FROM TRIANGULATED CATEGORIES TO CLUSTER ALGEBRAS — AFTER PALU

BASED ON THE TALK BY DONG YANG

Definition.

A triangulated category $\mathscr{C}$ is called 2-Calabi-Yau if there exist functorial isomorphisms

$$\mathrm{D}\,\mathrm{Hom}_{\mathscr{C}}(X,Y) \simeq \mathrm{Hom}_{\mathscr{C}}(Y,\Sigma^2 X)$$

for all objects X and Y of the category $\mathscr{C}$.

Assumption.

For the rest of the talk we assume that $\mathscr{C}$ is a fixed triangulated category, which is Hom-finite, Krull–Schmidt, and 2-Calabi–Yau.

Definition.

By a cluster character we mean every function $X : \mathrm{Ob}\,\mathscr{C} \to A$, where A is a commutative algebra, such that for all objects M and N of the category $\mathscr{C}$ the following conditions are satisfied:

(1) if $M \simeq N$, then $X_M = X_N$,
(2) $X_{M\oplus N} = X_M \cdot X_N$,
(3) if $\dim_k \mathrm{Hom}_{\mathscr{C}}(M,\Sigma N) = 1$, then $X_{M\oplus N} = X_E + X_F$, where E and F are defined by the following nonsplit triangles $N \to E \to M \to \Sigma N$ and $M \to F \to N \to \Sigma M$.

Assumption.

For the rest of the section we fix a cluster tilting object T in the category $\mathscr{C}$, and put $B := \mathrm{End}_{\mathscr{C}}(T)$ and $F := \mathrm{Hom}_{\mathscr{C}}(T,-)$.

Remark.

Keller and Reiten proved that the functor F induces an equivalence between the categories $\mathscr{C}/\Sigma T$ and $\mathrm{mod}\,B$.

Notation.

For B-modules M and N we put

$$\langle M,N\rangle := \dim_k \mathrm{Hom}_B(M,N) - \dim_k \mathrm{Ext}^1_B(M,N) - \dim_k \mathrm{Hom}_B(N,M) + \dim_k \mathrm{Ext}^1_B(N,M).$$

Proposition.

The linear form $\langle -,-\rangle$ descends to the Gorthendieck group of the category $\mathrm{mod}\,B$.

Date: 17.07.2009.

Remark.

Keller and Reiten proved that for each object M of the category $\mathscr{C}$ there exists a triangle $M \to \Sigma^2 T^0 \to \Sigma^2 T^1 \to \Sigma M$ with $T_0, T_1 \in \operatorname{add} T$.

Definition.

For an object M of the category $\mathscr{C}$ we define the coindex $\operatorname{coind} M$ as the sequence $\mathbf{m} \in \mathbb{Z}^{Q_0}$, where Q is the Gabriel quiver of the algebra B, such that $\mathbf{dim}\, FT^0 - \mathbf{dim}\, FT^1 = \sum_{i \in Q_0} m_i \,\mathbf{dim}\, P_i$, where T_0 and T_1 are defined as in the previous remark.

Notation.

For an object M of the category $\mathscr{C}$ we put

$$X_M^T := \mathbf{x}^{-\operatorname{coind} M} \cdot \sum_{\mathbf{e}} \chi(\operatorname{Gr}_{\mathbf{e}} FM) \cdot \prod_{i \in Q_0} x_i^{\langle \mathbf{e}_i, \mathbf{e} \rangle}.$$

Theorem (Palu).

The function X^T is a cluster character.