

REPRESENTATIONS OF THE ALGEBRAIC GROUP

$\text{Aut}(k[t]/t^p)$

BASED ON THE TALK BY MARKUS SEVERITT

Throughout the talk k will be a field.

By an algebraic group over a field k we mean a functor G from the category of k -algebras to the category of groups which is representable by a Hopf algebra, i.e. there exists a Hopf algebra, denoted $k[G]$, such that $G(R)$ is the set of k -algebra homomorphisms from $k[G]$ to R with the group structure determined by the comultiplication in $k[G]$.

Let \mathbb{G}_a denote the algebraic group which associates to a k -algebra R its additive group. In this case $k[G] = k[x]$ with the comultiplication

$$x \mapsto x \otimes 1 + 1 \otimes x.$$

Similarly, by \mathbb{G}_m we denote the algebraic group which associates to a k -algebra R its multiplicative group. In this case $k[G] = k[x, x^{-1}]$ with the comultiplication

$$x \mapsto x \otimes x.$$

Finally, if V is a finite dimensional vector space, then $\text{GL}(V)$ is the algebraic group which associates to a k -algebra R the group of R -automorphism of $V \otimes_k R$.

By a representation of an algebraic group G we mean a morphism $\rho : G \rightarrow \text{GL}(V)$ of algebraic groups for a finite dimensional vector space V . It follows that the category of representations of an algebraic group G coincides with the category of right $k[G]$ -comodules.

For an algebraic group G we define its Lie algebra $L(G)$ by

$$L(G) := \text{Der}_k(k[G], k).$$

If $G \rightarrow \text{GL}(V)$ is a representation of an algebraic group G , then we have the induced representation $L(G) \rightarrow \text{End}(V)$ of $L(G)$.

Let $C := k[t]/t^p$. By $G := \text{Aut}(C)$ we denote the algebraic group which associates to a k -algebra R the group of R -algebra automorphisms of $R[t]/t^p$.

Fix a k -algebra R . We may identify $\varphi \in G(R)$ with $\varphi(t)$. Under this identification

$$G(R) = \{a_0 + a_1 t + \cdots + a_{p-1} t^{p-1} \mid a_0, \dots, a_{p-1} \in R, a_0^p = 0, a_1 \in R^\times\}.$$

Put

$$G_-(R) := \{a_0 + t \mid a_0 \in R, a_0^p = 0\}, \quad G_0(R) := \{a_1 t \mid a_1 \in R^\times\},$$

and

$$G_+(R) := \{t + a_2 t + \cdots + a_{p-1} t^{p-1} \mid a_2, \dots, a_{p-1} \in R\}.$$

Observe that G_- is isomorphic to the first Frobenius kernel of \mathbb{G}_a . Similarly, G_0 is isomorphic to \mathbb{G}_m . Finally, as a variety G_+ is isomorphic to \mathbb{A}^{p-2} .

LEMMA.

The multiplication map

$$G_- \times G_0 \times G_+ \rightarrow G$$

gives an isomorphism of set valued functors.

The natural inclusion $G \subset \mathrm{GL}(C)$ gives C the structure of a representation of G , which we denote by C .

We introduce in C another structure of a representation of G , which we denote by ΩC (because of its connection with the Kähler module of differentials). If R is a k -algebra, $g \in G(R)$, and $f \in R[t]/t^p$, then

$$g * f := f(g) \cdot g'.$$

We have the morphism $d : C \rightarrow \Omega C$ of G -representations given by

$$d(f) := f'$$

for $f \in R[t]/t^p$ and a k -algebra R . More generally, let $r \in [0, p-1]$. Then by $(\Omega C)^{\otimes_{C^r}}$ we denote the structure of a G -representation in C given by the formula

$$g * f := f(g) \cdot g'^r,$$

where $g \in G(R)$ and $f \in R[t]/t^p$ for a k -algebra R . Let C_r be the subrepresentation of $(\Omega C)^{\otimes_{C^r}}$ generated by 1. Then

$$C_r = \begin{cases} k \cdot 1 & r = 0, \\ \mathrm{Im} d & r = 1, \\ (\Omega C)^{\otimes_{C^r}} & r > 1. \end{cases}$$

In particular, C_0 is isomorphic to the trivial representation of G .

Let $L : G \rightarrow \mathbb{G}_m$ be the representation of G given by

$$L(g) := (g'(0))^p$$

for $g \in G(R)$ and a k -algebra R .

If $n \in \mathbb{Z}$ and $r \in [0, p-1]$, then

$$C_{n \cdot p + r} := L^{\otimes_k n} \otimes_k C_r.$$

PROPOSITION.

The sequence

$$0 \rightarrow C_0 \rightarrow C \xrightarrow{d} \Omega C \xrightarrow{\text{res}} L \rightarrow 0,$$

where

$$\text{res}(t^i) = \delta_{i,p-1}, \quad i \in [0, p-1],$$

is an exact sequence of G -representations.

For $i \in \{-1\} \cup \mathbb{N}$, let ∂_i be the endomorphism of C given by

$$\partial_i(f) := t^{i+1} \cdot f'.$$

Observe that $[\partial_i, \partial_j] = (j-i)\partial_{i+j}$ for all $i, j \in \{-1\} \cup \mathbb{N}$. Obviously, $\partial_i = 0$ for $i \in \mathbb{N}$, $i \geq p$.

LEMMA.

Under the inclusion $L(G) \subset \text{End}(C)$ induced by the inclusion $G \subset \text{GL}(C)$, $L(G)$ has a basis formed by $\partial_{-1}, \dots, \partial_{p-2}$.

For a representation $\rho : G \rightarrow \text{GL}(V)$ and $m \in \mathbb{Z}$ we put

$$V_m := \{v \in V \mid \rho(g) * v = g^m \cdot v \text{ for all } g \in G_0\}.$$

LEMMA.

If $\rho : G \rightarrow \text{GL}(V)$ is a representation of G , then $\partial_i(V_n) \subset V_{n+i}$ for each $n \in \mathbb{Z}$ and $i \in \{-1\} \cup \mathbb{N}$.

If $\rho : G \rightarrow \text{GL}(V)$ is a representation of G and $n \in \mathbb{Z}$, then we say that ρ is pure of weight n if $\text{Ker } \partial_{-1} = V_n$ and V_n generates V as a representation of G .

PROPOSITION.

Let $\rho : G \rightarrow \text{GL}(V)$ be a representation of G . Then V is simple if and only if there exists $n \in \mathbb{Z}$ such that ρ is pure of weight n and $\dim_k V_n = 1$.

PROPOSITION.

Let $\rho : G \rightarrow \text{GL}(V)$ and $\phi : G \rightarrow \text{GL}(W)$ be representations of G , and $n \in \mathbb{Z}$. If ρ and ϕ are of pure weight n and $\dim_k V_n = 1 = \dim_k W_n$, then $\rho \simeq \phi$.

PROPOSITION.

If $n \in \mathbb{Z}$, then C_n is a simple representation of G of pure weight n .

THEOREM (DELIGNE).

The representations C_n , $n \in \mathbb{Z}$, form a complete list of pairwise non-isomorphic simple representations of G .

Let $\text{Rep}(G)$ be the K_0 -group of representations of G . Then $\text{Rep}(G)$ has a structure of a ring with the multiplication given by the tensor product over k .

COROLLARY.

$\text{Rep}(G)$ is generated, as a k -algebra, by $[L]$ and $[(\Omega C)^{\otimes c^r}]$, $r \in [1, p-1]$.

PROOF.

We use that $[C_0] = 1$, $[C_1] = [\Omega C] - [L]$, and $[C_r] = [(\Omega C)^{\otimes c^r}]$ for each $r \in [2, p-1]$.