# QUANTISED BORCHERDS ALGEBRAS AND HALL ALGEBRAS

BASED ON THE TALK BY DIETER VOSSIECK

Assumption.

Throughout the talk $\Gamma$ will be a fixed finite quiver without oriented cycles and $k$ a fixed finite field of cardinality $q$. Moreover, by $\operatorname{ind}\Gamma$ we denote a set of chosen representatives of the isomorphism classes of representations of $\Gamma$ over $q$.

Definition.

Put

$$\mathscr{H} := \mathbb{R}\operatorname{ind}\Gamma$$

Observe that

$$\mathscr{H} = \bigoplus_{\alpha\in\mathbb{N}\Gamma_0} \mathscr{H}_\alpha,$$

where

$$\mathscr{H}_\alpha := \mathbb{R}\{A \in \operatorname{ind}\Gamma \mid \mathbf{dim}\, A = \alpha\}.$$

In $\mathscr{H}$ we introduce the multiplication by

$$A \cdot B = q^{\langle \mathbf{dim}\, A, \mathbf{dim}\, B\rangle/2} \cdot \Big(\sum_{C\in\operatorname{ind}\Gamma} g^C_{A,B} \cdot C\Big),$$

where

$$g^C_{A,B} := \#\{X \subseteq C \mid C/X \simeq A \text{ and } X \simeq B\}$$

and $\langle -, -\rangle$ is the Euler homological form.

In $\mathscr{H}$ we also have the comultiplication $\delta : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ defined by

$$\delta(A) := \sum_{B,C\in\operatorname{ind}\Gamma} q^{\langle \mathbf{dim}\, B, \mathbf{dim}\, C\rangle} \cdot g^A_{B,C} \cdot \frac{|\operatorname{Aut} B| \cdot |\operatorname{Aut} C|}{|\operatorname{Aut} A|} \cdot (B \otimes C),$$

which is coassociative and have the counit $\varepsilon : \mathscr{H} \to \mathbb{R}$ given by

$$\varepsilon(A) := \begin{cases} 1 & A \simeq 0, \\ 0 & A \not\simeq 0. \end{cases}$$

Moreover, if we define the multiplication in $\mathscr{H} \otimes \mathscr{H}$ by

$$(A \otimes B) \cdot (C \otimes D) := q^{(\mathbf{dim}\, B, \mathbf{dim}\, C)/2} \cdot ((A \cdot C) \otimes (B \cdot D)),$$

where

$$(\alpha, \beta) := \langle \alpha, \beta\rangle + \langle \beta, \alpha\rangle,$$

then $\delta$ becomes a homomorphism of algebras.

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Next, if we define the symmetric pairing $\mathscr{H} \times \mathscr{H} \to \mathbb{R}$ by

$$(A \mid B) := \begin{cases} \frac{1}{|\operatorname{Aut} A|} & A \simeq B, \\ 0 & A \not\simeq B, \end{cases}$$

then

$$(A \mid B \cdot C) = (\delta(A) \mid B \otimes C),$$

where

$$(B' \otimes C' \mid B'' \otimes C'') := (B' \mid B'') \cdot (C' \mid C'').$$

For $\alpha \in \mathbb{N}\Gamma_0$, $\alpha \neq 0$, let $\mathscr{H}'_\alpha$ be the orthogonal complement in $\mathscr{H}_\alpha$ of $\sum \mathscr{H}_\beta \cdot \mathscr{H}_\gamma$ (with respect to the above pairing), where the sum runs over all $\beta, \gamma \in \mathbb{N}\Gamma_0$, $\beta, \gamma \neq 0$, such that $\beta + \gamma = \alpha$. Choose an orthonormal basis $(\theta_i)_{i \in I_\alpha}$ in $\mathscr{H}'_\alpha$. Let $I$ be the disjoint union of all $I_\alpha$. For $i, j \in I$ we put

$$(i, j) := (\alpha, \beta)$$

provided $i \in I_\alpha$ and $j \in I_\beta$.

Let $\mathscr{U}_+$ be the $\mathbb{R}$-algebra generated by $E_i$, $i \in I$, and relations

$$E_i \cdot E_j - E_j \cdot E_i = 0$$

for $i, j \in I$ such that $(i, j) = 0$, and

$$\sum_{l \in [0, 1-(i,j)]} (-1)^l \cdot \begin{Bmatrix} 1 - (i, j) \\ i \end{Bmatrix} \cdot E_i^l \cdot E_j \cdot E_i^{1-(i,j)-l} = 0$$

for $i, j \in I$ such that $(i, i) = 2$, where

$$\{n\} := \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}},$$

$$\{n\}! := \begin{cases} 1 & n = 0, \\ \{1\} \cdot \ldots \cdot \{n\} & n > 0, \end{cases}$$

and

$$\begin{Bmatrix} n \\ k \end{Bmatrix} := \frac{\{n\}!}{\{k\}! \cdot \{n-k\}!}.$$

Theorem.

The map $\mathscr{U}_+ \to \mathscr{H}$ given by $E_i \mapsto \theta_i$, $i \in I$, is an isomorphism of algebras.