THE AUSLANDER AND RINGEL/TACHIKAWA THEOREM FOR SUBMODULE EMBEDDINGS

BASED ON THE TALK BY AUDREY MOORE

Assumption.

Throughout the talk P is a fixed poset and Λ is a fixed artin algebra.

Definition.

Let A be an indecomposable object in $\operatorname{Rep}_\Lambda P$. A homomorphism $f : A \to B$ in $\operatorname{Rep}_\Lambda P$is said to be left almost split in $\operatorname{Rep}_\Lambda P$ if the following conditions are satisfied:

(1) f is not a split monomorphism,
(2) if $h : A \to X$ is a non-zero homomorphism in $\operatorname{Rep}_\Lambda P$ which is not a split monomorphism, then h factors through f.

Lemma.

If A is an indecomposable object in $\operatorname{rep}_\Lambda P$, then there exists a left almost split homomorphism $f : A \to B$ in $\operatorname{Rep}_\Lambda P$ with $B \in \operatorname{rep}_\Lambda P$.

Proposition.

If $\operatorname{rep}_\Lambda P$ is of finite representation type, then every indecomposable object in $\operatorname{Rep}_\Lambda P$ belongs to $\operatorname{rep}_\Lambda P$.

Proof.

Assume that there exists an indecomposable object X in $\operatorname{Rep}_\Lambda P$ which does not belong to $\operatorname{rep}_\Lambda P$. We obtain a contradiction by constructing a sequence $(A_n)_{n\in\mathbb{N}}$ of indecomposable objects in $\operatorname{rep}_\Lambda P$, a sequence $(g_n)_{n\in\mathbb{N}_+}$ of homomorphisms in $\operatorname{Rep}_\Lambda P$ with $g_n : A_{n-1} \to A_n$ for $n \in \mathbb{N}_+$, and a sequence $(h_n)_{n\in\mathbb{N}}$ of homomorphisms in $\operatorname{Rep}_\Lambda P$ with $h_n : A_n \to X$ for $n \in \mathbb{N}$, such that $h_n f_n \cdots f_1 \neq 0$ for each $n \in \mathbb{N}$.

First we take as A_0 an arbitrary simple subobject of X and denote by g_0 the corresponding canonical injection.

Now assume that $n \in \mathbb{N}_+$. Let $f : A_{n-1} \to B$ be a left almost split homomorphism in $\operatorname{Rep}_\Lambda P$ with $B \in \operatorname{rep}_\Lambda B$. By construction $h_{n-1} \neq 0$. Moreover, our assumption made on X implies that h_{n-1} is not a split monomorphism. Consequently, there exists a homomorphism $f' : B \to X$ in $\operatorname{Rep}_\Lambda P$ such that $h_{n-1} = f' \circ f$. Let $B = \bigoplus_{i\in[1,m]} B_i$ be a decomposition of B into a direct sum of indecomposable objects in $\operatorname{Rep}_\Lambda P$. If $f_i : A_{n-1} \to B_i$, $i \in [1, m]$, and $f_i' : B_i \to X$, $i \in [1, m]$, are the homomorphisms induced by the above decomposition and f and

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f', respectively, then

$$\sum_{i\in[1,m]} f'_i f_i g_{n-1}\cdots g_1 = h_{n-1}g_{n-1}\cdots g_1 \neq 0.$$

Consequently, there exists $i \in [1,m]$ such that $f'_i f_i g_{n-1}\cdots g_1 \neq 0$, and we put $A_n := B_i$, $g_n := f_i$, and $h_n := f'_i$.

THEOREM.

If $\operatorname{rep}_\Lambda P$ is of finite representation type, then every object in $\operatorname{Rep}_\Lambda P$ is a direct sum of objects from $\operatorname{rep}_\Lambda P$.

PROOF.

Let M be the direct sum of representatives of the isomorphism classes of the indecomposable objects in $\operatorname{rep}_\Lambda P$. Let S be the endomorphism algebra of M. Let $H := \operatorname{Hom}(M,X)$. Then $X = \sum_{h\in H}\operatorname{Im} h$, hence we get an exact sequence

$$\varepsilon : 0 \to K \to M^{(H)} \to X \to 0$$

in $\operatorname{Rep}_\Lambda P$, such that the sequence

$$\operatorname{Hom}(M,\varepsilon) : 0 \to \operatorname{Hom}(M,K) \to \operatorname{Hom}(M,M^{(H)}) \to \operatorname{Hom}(M,X) \to 0$$

is exact. Moreover, $\operatorname{Hom}(M,\varepsilon)$ is pure exact, i.e. for each finitely presented S-module N the sequence $\operatorname{Hom}_S(N,\operatorname{Hom}(M,\varepsilon))$ is exact. Next, $\operatorname{Hom}(M,M^{(H)}) \simeq \operatorname{Hom}(M,M)^{(H)}$, hence $\operatorname{Hom}(M,M^{(H)})$ is a projective S-module. Using that S is semiprimary, we get that $\operatorname{Hom}(M,X)$ is a projective S-module. It remains to show that $X \simeq \operatorname{Hom}(M,X)\otimes_S M$. It is obvious that the canonical maps $\operatorname{Hom}(M,X)\otimes_S M \to X$ is surjective. Now assume that $\sum_{i\in I}\phi_i(x_i) = 0$ for $\sum_{i\in I}\phi_i\otimes x_i \in \operatorname{Hom}(M,X)\otimes_S M$. Let R be the incidence algebra of P^*. There exists $\sum_{j\in J}\alpha_j\otimes m_j \in \operatorname{Hom}(M,R)\otimes_S M$ such that $\sum_{j\in J}\alpha_j(m_j) = 1$. For each $(i,j)\in I\times J$ define $\beta_{i,j}\in S$ by $\beta_{i,j}(m) := \alpha_j(m)x_i$. Then

$$\sum_{i\in I}\phi_i\otimes x_i = \sum_{i\in I}\sum_{j\in J}\phi_i\otimes\alpha_j(m_j)x_i = \sum_{j\in J}\sum_{i\in I}\phi_i\otimes\beta_{i,j}(m_j).$$

Now

$$\sum_{i\in I}\phi_i\beta_{i,j}(m) = \sum_{i\in I}\phi_i(\alpha_j(m)x_i) = \alpha_j(m)\sum_{i\in I}\phi_i(x_i) = 0$$

for each $m\in M$, and, consequently

$$\sum_{i\in I}\phi_i\otimes x_i = \sum_{j\in J}\sum_{i\in I}\phi_i\otimes\beta_{i,j}(m_j) = \sum_{j\in J}\sum_{i\in I}\phi_i\beta_{i,j}\otimes m_j = \sum_{j\in J}0\otimes m_j = 0.$$