MUTATION ORBITS OF TRIPLES WITH MARKOV CONSTANT 4

BASED ON THE TALK BY ANDRE BEINEKE

Notation.

We define maps $\mu_1, \mu_2, \mu_3 : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\mu_1(x, y, z) := (yz - x, y, z), \qquad \mu_2(x, y, z) := (x, xz - y, z),$$

and

$$\mu_3(x, y, z) := (x, y, xy - z).$$

Moreover, we denote by $\mathscr{M}$ the subgroup of $\mathfrak{S}(\mathbb{R}^3)$ generated by μ_1, μ_2, μ_3.

Definition.

We say that $(x, y, z) \in \mathbb{R}^3$ is *cyclic* if $x, y, z > 0$.

Definition.

We say that $(x, y, z) \in \mathbb{R}^3$ is *mutation cyclic* if the elements of $\mathscr{M} \cdot (x, y, z)$ are cyclic.

Notation.

We define maps $\pi_1, \pi_2, \pi_3 : \mathbb{R}^3 \to \mathbb{R}$ by

$$\pi_1(x, y, z) := x, \qquad \pi_2(x, y, z) := y, \qquad \pi_3(x, y, z) := z.$$

Lemma.

Let $(x, y, z) \in \mathbb{R}^3$.

(1) If $\min(x, y, z) < 2$, then (x, y, z) is not mutation cyclic.
(2) If $\min(x, y, z) \geq 2$, then there exists at most one $k \in \{1, 2, 3\}$ such that $\mu_k(x, y, z) < (x, y, z)$. Moreover, if this is the case, then $\pi_i(x, y, z) < \pi_k(x, y, z)$ for all $i \in \{1, 2, 3\}$, $i \neq k$.

Definition.

We say that $(x, y, z) \in \mathbb{R}^3$ is *minimal* if $\mu_k(x, y, z) \geq (x, y, z)$ for all $k \in \{1, 2, 3\}$.

Proposition.

If $(x, y, z) \in \mathbb{R}^3$ is minimal and cyclic, then (x, y, z) is mutation cyclic.

Date: 03.07.2009.

Definition.

For $(x,y,z) \in \mathbb{R}^3$ we define the *Markov constant* $C(x,y,z)$ by

$$C(x,y,z) := x^2 + y^2 + z^2 - 3xyz.$$

Lemma.

If $(x,y,z) \in \mathbb{R}^3$ is minimal, cyclic, and $x \geq y \geq z$, then

$$C(x,y,z) \leq -(z-2)y^2 + z^2 \leq -z^3 + 3z^2.$$

Proposition.

If $(x,y,z) \in \mathbb{R}^3$ and $x,y,z \geq 2$, then the following conditions are equivalent:

(1) (x,y,z) is mutation cyclic,
(2) $C(x,y,z) \leq 4$,
(3) $m^-(x,y) \leq z \leq m^+(x,y)$, where

$$m^\pm(x,y) := \frac{1}{2}(xy \pm \sqrt{(x^2-4)(y^2-4)}).$$

Corollary.

If $(x,y,z) \in \mathbb{R}^3$ is minimal, cyclic, and $C(x,y,z) = 4$, then up to a permutation (x,y,z) equals $(a,a,2)$ for some $a \geq 2$.

Lemma.

Let $(x,y,z) \in \mathbb{R}^3$ be mutation cyclic such that there is no triple in $\mathscr{M} \cdot (x,y,z)$ which is minimal. Then $\lim_{n\to\infty}(x_n,y_n,z_n) = (2,2,2)$ for a (unique) decreasing sequence (x_n,y_n,z_n) such that $(x_0,y_0,z_0) = (x,y,z)$ and, for each $n \in \mathbb{N}$, $(x_{n+1},y_{n+1},z_{n+1}) = \mu_k(x,y,z)$ for some $k \in \{1,2,3\}$.

Definition.

We define polynomials P_n, $n \in \mathbb{N}$, by

$$P_0 := 2, \qquad P_1 := X, \qquad P_{n+1} := X \cdot P_n - P_{n-1},\ n \in \mathbb{N}_+.$$

Lemma.

We have the following.

(1) $\deg P_n = n$ for each $n \in \mathbb{N}$.
(2) $P_{n+m} = P_n P_m - P_{n-m}$ for all $n,m \in \mathbb{N}$ such that $n \geq m$.
(3) $P_{n\cdot m} = P_m(P_n)$ for all $n,m \in \mathbb{N}$.

Proposition.

If $a \in \mathbb{R}$, $a \geq 2$, then the elements of $\mathscr{M} \cdot (a,a,2)$ are, up to a permutation, the triples $(P_{n+m}(a), P_n(a), P_m(a))$ for $n,m \in \mathbb{N}$ with $\gcd(n,m) = 1$.

Corollary.

Let $(x,y,z) \in \mathbb{R}$ be mutation cyclic with $C(x,y,z) = 4$. Then $\mathscr{M} \cdot (x,y,z)$ contains a minimal element if and only if $P_n(y) = P_m(z)$ for some $n,m \in \mathbb{N}$.