ON TENSOR PRODUCTS OF TOTALLY ACYCLIC COMPLEXES OF FINITELY GENERATED FREE MODULES

BASED ON THE TALK BY DAVID JORGENSEN

The talk was based on joint work with Lars W. Christiansen.

Throughout the talk R is always a noetherian commutative ring. By $(-)^*$ we denote the duality $\operatorname{Hom}_R(-, R)$.

A finitely generated R-module M is called totally reflexive if the following conditions are satisfied:

(1) The canonical map $M \to M^{**}$ is an isomorphism.
(2) $\operatorname{Ext}^i_R(M, R) = 0$ for each $i \in \mathbb{N}_+$.
(3) $\operatorname{Ext}^i_R(M^*, R) = 0$ for each $i \in \mathbb{N}_+$.

A complex A of finitely generated free R-modules is called totally acyclic if $H(A) = 0$ and $H(A^*) = 0$.

Fact.

A finitely generated R-module M is totally reflexive if and only if there exists a totally acyclic R-complex A such that $M = \operatorname{Im} d^A_0$.

Let A be a totally acyclic R-complex and $M := \operatorname{Im} d^A_0$. If N is an R-module, then we define the Tate Tor by

$$\hat{\operatorname{Tor}}^R_i(M, N) := H_i(A \otimes_R N)$$

for $i \in \mathbb{Z}$. Observe that

$$\hat{\operatorname{Tor}}^R_i(M, N) := \operatorname{Tor}^R_i(M, N)$$

for each $i \in \mathbb{N}$.

For complexes A and B of R-modules we define the tensor product $A \otimes_R B$ of A and B by

$$(A \otimes B)_n := \bigoplus_{\substack{p,q \in \mathbb{Z}\\ p+q=n}} A_p \otimes B_q$$

for $n \in \mathbb{Z}$, and

$$d^{A\otimes B}_n(a \otimes b) := d^A_p(a) \otimes b + (-1)^p \cdot a \otimes d^B_q(b),$$

for $n \in \mathbb{Z}$, $a \in A_p$, $b \in B_q$, and $p, q \in \mathbb{Z}$, such that $p + q = n$.

Date: 22.05.2009.

For a complex A of R-modules and $n \in \mathbb{Z}$ we denote by $A_{\leq n}$ and $A_{\geq n}$ the truncated complexes. Moreover, by ΣA we denote the shifted complex.

If A and B are complexes of R-modules, then we defined the complex $A \boxtimes_R B$ by the following conditions:

$$(A \boxtimes_R B)_{\geq 0} = A_{\geq 0} \otimes_R B_{\geq 0}, \qquad (A \boxtimes_R B)_{\leq 0} = \Sigma(A_{\leq -1} \otimes_R B_{\leq -1}),$$

and

$$d_0^{A \otimes B} := d_0^A \otimes d_0^B.$$

Lemma.

Let A and B be totally acyclic R-complexes, and $M := \operatorname{Im} d_0^A$ and $N := \operatorname{Im} d_0^B$. Then

$$\hat{\operatorname{Tor}}_i^R(M, N) = H_i(A \boxtimes_R B).$$

Theorem.

Let M and N be totally reflexive R-modules such that $\hat{\operatorname{Tor}}_i^R(M, N) = 0$ for all $i \in \mathbb{Z}$. Then $M \otimes_R N$ is totally reflexive.