EXCEPTIONAL COMPONENTS OF WILD HEREDITARY ALGEBRAS

BASED ON THE TALK BY NILS MAHRT

Throughout the talk k will be a fixed algebraically closed field and Q a wild quiver without oriented cycles.

An indecomposable representation X of Q is called regular if $\tau^n X \neq 0$ for all $n \in \mathbb{Z}$. A regular representation X of Q is called quasi-simple if the middle term of the Auslander–Reiten sequence ending at X is indecomposable. For each regular representation X of Q there exists a section path $X_1 \to \cdots \to X_m$ in $\Gamma(Q)$ such that $m \in \mathbb{N}_+$, $X_m = X$, and X_1 is quasi-simple. In the above situation we write $X_1[m] := X$. In this way we obtain a well-defined bijection between the isomorphism classes of regular representations of Q and the pairs (x, m), where x is an isomorphism class of a quasi-simple representation of Q and $m \in \mathbb{N}_+$.

A representation X of Q is called a brick if $\mathrm{End}_Q(X) = k$. A representation X of Q is called a stone if $\mathrm{End}_Q(X) = k$ and $\mathrm{Ext}^1_Q(X, X) = 0$.

Proposition.

Let X be a quasi-simple representation of Q and $m \in \mathbb{N}_+$, $m > 1$. Then the following conditions are equivalent:

(1) $X[m]$ is a brick.
(2) $X[m-1]$ is a stone.
(3) $X, \ldots, \tau^{-m+1}X$ are pairwise orthogonal stones.

Let X and Y be regular representations of Q. Baer proved that $\mathrm{Hom}_Q(X, \tau^r Y) \neq 0$ for all $r \gg 0$. Moreover, Kerner proved that $\mathrm{Hom}_Q(X, \tau^{-r} Y) = 0$ for all $r \gg 0$.

Let $\mathscr{C}$ be a regular component of $\Gamma(Q)$. We define the quasi-rank $\mathrm{rk}(\mathscr{C})$ of $\mathscr{C}$ by

$$\mathrm{rk}(\mathscr{C}) := \min\{n \in \mathbb{N}_+ \mid \mathrm{rad}(X, \tau^{n+l} X) \neq 0 \text{ for all } l \in \mathbb{N} \text{ and quasi-simple } X \in \mathscr{C}\}.$$

Let $\mathscr{C}$ be a regular component of $\Gamma(Q)$ containing a quasi-simple stone X. Then $\mathscr{C}$ is called exceptional if

$$\min\{m \in \mathbb{N}_+ \mid \mathrm{Hom}_Q(X, \tau^m X) \neq 0\} < \mathrm{rk}(\mathscr{C}).$$

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Theorem.

There is only a finite number of exceptional components in $\Gamma(Q)$.

Proposition (Happel/Ringel).

Let X and Y be indecomposable representations of Q. Assume in addition that $\operatorname{Hom}_Q(X, \tau Y) = 0$. If $f \in \operatorname{Hom}_Q(X, Y)$, $f \neq 0$, then either f is a monomorphism or f is an epimorphism.

Proof.

Observe that the map $\operatorname{Ext}^1_Q(\operatorname{Coker} f, X) \to \operatorname{Ext}^1_Q(\operatorname{Coker} f, \operatorname{Im} f)$ induced by $X \to \operatorname{Im} f$ is surjective, hence there exists an exact sequence

$$0 \to X \to Z \to \operatorname{Coker} f \to 0$$

whose push-out along $X \to \operatorname{Im} f$ equals

$$0 \to \operatorname{Im} f \to Y \to \operatorname{Coker} f \to 0.$$

Consequently, we get the exact sequence

$$0 \to X \to \operatorname{Im} f \oplus Z \oplus Y \to 0,$$

which splits, since $\operatorname{Hom}_Q(X, \tau Y) = 0$. Consequently, either $\operatorname{Im} f \simeq X$ or $\operatorname{Im} f \simeq Y$.

Proposition (Unger).

Let X and Y be nonisomorphic stones such that $\operatorname{Hom}_Q(X, \tau Y) = 0$. If $f : X \to Y$ is a monomorphism and $C := \operatorname{Coker} f$, then C is a brick and

$$\dim_k \operatorname{Hom}_Q(X, Y) = 1 + \dim_k \operatorname{Ext}^1_Q(C, C).$$

Proof.

We have the exact sequence

$$(*) \qquad 0 \to X \to Y \to C \to 0.$$

Applying the functor $\operatorname{Hom}_Q(Y, -)$ to this sequence we get the sequence

$$0 = \operatorname{Ext}^1_Q(Y, Y) \to \operatorname{Ext}^1_Q(Y, C) \to \operatorname{Ext}^2_Q(Y, X) = 0,$$

hence $\operatorname{Ext}^1_Q(Y, C) = 0$. Next, applying the functor $\operatorname{Hom}_Q(-, C)$ we get the sequence

$$0 \to \operatorname{End}_Q(C) \xrightarrow{\alpha} \operatorname{Hom}_Q(Y, C) \to \operatorname{Hom}_Q(X, C) \xrightarrow{\beta} \operatorname{Ext}^1_Q(C, C) \to 0,$$

hence, in particular, $\operatorname{Hom}_Q(Y, C) \neq 0$. Applying once more the functor $\operatorname{Hom}_Q(Y, -)$ we get the sequence

$$k = \operatorname{Hom}_Q(Y, Y) \to \operatorname{Hom}_Q(Y, C) \to \operatorname{Ext}^1_Q(Y, X) = 0,$$

hence $\dim_k \operatorname{Hom}_Q(Y, C) = 1$. In particular, α is an isomorphism and $\operatorname{End}_Q(C) = k$. Moreover, it implies that β is also an isomorphism. Finally, we apply the functor $\operatorname{Hom}_Q(X, -)$ and we get the sequence

$$0 \to \operatorname{End}_Q(X) \to \operatorname{Hom}_Q(X, Y) \to \operatorname{Hom}_Q(X, C) \to \operatorname{Ext}^1_Q(X, X) = 0,$$

thus

$$\begin{aligned}\dim_k \operatorname{Hom}_Q(X,Y) &= \dim_k \operatorname{End}_Q(X) + \dim_k \operatorname{Hom}_Q(X,C) \\ &= 1 + \dim_k \operatorname{Ext}^1_Q(C,C).\end{aligned}$$

Corollary.

Let X be a regular stone. If $m \in \mathbb{N}_+$ is such that $\operatorname{Hom}_Q(X, \tau^m X) \neq 0$ and $\operatorname{Hom}_Q(X, \tau^{m+1} X) = 0$, then $\dim_k \operatorname{Hom}_Q(X, \tau^m X) = 1$.

Proof.

Fix $f \in \operatorname{Hom}_Q(X, \tau^m X)$, $f \neq 0$. Without loss of generality we may assume that f is a monomorphism. Put $C := \operatorname{Coker} f$. It is sufficient to show that C is preinjective, i.e. there exists $n \in \mathbb{N}_+$ such that $\tau^{-n} C = 0$. If this is not the case, then for each $n \in \mathbb{N}_+$ we have an exact sequence

$$0 \to \tau^{-mn} X \to \tau^{-m(n-1)} X \to \tau^{-m(n-1)} C \to 0,$$

hence $\dim_k \tau^{-m(n-1)} X > \dim_k \tau^{-mn} X$ for each $n \in \mathbb{N}_+$, contradiction.