CONCEALED ALGEBRAS

BASED ON THE TALK BY OTTO KERNER

Throughout the talk k is a fixed algebraically closed field. All considered quivers have no oriented cycles. For a vertex x of a quiver Q we denote by P_x the corresponding projective representation of Q.

A vertex x of a quiver Q is called a tip if there is precisely one arrow adjacent to x.

Let P be an indecomposable projective representation of a quiver Q of infinite representation type. We define the orbit $\mathscr{O}(P)$ by

$$\mathscr{O}(P) := \{\tau^{-n}P \mid n \in \mathbb{N}\}.$$

We call $\mathscr{O}(P)$ a monoorbit if each nonzero $f \in \operatorname{Hom}(X, Y)$, with $X, Y \in \mathscr{O}(P)$, is injective.

Theorem (Kerner/Takane).

The following conditions are equivalent for a connected quiver Q of infinite representation type.

(1) All preprojective tilting representations are slice modules.
(2) $\mathscr{O}(P)$ is a monoorbit for each indecomposable projective representation of Q.
(3) Q has no tips.

Proof ((1) $\Rightarrow$ (3)).

Let x be a vertex of Q. If x is a tip, then $\operatorname{Hom}(P_x, \tau^{-}P_x) = 0$, hence $P_x \oplus \tau^{-2}P_x$ is a partial tilting representation, which can be extended to a nonslice preprojective tilting representation.

Lemma.

Let P be an indecomposable projective representation of a quiver Q of infinite representation type. If $\mathscr{O}(P)$ is a monoorbit, then the following conditions are satisfied:

(1) each nonzero $f \in \operatorname{Hom}(X, Y)$, with $X \in \mathscr{O}(P)$ and Y preprojective, is injective,
(2) for each nonzero $f \in \operatorname{Hom}(X, Y)$, with $X, Y \in \mathscr{O}(P)$, $\operatorname{Coker} f$ is regular.

Proof.

(1) Let $f \in \operatorname{Hom}(X, Y)$ be nonzero for $X \in \mathscr{O}(P)$ and Y preprojective. Let $g : X \to \operatorname{Im} f$ be the restriction of f. There exists $m \in \mathbb{N}$ such that

Date: 08.05.2009.

$\mathrm{Hom}(U, \tau^{-m}X) \neq 0$. Fix $h \in \mathrm{Hom}(U, \tau^{-m}X)$, $h \neq 0$. By assumption $h \circ g$ is injective, hence g is injective, and the claim follows.

(2) Let $f \in \mathrm{Hom}(X, Y)$ be nonzero for $X, Y \in \mathscr{O}(P)$. Write $\mathrm{Coker}\, f = P \oplus R \oplus Q$ for preprojective P, regular R, and preinjective Q. By (1) there are no proper nonzero epimorphisms $Y \to Z$ with Z preprojective, hence $P = 0$. Next, if $Q \neq 0$, then by applying τ^- we may assume that $\nu^- Q \neq 0$. Then $\mathrm{Ker}\, \tau^- f = \nu^- Q \neq 0$, hence $\tau^- f$ is not injective — contradiction.

For a connected wild quiver Q we denote put

$$\rho_Q := \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } \Phi_Q\},$$

where Φ_Q is the Coxter transformation. Then ρ_Q is an algebraically simple eigenvalue of Φ_Q which possesses a strictly positively eigenvector.

Theorem (Kerner/Takane).

Let x be a unique sink in a wild connected quiver Q. Then $\mathscr{O}(P_x)$ is a monoorbit if and only if $\eta_x < \eta_y$ for each vertex $y \neq x$, where η is a strictly positively eigenvector of Φ_Q for ρ_Q.

Corollary.

Let x be a unique sink in a connected quiver Q which is either wild or of type $\tilde{\mathbb{A}}$. Then $\mathscr{O}(P_x)$ is a monoorbit if and only if every for each $f \in \mathrm{Hom}(P_x, P_y)$, with y a vertex of Q, $\mathrm{Coker}\, f$ has no nonzero preinjective direct summands.

Proof.

The claim is obvious if Q of type $\tilde{\mathbb{A}}$, hence assume that Q is wild. Take $f \in \mathrm{Hom}(P_x, P_y)$ for a vertex y of Q. Without loss of generality we may assume that $y \neq x$ and $f \neq 0$. Then f is injective and $\mathrm{Coker}\, f$ is indecomposable. Let η be a strictly positively eigenvector of Φ_Q for ρ_Q. Then

$$\langle [\mathrm{Coker}\, f], \eta \rangle = \langle [P_y], \eta \rangle - \langle [P_x], \eta \rangle = \eta_y - \eta_x > 0,$$

and this implies that $\mathrm{Coker}\, f$ is not preinjective.

Let P be an indecomposable representation of a quiver Q of infinite representation type. We say that $\mathscr{O}(P)$ is a strict monoorbit if each exact sequence $0 \to U \to \tau^{-m}P \to V \to 0$, with preprojective V, splits. Observe that each strict monoorbit is obviously a monoorbit.

Lemma.

If x is a vertex of wild connected quiver Q such that $Q \setminus \{x\}$ is representation finite, then $\mathscr{O}(P_x)$ is a strict monoorbit.

Proof.

Put $Q' := Q \setminus \{x\}$. There exists a tilting representation T of Q' which is regular as representation of Q. Fix $m \in \mathbb{N}$. Then

$$(\tau^{-m-1}T)^{\perp} = \mathrm{add}(\tau^{-m}P_x),$$

where for a representation M of Q we put

$$M^{\perp} := \{N \in \operatorname{rep} Q \mid \operatorname{Hom}(M,N) = 0 = \operatorname{Ext}^1(M,N)\}.$$

Let $0 \to U \to \tau^{-m}P \to V \to 0$ be an exact sequence with V preprojective. One easily checks that $U, V \in (\tau^{-m-1}T)^{\perp}$, hence the claim follows.

Proof ((3) $\Rightarrow$ (2)).

Our easily observes that our assumption implies that Q is either wild or of type $\tilde{\mathbb{A}}$. Let x be a vertex of Q. Without loss of generality we may assume that x a unique sink in Q. We show that if $f \in \operatorname{Hom}(P_x, P_y)$, with y a vertex of Q, then $\operatorname{Coker} f$ has no nonzero preinjective direct summands. According to Corollary this will imply our claim.

Let $Q'^{(1)}, \dots, Q'^{(s)}$ be the connected components of $Q \setminus \{x\}$. Moreover, for each $i = 1, \dots, s$ let $Q^{(i)}$ be the full subquiver of Q generated by $Q'^{(i)} \cup \{x\}$. It is sufficient to show for each $i = 1, \dots, s$ we show that if $f \in \operatorname{Hom}(P_x, P_y)$, with y a vertex of $Q^{(i)}$, then $\operatorname{Coker} f$ has no nonzero direct summands, which are preinjective over $Q^{(i)}$.

Fix $i \in \{1, \dots, s\}$. If $Q'^{(i)}$ is a tree, then it follows that $Q^{(i)}$ is of type $\tilde{\mathbb{A}}$, and the claim follows. Otherwise, $Q'^{(i)}$ is of infinite representation type and the claim follows from the previous lemma and Corollary.

Proof ((2) $\Rightarrow$ (1)).

Observe that our assumption implies that $\operatorname{Hom}(P, \tau^{-}mP) \neq 0$ for each indecomposable projective representation P of Q and $m \in \mathbb{N}$. Consequently, if T is a preprojective tilting representation of Q, then $\mathscr{O}(P)$ contains precisely one indecomposable direct summand of T. Using once more our assumption one checks that T must be a slice module.

Remark (Ringel).

There is an easy proof of the implication (3) $\Rightarrow$ (1), which is valid over an arbitrary field.

Observe that this is sufficient to show that all irreducible maps in the preprojective component are injective. This is obviously true for the irreducible maps between projective modules. Moreover, this also follows by induction for the remaining irreducible maps, since the Auslander–Reiten sequences in the preprojective component have decomposable middle terms.