HAMMOCKS AND MORE HAMMOCKS

BASED ON THE TALK BY NILS MAHRT

Definition.

A finite directed translation quiver H with a unique source ω is called a hammock if there exists function $h : H_0 \to \mathbb{N}_+$ such that the following conditions are satisfied:

(h1) $h(\omega) = 1$.

(h2) if x is a projective vertex of H different from ω, then

$$h(x) = \sum_{\substack{\alpha \in H_1 \\ t\alpha = x}} h(s\alpha).$$

(h3) if x is a vertex of H which is neither projective nor injective, then

$$h(x) + h(\tau x) = \sum_{\substack{\alpha \in H_1 \\ t\alpha = x}} h(s\alpha).$$

(h4) if x is an injective vertex of H, then

$$h(x) \geq \sum_{\substack{\alpha \in H_1 \\ s\alpha = x}} h(t\alpha).$$

Remark.

If H is a hammock, then there exists a unique function $h : H_0 \to \mathbb{N}_+$ satisfying the conditions (h1)–(h4) and we call h the hammock function of H.

Notation.

Throughout the rest of the talk A will be a fixed path algebra of a bound quiver. We assume that A is representation directed. We also fix two (not necessarily different) vertices a and b of the quiver and non-zero $w \in bAa$. Next, by C_w we denote the cokernel of the map $P(b) \to P(a)$ induced by the multiplication by w. Finally, we denote by H_w the full subquiver of Γ_A with the set of vertices

$$\{X \in \operatorname{ind} A \mid X(w) \neq 0\}$$

and we define $h_w : \operatorname{mod} A \to \mathbb{N}$ by

$$h_w(M) := \dim_k \operatorname{Hom}_A(P_a, M) - \dim_k \operatorname{Hom}_A(C_w, M).$$

Date: 20.06.2008.

Remark.

Either $C_w = 0$ or $C_w \in \operatorname{ind} A$. Moreover, if $C_w \neq 0$, then C_w is not projective.

Lemma.

If $X \in \operatorname{ind} A$, then $X(w) \neq 0$ if and only if $h_w(X) \neq 0$.

Proof.

We have the following exact sequence

$$0 \to \operatorname{Hom}_A(C_w, X) \to \operatorname{Hom}_A(P_a, X) \xrightarrow{X(w)} \operatorname{Hom}_A(P_b, X),$$

which implies the claim.

Lemma.

If X is an indecomposable projective A-module such that $X \not\simeq P_a$, then

$$h_w(X) = h_w(\operatorname{rad} X).$$

Proof.

Since $X \not\simeq C_w$, the claim follows.

Lemma.

If $X \in \operatorname{ind} A$ is not a projective A-module such that $X \not\simeq C_w$, then

$$h_w(X) + h_w(\tau_A X) = h_w(M),$$

where M is the middle term of the Auslander–Reiten sequence ending at X.

Proof.

Obvious.

Theorem.

The quiver H_w is a hammock with the hammock function h_w.

Proof.

First observe that P_a is a unique source in H_w and $h_w(P_a) = 1$.

Next we show that neither C_w nor $\tau_A C_w$ is a vertex of H_w. Indeed, since A is representation finite, $\dim_k \operatorname{Hom}_A(P_b, P_a) = 1$, and consequently $C_w(b) = 0$. In particular, $C_w(w) = 0$. Dually, $(\tau_A C_w)(a) = 0$ and $(\tau_A C_w)(w) = 0$.

Since C_w is not a vertex of H_w, the above lemmas imply that the conditions (h2)–(h3) are satisfied. It remains to show that if Y is an injective vertex of H_w, then $h_w(Y) \geq h_w(M)$, where $Y \to M$ is a minimal left almost split map. If Y is not an injective A-module, then $Y = \tau_A X$ for $X \in \operatorname{ind} A$. Since $X \not\simeq C_w$, the claim in this case follows from the previous lemma. Finally, assume that Y is an injective module. Then we have an exact sequence $0 \to S \to I \to M \to 0$ for a simple A-module S. Consequently,

$$\dim_k \operatorname{Hom}_A(P_a, I) = \dim_k \operatorname{Hom}_A(P_a, M) + \dim_k \operatorname{Hom}_A(P_a, S)$$

and

$$\dim_k \operatorname{Hom}_A(C_w, I) \leq \dim_k \operatorname{Hom}_A(C_w, M) + \dim_k \operatorname{Hom}_A(C_w, S).$$

Moreover, $\dim_k \operatorname{Hom}_A(C_w, S) \leq \dim_k \operatorname{Hom}_A(P_a, S)$ and the claim follows.

Theorem.

If $X \in \operatorname{ind} A$, then

$$\begin{aligned} h_w(X) &= \dim_k \operatorname{Hom}_A(X, I_b) - \dim_k \operatorname{Hom}_A(X, I_a) = \operatorname{rk} X(w) \\ &= \min\{\dim_k \operatorname{Hom}_A(P_a, X), \dim_k \operatorname{Hom}_A(P_b, X)\}. \end{aligned}$$

Corollary.

$H_w = H_a \cap H_b$.