GEOMETRIC REALIZATIONS OF CLUSTER ALGEBRAS OF FINITE TYPE

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Assumption.

Throughout the talk n is a fixed positive integer.

Definition.

We say that $B \in \mathbb{M}_n(\mathbb{Z})$ is called skew-symmetrizable if there exists $\mathbf{d} \in \mathbb{N}_+^n$ such that $d_i b_{i,j} + d_j b_{j,i} = 0$ for all $i, j \in [1, n]$.

Notation.

We denote by $\mathscr{B}$ the set of all skew-symmetrizable $n \times n$-matrices.

Notation.

If $b \in \mathbb{Z}$, then $[b]_+ := \max(b, 0)$.

Definition.

For $k \in [1, n]$ we define the mutation $\mu_k : \mathscr{B} \to \mathscr{B}$ in direction k by $\mu_k B := B'$, where

$$b'_{i,j} := \begin{cases} -b_{i,j} & \text{if } i = k \text{ or } j = k, \\ b_{i,j} + [b_{i,k}]_+ \cdot [b_{k,j}]_+ - [-b_{i,k}]_+ \cdot [-b_{k,j}]_+ & \text{if } i \neq k \neq j, \end{cases}$$

for $i, j \in [1, n]$ and $B \in \mathscr{B}$.

Remark.

$\mu_k^2 B = B$ for each $B \in \mathscr{B}$.

Definition.

We call $B, B' \in \mathscr{B}$ mutation equivalent if there exist $k_1, \ldots, k_l \in [1, n]$ such that $B' = \mu_{k_l} \cdots \mu_{k_1} B$.

Remark.

With a skew-symmetric matrix B we associate the quiver Q_B with the set of vertices $[1, n]$ and $[b_{i,j}]_+$ arrows from j to i for $i, j \in [1, n]$. This assignment induces a bijection between the set of skew-symmetric matrices and the set of quivers with the set of vertices $[1, n]$ which have no loops and no cycles of length 2.

Definition.

A finitely generated free abelian multiplicative group $\mathbb{P}$ is called a semifield if it possesses an auxiliary addition $\oplus$ which is commutative, associative and distributive.

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Example.

If $\mathbb{P}$ is the free abelian multiplicative group generated by $p_1, \ldots, p_l$, then $\mathbb{P}$ together with the auxiliary addition $\oplus$ defined by

$$\prod_{j\in[1,n]} p_j^{a_j} \oplus \prod_{j\in[1,n]} p_j^{b_j} := \prod_{j\in[1,n]} p_j^{\min(a_j,b_j)},$$

for $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$, is a semifield, which we denote $\operatorname{Trop}(p_1, \ldots, p_l)$.

Definition.

By a Y-seed in a semifield $\mathbb{P}$ we mean a pair $(B, \mathbf{y})$ consisting of $B \in \mathscr{B}$ and $\mathbf{y} \in \mathbb{P}^n$.

Notation.

Let $\mathscr{Y}(\mathbb{P})$ denote the set of all Y-seeds in a semifield $\mathbb{P}$.

Definition.

Let $\mathbb{P}$ be a semifield and $\mathscr{Y} := \mathscr{Y}(\mathbb{P})$. For $k \in [1, n]$ we define the mutation $\mu_k : \mathscr{Y} \to \mathscr{Y}$ in direction k by $\mu_k(B, \mathbf{y}) := (B', \mathbf{y}')$, where $B' := \mu_k B$ and

$$y'_j := \begin{cases} y_k^{-1} & \text{if } j = k, \\ y_j y_k^{[b_{k,j}]_+} (y_k \oplus 1)^{-b_{k,j}} & \text{if } j \neq k, \end{cases}$$

for $j \in [1, n]$ and $(B, \mathbf{y}) \in \mathscr{Y}$.

Remark.

If $k \in [1, n]$, then μ_k^2 is the identity and we define the mutation equivalence relation in the same way as before.

Remark.

If $\mathbb{P} = \operatorname{Trop}(p_1, \ldots, p_l)$, then with $(B, \mathbf{y}) \in \mathscr{Y}(\mathbb{P})$ we may associate $C \in \mathbb{M}_{(n+l)\times n}(\mathbb{Z})$ by

$$c_{i,j} := \begin{cases} b_{i,j} & \text{if } i \in [1, n], \\ a_{i-n,j} & \text{if } i \in [n+1, n+l], \end{cases}$$

for $i \in [1, n+l]$ and $j \in [1, n]$, provided $y_j = \prod_{i\in[1,l]} p_i^{a_{i,j}}$ for each $j \in [1, n]$. If μ'_k denotes the operation induced by μ_k on the set of matrices obtained in this way, then $\mu'_k C = C'$, where

$$c'_{i,j} := \begin{cases} -c_{i,j} & \text{if } i = k \text{ or } j = k, \\ c_{i,j} + [c_{i,k}]_+ \cdot [c_{k,j}]_+ - [-c_{i,k}]_+ \cdot [-c_{k,j}]_+ & \text{if } i \neq k \neq j, \end{cases}$$

for $i \in [1, n+l]$ and $j \in [1, n]$.

Definition.

A Y-seed $(B, \mathbf{y})$ in a semifield $\mathbb{P}$ is called principal provided $\mathbb{P} \simeq \operatorname{Trop}(y_1, \ldots, y_n)$.

Conjecture.

Let $(B,\mathbf{y})$ be a principal Y-seed in a semifield $\mathbb{P}$. If $(B',\mathbf{y}')$ is a Y-seed which is mutation equivalent to $(B,\mathbf{y})$, then $a_{i,l}a_{j,l} \geq 0$ for all $i,j,l \in [1,n]$ provided $y'_l = \prod_{i\in[1,n]} y_i^{a_{i,l}}$ for each $l \in [1,n]$.

Remark.

The above conjecture has been verified for skew-symmetric matrices.

Definition.

By a seed over a semifield $\mathbb{P}$ we mean a triple $(B,\mathbf{y},\mathbf{x})$ such that $(B,\mathbf{y})$ is a Y-seed in $\mathbb{P}$ and $x_1, \ldots, x_n$ are algebraically independent generators of the pure transcendental extension of $\mathbb{Q}(\mathbb{P})$ of degree n, where $\mathbb{Q}(\mathbb{P})$ is the field of fractions of the group ring $\mathbb{Z}\mathbb{P}$.

Definition.

If $(B,\mathbf{y},\mathbf{x})$ is a seed, then $x_1, \ldots, x_n$ are called the cluster variables of $(B,\mathbf{y},\mathbf{x})$.

Notation.

Let $\mathscr{S}(\mathbb{P})$ denote the set of all seeds over a semifield $\mathbb{P}$.

Definition.

Let $\mathbb{P}$ be a semifield and $\mathscr{S} := \mathscr{S}(\mathbb{P})$. For $k \in [1,n]$ we define the mutation $\mu_k : \mathscr{S} \to \mathscr{S}$ in direction k by $\mu_k(B,\mathbf{y},\mathbf{x}) := (B',\mathbf{y}',\mathbf{x}')$, where $(B',\mathbf{y}') := \mu_k(B,\mathbf{y})$ and

$$x'_j := \begin{cases} x_j & \text{if } j \neq k, \\ \frac{y_k \prod_{i\in[1,n]} x_i^{[b_{i,k}]_+} + \prod_{i\in[1,n]} x_i^{[-b_{i,k}]_+}}{(y_k \oplus 1)x_k} & \text{if } j = k, \end{cases}$$

for $j \in [1,n]$ and $(B,\mathbf{y},\mathbf{x}) \in \mathscr{S}$.

Remark.

If $k \in [1,n]$, then μ_k^2 is the identity and we define the mutation equivalence relation in the same way as before.

Notation.

For a seed $(B,\mathbf{y},\mathbf{x})$ we denote by $\mathscr{X}(B,\mathbf{y},\mathbf{x})$ the union of all cluster variables of seeds which are mutation equivalent to $(B,\mathbf{y},\mathbf{x})$.

Notation.

For a seed $(B,\mathbf{y},\mathbf{x})$ over a semifield $\mathbb{P}$ we put

$$\mathscr{A}(B,\mathbf{y},\mathbf{x}) := \mathbb{Z}\mathbb{P}[\mathscr{X}(B,\mathbf{y},\mathbf{x})].$$

Definition.

By a cluster algebra we mean an algebra of the form $\mathscr{A}(B,\mathbf{y},\mathbf{x})$ for a seed $(B,\mathbf{y},\mathbf{x})$.

Theorem (Laurant Phenomena).

If $(B, \mathbf{y}, \mathbf{x})$ is a seed over a semifield $\mathbb{P}$, then

$$\mathscr{A}(B, \mathbf{y}, \mathbf{x}) \subseteq \mathbb{Z}\mathbb{P}[x_1^{\pm}, \ldots, x_n^{\pm}].$$

Theorem.

If $(B, \mathbf{y}, \mathbf{x})$ is a seed over $\operatorname{Trop}(p_1, \ldots, p_l)$, then

$$\mathscr{A}(B, \mathbf{y}, \mathbf{x}) \subseteq \mathbb{Z}[p_1^{\pm}, \ldots, p_l^{\pm}, x_1^{\pm}, \ldots, x_n^{\pm}].$$

Theorem.

If $(B, \mathbf{y}, \mathbf{x})$ is a seed over a semifield $\mathbb{P}$ such that $(B, \mathbf{y})$ is principal, then

$$\mathscr{A}(B, \mathbf{y}, \mathbf{x}) \subseteq \mathbb{Z}[y_1, \ldots, y_n, x_1^{\pm}, \ldots, x_n^{\pm}].$$

Theorem.

If $(B, \mathbf{y}, \mathbf{x})$ is a seed over a semifield $\mathbb{P}$ such that $(B, \mathbf{y})$ is principal, then for each $z \in \mathscr{X}(B, \mathbf{y}, \mathbf{x})$ there exist uniquely determined $\mathbf{g}_z \in \mathbb{Z}^n$ and $F_z \in \mathbb{Z}[T_1, \ldots, T_n]$ such that $z = \mathbf{x}^{\mathbf{g}_z} F_z(\hat{\mathbf{y}})$, where $\hat{y}_j := y_j \prod_{i \in [1,n]} x_i^{b_{i,j}}$ for $j \in [1, n]$.

Remark.

If $(B, \mathbf{y}, \mathbf{x})$ is a seed over a semifield $\mathbb{P}$ such that $(B, \mathbf{y})$ is principal and $z \in \mathscr{X}(B, \mathbf{y}, \mathbf{x})$, then $T_i \nmid F_z$ for each $i \in [1, n]$.

Remark.

If $(B, \mathbf{y}, \mathbf{x})$ is a seed over a semifield $\mathbb{P}$ such that $(B, \mathbf{y})$ is principal, then for each $z \in \mathscr{X}(B, \mathbf{y}, \mathbf{x})$ there exist $G, H \in \mathbb{N}[T_1, \ldots, T_n]$ such that $F_z = \frac{G}{H}$. Consequently, if $\mathbf{y}' \in \mathbb{P}'^n$ for a semifield $\mathbb{P}'$, then we may define $F_z(\mathbf{y}') \in \mathbb{P}'$.

Theorem.

Let $(B, \mathbf{y}, \mathbf{x})$ be a seed over a semifield $\mathbb{P}$ such that $(B, \mathbf{y})$ is principal. If $(B, \mathbf{y}', \mathbf{x}')$ is a seed over a semifield $\mathbb{P}'$, then

$$\mathscr{X}(B, \mathbf{y}', \mathbf{x}') = \left\{ \mathbf{x}'^{\mathbf{g}_z} \frac{F_z(\hat{\mathbf{y}'})}{F_z(\mathbf{y}')} \mid z \in \mathscr{X}(B, \mathbf{y}, \mathbf{x}) \right\},$$

where $\hat{y}_j := y_j \prod_{i \in [1,n]} x_i^{b_{i,j}}$ for $j \in [1, n]$.

Example.

Let $(B, \mathbf{y}, \mathbf{x})$ be a seed over a semifield $\mathbb{P}$ such that $(B, \mathbf{y})$ is principal. If $k \in [1, n]$ and $(B', \mathbf{y}', \mathbf{x}') := \mu_k(B, \mathbf{y}, \mathbf{x})$, then

$$\mathbf{g}_{x'_k} = ([-b_{1,k}], \ldots, [-b_{k-1,k}], -1, [-b_{k+1,k}], \ldots, [-b_{n,k}])$$

and $F_{x'_k} = T_k + 1$.

Remark.

Let $(B, \mathbf{y}, \mathbf{x})$ be a seed over a semifield $\mathbb{P}$ such that $(B, \mathbf{y})$ is principal. If B is skew-symmetric, then with every $z \in \mathscr{X}(B, \mathbf{y}, \mathbf{x})$ one may associate an indecomposable representation M_z of Q_B.

Definition.

If M is a representation of a quiver Q with the set of vertices $[1, n]$, then we put

$$F_M := \sum_{\mathbf{e} \in \mathbb{N}^n} \chi(\mathrm{Gr}_{\mathbf{e}}(M)) T_1^{e_1} \cdots T_n^{e_n},$$

where for $\mathbf{e} \in \mathbb{N}^n$ we denote by $\mathrm{Gr}_{\mathbf{e}}(M)$ the variety of subrepresentations of M of dimension vector $\mathbf{e}$.

Theorem.

Let $(B, \mathbf{y}, \mathbf{x})$ be a seed over a semifield $\mathbb{P}$ such that $(B, \mathbf{y})$ is principal. If B is skew-symmetric, then $F_z = F_{M_z}$ for all $z \in \mathscr{X}(B, \mathbf{y}, \mathbf{x})$.

Remark.

There is a similar interpretation of g-vectors for skew-symmetric matrices.