AUSLANDER-REITEN THEORY FOR MODULES OF FINITE COMPLEXITY OVER SELFINJECTIVE ALGEBRAS

BASED ON THE TALK BY DAN ZACHARIA

The talk was based on joint work with Ed Green.

Throughout the talk R is a selfinjective algebra over a field K.

Let M be a finitely generated R-module. For $i \in \mathbb{N}$ we define the i-th Betti number $\beta_i(M)$ of M by

$$\beta_i(M) := \dim_K \operatorname{Ext}^i_R(M, R/\operatorname{rad} R).$$

We define the complexity $\operatorname{cx} M$ of M by

$$\operatorname{cx} M := \inf\{d \in \mathbb{N} \mid \text{there exists } c \in \mathbb{R} \text{ such that } \beta_i(M) \leq c \cdot i^{d-1} \text{ for all } i \gg 0\},$$

where the infimum of the empty set equals ∞.

Observe that $\operatorname{cx} M = 0$ if and only if M is projective. Next, $\operatorname{cx} M = 1$ if and only if the Betti numbers of M are bounded. Moreover, if M is either Ω or τ-periodic, then $\operatorname{cx} M = 1$. If R is the group algebra of a finite group, then $\operatorname{cx} M < \infty$ for each R-module M.

It is known that $\operatorname{cx}(\Omega M) = \operatorname{cx} M$ and $\operatorname{cx}(\tau M) = \operatorname{cx} M$ for each R-module M. If $0 \to A_1 \to A_2 \to A_3 \to 0$ is an exact sequence, then

$$\operatorname{cx} A_i \leq \max\{\operatorname{cx} A_j \mid j \in \{1, 2, 3\} \setminus \{i\}\}$$

for each $i \in \{1, 2, 3\}$. The above properties imply, that if $\mathscr{C}$ is a connected component of the Auslander–Reiten quiver of R, then there exists $d \in \mathbb{N}$ such that $\operatorname{cx} M = d$ for all M in $\mathscr{C}$ which are nonprojective.

The following example is due to Rainer Schulz. Let

$$A := k\langle x, y\rangle/(x^2, y^2, xy + qyx),$$

where $q \neq 0$ and q is not a root of 1. If $M := A/(x + qy)$, then $\beta_i(M) = 1$ for all $i \in \mathbb{N}$ and M is not Ω-periodic (but M is τ-periodic).

Conjecture.

Assume that R is local.

(1) If $\operatorname{cx} M = 1$ for an R-module M, then the Betti numbers of M are eventually periodic.

(2) If the Betti numbers of an R-module M are eventually periodic, then they are eventually constant.

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THEOREM.

Let $\mathscr{C}$ be a regular component of the Auslander–Reiten quiver of R. If there exists M in $\mathscr{C}$ such that $\operatorname{cx} M = 1$, then $\mathscr{C}$ is of type $\mathbb{Z}\mathbb{A}_\infty/\langle\tau^i\rangle$ for some $i \in \mathbb{N}$.

THEOREM.

Let $\mathscr{C}$ be a regular component of the Auslander–Reiten quiver of R of type $\mathbb{Z}\mathbb{A}_\infty/\langle\tau^i\rangle$ for some $i \in \mathbb{N}$. If there exists M in $\mathscr{C}$ such that $\beta_i(M) = b$ for some $b \in \mathbb{N}$ and all $i \gg 0$, then for each $t \in \mathbb{N}_+$ there exists N in $\mathscr{C}$ such that $\beta_i(N) = t \cdot b$ for all $i \gg 0$.

For a nonprojective R-module M we denote by $\alpha(M)$ the number of nonprojective indecomposable direct summands of the almost split sequence ending in M.

THEOREM.

Assume that R has no periodic simple modules. If $\operatorname{cx} M < \infty$ for an R-module M, then $\alpha(M) \leq 4$.

For each homomorphism $f : M \to N$ we choose $\Omega f : \Omega M \to \Omega N$ using the isomorphism $\underline{\operatorname{Hom}}(M, N) \simeq \underline{\operatorname{Hom}}(\Omega M, \Omega N)$. Observe that if f is an irreducible homomorphism, then Ωf is irreducible as well. We say that a homomorphism f is Ω-perfect if either $\Omega^n f$ is an epimorphism for each $n \in \mathbb{N}$ or $\Omega^n f$ is a monomorphism for each $n \in \mathbb{N}$.

LEMMA.

Let $f : B \to A$ be an irreducible epimorphism. Then f is Ω-perfect if and only if $\Omega^n \operatorname{Ker} f$ is not simple for each $n \in \mathbb{N}$.

PROOF.

Put $C := \operatorname{Ker} f$.

If C is simple, then one easily shows that Ωf is a monomorphism.

Now assume that C is not simple. It suffices to show that $\operatorname{rad} C = \operatorname{rad} B \cap C$. Obviously, $\operatorname{rad} C \subset \operatorname{rad} B \cap C$. In order to prove the reverse inclusion is suffices to show for each indecomposable direct summand S of $C/\operatorname{rad} C$ that $\pi \circ \gamma = 0$, where $\gamma : \operatorname{rad} B \cap C \to C$ is the inclusion map and $\pi : C \to S$ is the projection map. Let $\iota : C \to B$ be the inclusion map. Since f is irreducible either there exists $g : S \to B$ such that $\iota = g \circ \pi$ or there exists $h : B \to S$ such that $\pi = h \circ \iota$. However, the former possibility cannot hold, since ι is a monomorphism, while π is not (note that S is simple and C is not). Consequently,

$$\pi \circ \gamma = h \circ \iota \circ \gamma = h \circ \beta \circ \iota',$$

where $\iota' : \operatorname{rad} B \cap C \to \operatorname{rad} B$ and $\beta : \operatorname{rad} B \to B$ are the inclusion maps. Observe that $h \circ \beta = 0$, since S is simple, hence the claim follows.