JORDAN TYPES AND AR-SEQUENCES: BACKGROUND AND EXAMPLES

BASED ON THE TALK BY ROLF FARNSTEINER

Assumption.

Throughout the talk k will be an algebraically closed field of positive characteristic p.

Let $\mathscr{G}$ be a finite group scheme (we usually assume that either $\mathscr{G} = kG$ for a finite group G or $\mathscr{G} = \mathscr{U}_0(\mathfrak{g})$ for a restricted Lie algebra $\mathfrak{g}$). Then $\mathscr{G}$ is a Hopf algebra, hence in particular $\operatorname{mod}\mathscr{G}$ has tensor products. Put

$$H^\circ(\mathscr{G}, k) := \bigoplus_{n\in\mathbb{N}} H^{2n}(\mathscr{G}, k).$$

For $M \in \operatorname{mod}\mathscr{G}$ we define $\Phi_M : H^\circ(\mathscr{G}, k) \to \operatorname{Ext}^*_{\mathscr{G}}(M, M)$ by $[f] \mapsto [f \otimes M]$. Friedlander and Suslin proved that $H^\circ(\mathscr{G}, k)$ is a finitely generated k-algebra and $\operatorname{Ext}^*_{\mathscr{G}}(M, M)$ is a finitely generated $H^\circ(\mathscr{G}, k)$-module for each $M \in \operatorname{mod}\mathscr{G}$. Consequently, for $M \in \operatorname{mod}\mathscr{G}$ we define the support variety $\mathscr{V}_{\mathscr{G}}(M)$ of M as the zero set of $\operatorname{Ker}\Phi_M$ inside the maximal spectrum of $H^\circ(\mathscr{G}, k)$. It is known that $\dim \mathscr{V}_{\mathscr{G}}(M)$ equals the complexity $\operatorname{cx}_{\mathscr{G}}(M)$ of M. If M and N belong to the same component of the stable Auslander–Reiten quiver $\Gamma_s(\mathscr{G})$ of $\mathscr{G}$, then $\mathscr{V}_{\mathscr{G}}(M) = \mathscr{V}_{\mathscr{G}}(N)$. Consequently, we may define $\mathscr{V}_{\mathscr{G}}(\Theta)$ for a component Θ of $\Gamma_s(\mathscr{G})$. It is known that Θ is a tube provided $\dim \mathscr{V}_{\mathscr{G}}(\Theta) = 1$ and Θ is infinite. Moreover, $\Theta \simeq \mathbb{Z}\mathbb{A}_\infty$ if $\dim \mathscr{V}_{\mathscr{G}}(\Theta) \geq 3$.

Notation.

For an element x of a Lie algebra $\mathfrak{g}$ we define $\operatorname{ad}_x : \mathfrak{g} \to \mathfrak{g}$ by $\operatorname{ad}_x(y) := [x, y]$ for $y \in \mathfrak{g}$.

Definition.

Let $\mathfrak{g}$ be a Lie algebra. A map $(-)^{[p]} : \mathfrak{g} \to \mathfrak{g}$ is called a p-map if the following conditions are satisfied:

(1) $\operatorname{ad} x^{[p]} = (\operatorname{ad} x)^p$ for each $x \in \mathfrak{g}$.
(2) $(\alpha x)^{[p]} = \alpha^p x^{[p]}$ for each $\alpha \in k$ and $x \in \mathfrak{g}$.
(3) for each $x, y \in \mathfrak{g}$,

$$(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i\in[1,p-1]} s_i(x, y),$$

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where $s_1(x,y)$, ..., $s_{p-1}(x,y)$ are products of length p involving only x and y.

Definition.

By a restricted Lie algebra we mean a Lie algebra $\mathfrak{g}$ together with a p-map $(-)^{[p]} : \mathfrak{g} \to \mathfrak{g}$.

Example.

If Λ is an associative algebra, then the map $\Lambda^- \to \Lambda^-$, $x \mapsto x^p$, is a p-map in the commutator algebra Λ^-. Any Lie subalgebra $\mathfrak{g}$ of Λ^- such that $x^p \in \mathfrak{g}$ for each $x \in \mathfrak{g}$ is a restricted Lie algebra. As examples of restricted Lie algebras obtained in this way one gets: $\mathfrak{gl}(n)$, $\mathfrak{sl}(n)$, the algebra $\mathrm{Upp}(n)$ of upper triangular algebras, and the algebra $\mathrm{Upp}^+(n)$ of strictly upper triangular matrices.

Remark.

By Theorem of Jacobson p-maps may be defined on a basis of a Lie algebra. Namely, if $(x_i)_{i\in I}$ is a basis of a Lie algebra $\mathfrak{g}$ and $(y_i)_{i\in I}$ are elements of $\mathfrak{g}$ such that $(\operatorname{ad} x_i)^p = \operatorname{ad} y_i$ for each $i \in I$, then there is a unique p-map $(-)^{[p]} : \mathfrak{g} \to \mathfrak{g}$ such that $x_i^{[p]} = y_i$ for each $i \in I$.

Definition.

For a restricted Lie algebra $\mathfrak{g}$ we define the restricted enveloping algebra by

$$\mathscr{U}_0(\mathfrak{g}) := \mathscr{U}(\mathfrak{g})/(x^p - x^{[p]} \mid x \in \mathfrak{g}),$$

where $\mathscr{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$.

Remark.

Let $\iota : \mathfrak{g} \to \mathscr{U}_0(\mathfrak{g})$ be the canonical map. If $(x_1, \ldots, x_n)$ is a basis of $\mathfrak{g}$, then

$$(\iota(x_1)^{a_1} \cdots \iota(x_n)^{a_n} \mid a_1, \ldots, a_n \in [0, p-1])$$

is a basis of $\mathscr{U}_0(\mathfrak{g})$. In particular, ι is injective and $\mathfrak{g}$ is a restricted subalgebra of $\mathscr{U}_0(\mathfrak{g})^-$. Moreover, if $\mathfrak{h}$ is a restricted subalgebra of $\mathfrak{g}$, then $\mathscr{U}_0(\mathfrak{h})$ is a subalgebra of $\mathscr{U}_0(\mathfrak{g})$ and $\mathscr{U}_0(\mathfrak{g})$ is a free left and right $\mathscr{U}_0(\mathfrak{h})$-module.

Example.

Let $\mathfrak{h}$ be the Heisenberg algebra, i.e. $\mathfrak{h} = kx \oplus ky \oplus kz$, $[x,y] = z$, and $[x,z] = 0 = [y,z]$ (in other words $\mathfrak{h}$ is the algebra of 3×3 upper triangular matrices). Each of the following formulas define a p-map on $\mathfrak{h}$:

(1) $x^{[p]} = y^{[p]} = z^{[p]} = 0$.

(2) $x^{[p]} = z = y^{[p]}$ and $z^{[p]} = 0$.

(3) $x^{[p]} = 0 = y^{[p]}$ and $z^{[p]} = z$.

In the first two cases $\mathscr{U}_0(\mathfrak{h})$ is local, while in the last case $\mathscr{U}_0(\mathfrak{h})$ has p-blocks.

Definition.

For a restricted Lie algebra $\mathfrak{g}$ we define the null cone $\mathscr{V}_{\mathfrak{g}}$ by

$$\mathscr{V}_{\mathfrak{g}} := \{x \in \mathfrak{g} \mid x^{[p]} = 0\}.$$

If M is a $\mathscr{U}_0(\mathfrak{g})$-module, then the rank variety $\mathscr{V}_{\mathfrak{g}}(M)$ of M is defined by

$$\mathscr{V}_{\mathfrak{g}}(M) := \{x \in \mathscr{V}_{\mathfrak{g}} \mid M|_{k[x]} \text{ is not projective}\} \cup \{0\}.$$

Theorem (Jantzen/Friedlander/Parshall).

If M is a $\mathscr{U}_0(\mathfrak{g})$-module for a restricted Lie algebra $\mathfrak{g}$, then $\mathscr{V}_{\mathfrak{g}}(M)$ is homeomorphic with the support variety of M.

Example.

If $p \geq 3$, then

$$\mathscr{V}_{\mathfrak{sl}(2)} = \left\{ \left[\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right] \mid a^2 + bc = 0 \right\}.$$

Definition.

Let $\mathscr{G}$ be a finite group scheme and $\mathfrak{A} := k[T]/T^p$. For a homorphism $\alpha : \mathfrak{A} \to \mathscr{G}$ we denote by α^* the pull-back functor $\operatorname{mod}\mathscr{G} \to \operatorname{mod}\mathfrak{A}$. A homomorphism $\alpha : \mathfrak{A} \to \mathscr{G}$ is called a p-point if $\alpha^*(\mathscr{G})$ is a projective $\mathfrak{A}$-module and there exists a unipotent abelian subgroup $\mathscr{U}$ of $\mathscr{G}$ such that $\operatorname{Im}\alpha \subseteq \mathscr{U}$. Two p-points α and β are said to be equivalent if for each $M \in \operatorname{mod}\mathscr{G}$, $\alpha^*(M)$ is projective if and only if $\beta^*(M)$ is projective. We denote by $P(\mathscr{G})$ the set of equivalences classes of p-points. Finally, for $M \in \operatorname{mod}\mathscr{G}$ we define the p-support $P(\mathscr{G})_M$ of M by

$$P(\mathscr{G})_M := \{[\alpha] \mid \alpha^*(M) \text{ is not projective}\}.$$

Theorem (Friedlander/Pevtsova).

Let $\mathscr{G}$ be a finite group scheme.

(1) The sets $\{P(\mathscr{G})_M \mid M \in \operatorname{mod}\mathscr{G}\}$ give a structure of a noetherian topological space on $P(\mathscr{G})$ as the closed subsets.

(2) There exists a homeomorphism $\psi_{\mathscr{G}} : P(\mathscr{G}) \to \operatorname{Proj}(\mathscr{V}_{\mathscr{G}}(k))$ such that $\psi_{\mathscr{G}}(P(\mathscr{G})_M) = \operatorname{Proj}(\mathscr{V}_{\mathscr{G}}(M))$ for each $M \in \operatorname{mod}\mathscr{G}$.

Definition.

Let $\mathscr{G}$ be a finite group scheme. For $M \in \operatorname{mod}\mathscr{G}$ we define the set $\operatorname{Jt}(M)$ of Jordan types of M as the set of isomorphism classes of $\alpha^*(M)$ for p-points α. We say that $M \in \operatorname{mod}\mathscr{G}$ is of constant Jordan type if $\operatorname{Jt}(M)$ is a singleton.

Notation.

For each $i \in [1, p]$ we denote by $[i]$ the isomorphism class of the indecomposable $\mathfrak{A}$-module of dimension i.

Remark.

A module M over a finite group scheme $\mathscr{G}$ is projective if and only if $\operatorname{Jt}(M) = \{n[p]\}$ for some $n \in \mathbb{N}$.

REMARK.

For modules M and N over a finite group scheme $\mathscr{G}$, $\mathrm{Hom}_k(M, N)$ has a structure of a $\mathscr{G}$-module.

DEFINITION.

A module M over a finite group scheme is called endo-trivial if $\mathrm{End}_k(M)$ is a direct sum of the trivial $\mathscr{G}$-module and a projective $\mathscr{G}$-module.

PROPOSITION.

Let Θ be a component of $\Gamma_s(\mathfrak{sl}(2))$.

(1) If $\dim_k P(\mathfrak{sl}(2))_\Theta = 1$, then every M in Θ has constant Jordan type. Moreover, there exists $s(\Theta) \in [1, p-1]$ such that

$$\mathrm{Jt}(M) = \left\{[s(\Theta)] \oplus \frac{\dim_k M - s(\Theta)}{p}[p]\right\}$$

for each $M \in \Theta$.

(2) If $\dim_k P(\mathfrak{sl}(2))_\Theta = 0$, then Θ is a tube and there exists $i(\Theta) \in [1, \frac{p-1}{2}]$ such that

$$\mathrm{Jt}(M) = \{\mathrm{ql}(M)[p], [i(\Theta)] \oplus [p - i(\Theta)] \oplus (\mathrm{ql}(M) - 1)[p]\}.$$

for each $M \in \Theta$.