$ in the obvious way.

§2. Differential graded modules

Definition.

By a differential graded module over a differential algebra A we mean

Date: 24.07.2009.

a graded module A together with a differential $d : M \to M$ such that the induced map $M \otimes A \to A$ is a chain map.

Notation.

If M is a differential graded module, then we define its shift $M[1]$ by $M[1]^i := M^{i+1}$ for $i \in \mathbb{Z}$ and $d_{M[1]} := -d_M$.

Notation.

If M and N are differential graded modules, then by $\mathcal{H}om_A(M, N)$ we denote the complex such that $\mathcal{H}om_A(M, N)^i := \mathrm{Hom}_{\mathrm{Grmod}\, A}(M, N[i])$ for $i \in \mathbb{Z}$ and the differential d is given by the formula $d(f) := d_N \circ f - (-1)^{|f|} f \circ d_M$.

Example.

If M is a differential graded module over a differential graded algebra A, then the complexes $\mathcal{H}om_A(A, M)$ and M are isomorphic.

Notation.

For a differential graded algebra A we denote by $\mathcal{D}\mathit{iff}\, A$ the category whose objects are the differential graded A-modules and the morphism spaces are given by $\mathcal{H}om_A$.

Lemma.

If A is a differential graded algebra, then the category $\mathcal{D}\mathit{iff}\, A$ is a differential graded category and the functor $[1] : \mathcal{D}\mathit{iff}\, A \to \mathcal{D}\mathit{iff}\, A$ is a differential graded functor.

Notation.

For a differential graded algebra A we put $\mathscr{C}(A) := Z^0\, \mathcal{D}\mathit{iff}\, A$ and $\mathscr{H}(A) := H^0\, \mathcal{D}\mathit{iff}\, A$.

Remark.

If M is a differential graded module over a differential graded algebra A, then $\mathrm{Hom}_{\mathscr{C}(A)}(A, M) = Z^0 M$ and $\mathrm{Hom}_{\mathscr{H}(A)}(A, M) = H^0 M$.

Lemma.

If A is a differential graded algebra, then the category $\mathscr{C}(A)$ is a Frobenius category and $\mathscr{H}(A) = \underline{\mathscr{C}(A)}$. In particular, the category $\mathscr{H}(A)$ is triangulated with the suspension functor given by [1].

§3. The derived category

Definition.

A differential graded module is called acyclic if $H^i N = 0$ for all $i \in \mathbb{Z}$.

Notation.

For a differential graded algebra A we denote by $\mathrm{acyc}(A)$ the full subcategory of the category $\mathscr{H}(A)$ formed by the acyclic modules.

Remark.

If A is a differential graded algebra, then the category $\operatorname{acyc}(A)$ is a triangulated subcategory of the category $\mathscr{H}(A)$.

Notation.

For a differential graded algebra A, we define its derived category $\mathscr{D}(A)$ by $\mathscr{D}(A) := \mathscr{H}(A)/\operatorname{acyc}(A)$.

Remark.

If A is a differential graded algebra, then the category $\mathscr{D}(A)$ is triangulated and the canonical projection $\mathscr{H}(A) \to \mathscr{D}(A)$ is a triangle functor.

Remark.

If A is a differential graded algebra, then the canonical projection $\mathscr{H}(A) \to \mathscr{D}(A)$ has a left adjoint $\mathscr{D}(A) \to \mathscr{H}(A)$.

Remark.

If A is a differential graded algebra, then the category $\mathscr{D}(A)$ has arbitrary direct sums.

§4. Morita theorem for triangulated categories

Definition.

An object T of a triangulated category $\mathscr{T}$ with arbitrary direct sums is called compact if the canonical map

$$\operatorname{Hom}_{\mathscr{T}}(T, \bigoplus_{i\in I} M_i) \to \bigoplus_{i\in I} \operatorname{Hom}_{\mathscr{T}}(T, M_i)$$

is an isomorphism for all objects M_i, $i \in I$, of the category $\mathscr{T}$.

Definition.

An object T of a triangulated category $\mathscr{T}$ with arbitrary direct sums is called a compact generator if the object T is compact and $\{X \in \mathscr{T} \mid \forall i \in \mathbb{Z} : \operatorname{Hom}_{\mathscr{T}}(T, \Sigma^i X) = 0\} = 0$.

Lemma.

If A is a differential graded algebra, then the differential graded module A is a compact generator of the category $\mathscr{D}(A)$.

Theorem.

Let $\mathscr{T}$ be an algebraic triangulated category with arbitrary direct sums. If T is a compact generator of the category $\mathscr{T}$, then there exists a differential graded algebra A and a triangle equivalence $F : \mathscr{T} \to \mathscr{D}(A)$ such that $FT = A$.

§5. The standard functor

Lemma.

Let A and B be differential graded algebras and $F : \mathscr{D}(B) \to \mathscr{D}(A)$ a triangle functor which commutes with arbitrary direct sums. Then the functor F is an equivalence if and only if the object FB is a compact generator of the category $\mathscr{D}(A)$.

Notation.

Let A and B be differential graded algebras. For a differential graded B-A-bimodule M we put

$$- \otimes_B^{\mathbb{L}} M := \pi \circ (- \otimes_B M) \circ \mathbf{p} : \mathscr{D}(B) \to \mathscr{D}(A),$$

where $\pi : \mathscr{H}(A) \to \mathscr{D}(A)$ is the canonical projection and $\mathbf{p} : \mathscr{D}(B) \to \mathscr{H}(B)$ is the left adjoint to the canonical projection $\mathscr{H}(B) \to \mathscr{D}(B)$.

Definition.

A differential graded module M over a differential graded algebra A is called K-projective if the canonical projection $\mathscr{H}(A) \to \mathscr{D}(A)$ induces an isomorphism $\mathrm{Hom}_{\mathscr{H}(A)}(M, -) \to \mathrm{Hom}_{\mathscr{D}(A)}(M, -)$.

Lemma.

Let M be a K-projective differential graded module over a differential graded algebra A. Then the functor $- \otimes_B^{\mathbb{L}} M : \mathscr{D}(B) \to \mathscr{D}(A)$ is an equivalence if and only if the module M is a compact generator of the category $\mathscr{D}(A)$ and the canonical map $B \to \mathcal{H}om_A(M, M)$ is a quasi-isomorphism.

Corollary.

Let M be a K-projective differential graded module over a differential graded algebra A. If the module M is a compact generator of the category $\mathscr{D}(A)$, then the functor

$$- \otimes_{\mathcal{H}om_A(M,M)}^{\mathbb{L}} M : \mathscr{D}(\mathcal{H}om_A(M, M)) \to \mathscr{D}(A)$$

is an equivalence.

Corollary.

If there exists a quasi-isomorphism $B \to A$ is of differential graded algebras, then the functor $- \otimes_B^{\mathbb{L}} A : \mathscr{D}(B) \to \mathscr{D}(A)$ is an equivalence.

§6. Rickard's theorem

Theorem.

Let $\mathscr{T}$ be an algebraic triangulated category with arbitrary direct sums and let T be a compact generator of the category $\mathscr{T}$ such that $\mathrm{Hom}_{\mathscr{T}}(T, \Sigma^i T) = 0$ for all $i \in \mathbb{Z}$, $i \neq 0$. Then the categories $\mathscr{T}$ and $\mathscr{D}(\mathrm{End}_{\mathscr{T}}(T))$ are triangle equivalent.