# THE DUAL OF THE DIRECT PRODUCT MIGHT BE THE DIRECT SUM

BASED ON THE TALK BY LUTZ HILLE

DEFINITION.

A ring $R$ is called slender if the canonical map

$$\Phi_R : \mathrm{Hom}_R(R^{\mathbb{N}}, R) \to R^{\mathbb{N}},\ \varphi \mapsto (\varphi(e_n)),$$

is a monomorphism whose image is $R^{(\mathbb{N})}$.

THEOREM.

Let $R$ be a PID. If $R$ is not local, then $R$ is slender.

REMARK.

Let $R$ be a PID. Then $R$ is slender if and only if $R$ is not complete local.

NOTATION.

Let $R$ be a ring. For $f \in R^{\mathbb{N}}$ and $n \in \mathbb{N}$ we define $f^{\leq n}, f^{\geq n} \in R^{\mathbb{N}}$ by

$$(f^{\leq n})_m := \begin{cases} m & m \in [0,n], \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad (f^{\geq n})_m := \begin{cases} 0 & m \in [0,n], \\ f_m & \text{otherwise.} \end{cases}$$

LEMMA.

Let $p$ be a prime element of a PID $R$. If $\varphi \in \mathrm{Hom}_R(R^{\mathbb{N}}, R)$ is such that $\varphi(e_n) = 0$ for each $n \in \mathbb{N}$, then $\varphi((a_n p^n)) = 0$ for each $a \in R^{\mathbb{N}}$.

PROOF.

Let $f := ((a_n p^n))$. If $n \in \mathbb{N}_+$, then our assumptions imply that $\varphi(f^{\leq (n-1)}) = 0$, hence $\varphi(f) = \varphi(f^{\geq n})$. Moreover, $p^n \mid (f^{\geq n})_m$ for each $m \in \mathbb{N}$ and $n \in \mathbb{N}_+$ , hence $p^n \mid \varphi(f)$ for each $n \in \mathbb{N}_+$, thus the claim follows.

PROOF (PROOF OF THEOREM).

Since $R$ is not a local ring, there exists prime elements $p$ and $q$ of $R$ such that $Rp \neq Rq$.

First we prove that $\Phi_R$ is a monomorphism. Take $\varphi \in \mathrm{Hom}_R(R^{\mathbb{N}}, R)$ such that $\varphi(e_n) = 0$ for each $n \in \mathbb{N}$. If $f \in R^{\mathbb{N}}$, then there exist $a, b \in R^{\mathbb{N}}$ such that $f_n = a_n p^n + b_n q^n$. Consequently the previous lemma implies that $\varphi(f) = 0$, hence the claim follows.

Now we show that $\mathrm{Im}\, \Phi_R = R^{(\mathbb{N})}$. Assume that $\Phi_R(\varphi) \notin R^{(\mathbb{N})}$ for some $\varphi \in \mathrm{Hom}_R(R^{\mathbb{N}}, R)$. Without loss of generality we may assume that

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$\varphi(e_n) \neq 0$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ let $k_n$ be the maximal $k \in \mathbb{N}$ such that $p^k \mid \varphi(e_n)$. Assume that there exists $f \in R^{\mathbb{N}}$ such that the following conditions are satisfied:

(1) $q \nmid f_0$,
(2) $qp^{n-1} \mid f_n$ for each $n \in \mathbb{N}_+$,
(3) $p^{n+k_{n+1}} \mid \varphi(f^{\leq n})$ for each $n \in \mathbb{N}$.

Then $p^n \mid \varphi(f) = \varphi(f^{\leq n}) + \varphi(f^{\geq(n+1)})$ for each $n \in \mathbb{N}$, hence $\varphi(f) = 0$. However $q \nmid \varphi(f) = \varphi(f^{\leq 0}) + \varphi(f^{\geq 1})$ — contradiction.

It remains to show the existence of $f \in R^{\mathbb{N}}$ satisfying the above conditions. First, we put $f_0 := p^{\max(0,k_1-k_0)}$. Now assume that $n \in \mathbb{N}_+$ and $f_0, \ldots, f_{n-1}$ are defined. Let $l := (n + k_{n+1}) - (n - 1 + k_n)$. If $l < 0$, then we put $f_n := 0$, thus assume that $l \geq 0$. There exists $a, b \in R$ such that

$$\sum_{m\in[0,n-1]} f_m\varphi(e_m) = a \cdot p^{n-1+k_n} \qquad \text{and} \qquad \varphi(e_n) = b \cdot p^{k_n}.$$

Moreover, $b \cdot q$ and $p$ are coprime, hence there exists $c \in R$ such that $a = c \cdot (b \cdot q) \pmod{p^l}$. Then we put $f_n := -c \cdot q \cdot p^{n-1}$.