CONTRAVARIANTLY FINITE SUBCATEGORIES CLOSED UNDER PREDECESSORS

BASED ON THE TALK BY FLÁVIO ULHOA COELHO

The talk was based on a joint work with Assem and Trepode.

For an Artin algebra A we define the classes $\mathscr{L}_A$ and $\mathscr{R}_A$ of indecomposable A-modules by

$$\mathscr{L}_A := \{X \in \operatorname{ind} A \mid \operatorname{pd}_A Y \leq 1 \text{ for each predecessor } Y \text{ of } X\}$$

and

$$\mathscr{R}_A := \{X \in \operatorname{ind} A \mid \operatorname{id}_A Y \leq 1 \text{ for each successor } Y \text{ of } X\}.$$

These classes may be used in order to characterize some classes of algebras. Namely, we have the following statements:

- an Artin algebra A is quasi-tilted if and only if $\mathscr{L}_A \cup \mathscr{R}_A = \operatorname{ind} A$ and $\operatorname{gl.dim} A \leq 2$;
- an Artin algebra A is shod if $\mathscr{L}_A \cup \mathscr{R}_A = \operatorname{ind} A$ (consequently, $\operatorname{gl.dim} A \leq 3$);
- an Artin algebra A is weakly shod if $\mathscr{L}_A \cup \mathscr{R}_A$ is cofinite in $\operatorname{ind} A$ and each indecomposable A-module X not in $\mathscr{L}_A \cup \mathscr{R}_A$ is directed;
- an Artin algebra A is a laura algebra if $\mathscr{L}_A \cup \mathscr{R}_A$ is cofinite in $\operatorname{ind} A$.

Let $\mathscr{C}$ be a class of indecomposable modules over an Artin algebra A. By a right $\mathscr{C}$-approximation of an A-module M we mean a morphism $f \in \operatorname{Hom}_A(C, M)$ with $C \in \operatorname{add} \mathscr{C}$ such that $\operatorname{Hom}_A(C', f)$ is onto for each $C' \in \mathscr{C}$. We say that $\mathscr{C}$ is contravariantly finite if each A-module M has a right $\mathscr{C}$-approximation. Dually, we define left $\mathscr{C}$-approximations and covariantly finite classes of indecomposable modules.

If A is a laura algebra which is not quasi-titled, then $\mathscr{L}_A$ is contravariantly finite and $\mathscr{R}_A$ is covariantly finite. An Artin algebra A is called left supported if $\mathscr{L}_A$ is contravariantly finite.

Let A be an Artin algebra A. For a class $\mathscr{C}$ of indecomposable A-modules we denote by $E_{\mathscr{C}}$ the direct sum of Ext-injective objects in $\mathscr{C}$. Similarly, in the above situation $F_{\mathscr{C}}$ denotes the direct sum of the indecomposable projective A-modules which do not belong to $\mathscr{C}$. Finally, by A_λ we denote the endomorphism ring of the direct sum of

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the indecomposable projective A-modules which belong to $\mathscr{L}_A$. It is known that A_λ is a product of quasi-titled algebras.

THEOREM (ASSEM/COELHO/TREPODE).

For an Artin algebra A the following conditions are equivalent.

- A is left supported.
- $E_{\mathscr{L}_A} \oplus F_{\mathscr{L}_A}$ is a tilting module.
- A_λ is a product of tilted algebras and $E_{\mathscr{L}_A}$ restricted to A_λ is the union of complete slices.

A class $\mathscr{C}$ of indecomposable modules over an Artin algebra A is called resolving if the following conditions are satisfied:

- $\mathscr{C}$ is closed under extensions;
- $\mathscr{C}$ is closed under kernels of epimorphisms;
- $\mathscr{C}$ contains all projective A-modules.

Let A be an Artin algebra. For an A-module M we put

$$M^\perp := \{X \in \operatorname{ind} A \mid \operatorname{Ext}_A^i(X, M) = 0 \text{ for all } i \in \mathbb{N}_+\}.$$

Similarly, we define $\mathscr{C}^\perp$ and ${}^\perp\mathscr{C}$ for a class $\mathscr{C}$ of indecomposable A-modules. Moreover, in the above situation we denote by $\check{\mathscr{C}}$ the class of A-modules M such that there exists an exact sequence

$$0 \to C_n \to \cdots \to C_0 \to M \to 0$$

with $C_0, \ldots, C_n \in \operatorname{add} \mathscr{C}$.

THEOREM (AUSLANDER/REITEN).

Let A be an Artin algebra.

The map $M \mapsto M^\perp$ induces a bijection between the isomorphism classes of multiplicity free (generalized) cotilting A-modules and the contravariantly finite resolving subclasses $\mathscr{C}$ of indecomposable A-modules such that $M \in \check{\mathscr{C}}$ for each A-module M. The inverse map is given by $\mathscr{C} \mapsto E_{\mathscr{C}}$.

THEOREM (ASSEM/COELHO/TREPODE).

The following conditions are equivalent for a class $\mathscr{C}$ of indecomposable modules over an Artin algebra A closed under predecessors.

- $\mathscr{C}$ is contravariantly finite and resolving.
- $E_{\mathscr{C}}$ is a (generalized) cotilting module.
- $\mathscr{C}^\perp$ is covariantly finite and $\mathscr{C} = {}^\perp(\mathscr{C}^\perp)$.
- $\operatorname{add} \mathscr{C} = \operatorname{Supp} \operatorname{Hom}_A(-, E_{\mathscr{C}})$ and $E_{\mathscr{C}}$ is sincere.

Moreover, if the above conditions are satisfied, then $\mathscr{C} = {}^\perp E_{\mathscr{C}}$. Finally, if

$$\max\{\operatorname{pd}_A C \mid C \in \mathscr{C}\} < \infty,$$

then the above conditions are equivalent to the condition

- $E_{\mathscr{C}}$ is a (generalized) tilting module.