DIMENSIONS AND VANISHING OF EXTENSIONS

BASED ON THE TALK BY JEAN-MARIE BOIS

The talk was based on the paper *Lower bounds for Auslander's representation dimension* by Steffen Oppermann.

Assumption.
Throughout the talk Λ is an Artin algebra.

Definition.
Let $M \in \operatorname{mod} \Lambda$. For $X \in \operatorname{mod} \Lambda$ we define the M-resolution dimension $M\text{-res.}\dim(X)$ of X as the minimal $n \in \mathbb{N}$ such that there exists a complex

$$0 \to M_n \to \cdots \to M_0 \to X \to 0$$

such that $M_0, \ldots, M_n \in \operatorname{add} M$ and the sequence

$$0 \to \operatorname{Hom}_\Lambda(M, M_n) \to \cdots \to \operatorname{Hom}_\Lambda(M, M_0) \to \operatorname{Hom}_\Lambda(M, X) \to 0$$

is exact.

Definition.
Let $M \in \operatorname{mod} \Lambda$. For a subcategory $\mathscr{X}$ of $\operatorname{mod} \Lambda$ we define the M-resolution dimension $M\text{-res.}\dim(\mathscr{X})$ of $\mathscr{X}$ by

$$M\text{-res.}\dim(\mathscr{X}) := \sup\{M\text{-res.}\dim(X) \mid X \in \mathscr{X}\}.$$

Definition.
For a subcategory $\mathscr{X}$ of $\operatorname{mod} \Lambda$ we define the resolution dimension $\text{res.}\dim(\mathscr{X})$ of $\mathscr{X}$ by

$$\text{res.}\dim(\mathscr{X}) := \min\{M\text{-res.}\dim(\mathscr{X}) \mid M \in \operatorname{mod} \Lambda\}.$$

Definition.
Let $M \in \operatorname{mod} \Lambda$. For $X \in \operatorname{mod} \Lambda$ we define the M-weak resolution dimension $M\text{-w.res.}\dim(X)$ of X as the minimal $n \in \mathbb{N}$ such that there exists an exact sequence

$$0 \to M_n \to \cdots \to M_0 \to X \to 0$$

with $M_0, \ldots, M_n \in \operatorname{add} M$.

Definition.
Let $M \in \operatorname{mod} \Lambda$. For a subcategory $\mathscr{X}$ of $\operatorname{mod} \Lambda$ we define the M-weak resolution dimension $M\text{-w.res.}\dim(\mathscr{X})$ of $\mathscr{X}$ by

$$M\text{-res.}\dim(\mathscr{X}) := \sup\{M\text{-w.res.}\dim(X) \mid X \in \mathscr{X}\}.$$

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Definition.

For a subcategory $\mathscr{X}$ of $\operatorname{mod}\Lambda$ we define the weak resolution dimension $\operatorname{w.res.dim}(\mathscr{X})$ of $\mathscr{X}$ by

$$\operatorname{w.res.dim}(\mathscr{X}) := \min\{M\text{-}\operatorname{w.res.dim}(\mathscr{X}) \mid M \in \operatorname{mod}\Lambda\}.$$

Remark.

If M is a generator of $\operatorname{mod}\Lambda$, then $M\text{-}\operatorname{w.res.dim}(X) \leq M \operatorname{res.dim}(X)$ for each $X \in \operatorname{mod}\Lambda$.

Definition.

For a subcategory $\mathscr{C}$ of $\mathscr{D}^b(\Lambda)$ we define the dimension $\dim\mathscr{C}$ of $\mathscr{C}$ by

$$\dim\mathscr{C} := \min\{\min\{n \in \mathbb{N} \mid \mathscr{C} \subseteq \langle M\rangle_{n+1}\} \mid M \in \mathscr{C}\}.$$

Similarly we define $\dim \underline{\operatorname{mod}}\,\Lambda$.

Theorem.

We have the following inequalities:

$$\dim\operatorname{mod}\Lambda \leq \operatorname{w.res.dim}\Lambda, \dim\mathscr{D}^b(\Lambda),$$
$$\operatorname{w.res.dim}\Lambda \leq \operatorname{rep.dim}\Lambda - 2, \operatorname{gl.dim}\Lambda, \operatorname{LL}(\Lambda) - 1,$$
$$\dim\mathscr{D}^b(\Lambda) \leq \operatorname{rep.dim}\Lambda, \operatorname{gl.dim}\Lambda, \operatorname{LL}(\Lambda) - 1.$$

Moreover, if Λ is selfinjective, then

$$\dim\underline{\operatorname{mod}}\,\Lambda \leq \dim\operatorname{mod}\Lambda.$$

Assumption.

Let $R := k[X_1, \dots, X_d]/I$ for a field k and a prime ideal I.

Lemma.

If $M \in \operatorname{mod} R$, then there is a nonempty open subset Ω of $\operatorname{MaxSpec}(R)$ such that $R_\mathfrak{m} \otimes_R M$ is a free $R_\mathfrak{m}$-module for all $\mathfrak{m} \in \Omega$.

Lemma.

If $\mathfrak{m} \in \operatorname{MaxSpec}(R)$, then the canonical functor $\mathscr{D}^b(R_\mathfrak{m}) \to \mathscr{D}^b(R)$ is full.

Proof.

Let X and Y be complexes over $R_\mathfrak{m}$ and $f \in \operatorname{Hom}_{\mathscr{D}^b(R)}(X, Y)$. Then $f = b^{-1} \circ a$ for a morphism $a : X \to Z$ and a quasi-isomorphism $b : Y \to Z$, where Z is a complex over R. For a complex U over R let $\eta_U : U \to R_\mathfrak{m} \otimes_R U$ be the canonical morphism. Obviously η_U is an isomorphism for a complex U over $R_\mathfrak{m}$. Observe that $\eta_Z b = (\operatorname{Id}_{R_\mathfrak{m}} \otimes b)\eta_Y$, hence $\eta_Z b$ is a quasi-isomorphism. Consequently, $f = (\eta_Z b)^{-1} \circ (\eta_Z a)$ and the claim follows.

Proposition.

If $M \in \mathscr{D}^b(R)$, then there is a nonempty open subset Ω of $\operatorname{MaxSpec}(R)$ such that

$$\operatorname{Hom}_{\mathscr{D}^b(R)}(X_1, X_2[1]) \operatorname{Hom}_{\mathscr{D}^b(R)}(M, X_1) = 0$$

for all $X_1, X_2 \in \operatorname{mod} R_{\mathfrak{m}}$ and $\mathfrak{m} \in \Omega$.

PROOF.

Let P be the projective resolution of M (observe that P may be not bounded above). Then

$$\operatorname{Hom}_{\mathscr{D}^b(R)}(M, N) \simeq \operatorname{Hom}_{\mathscr{K}(R)}(P, N)$$

for each $N \in \mathscr{D}^b(R)$. Let

$$Q : \cdots \to 0 \to P_0 / \operatorname{Im} \partial_P^1 \to P_{-1} \to P_{-2} \to \cdots ,$$

and $Q_{\mathfrak{m}} := R_{\mathfrak{m}} \otimes_R Q$ for $\mathfrak{m} \in \operatorname{MaxSpec}(R)$. The above isomorphism implies that the canonical map $P \to Q_{\mathfrak{m}}$ induces an epimorphism

$$\operatorname{Hom}_{\mathscr{D}^b(R)}(Q_{\mathfrak{m}}, X) \to \operatorname{Hom}_{\mathscr{D}^b(R)}(M, X)$$

for each $X \in \operatorname{mod} R_{\mathfrak{m}}$ and $\mathfrak{m} \in \operatorname{MaxSpec}(R)$. Using in addition the previous lemma, it suffices to show that there exists an open subset Ω of $\operatorname{MaxSpec}(R)$ such that

$$\operatorname{Hom}_{\mathscr{D}^b(R_{\mathfrak{m}})}(X_1, X_2[1]) \operatorname{Hom}_{\mathscr{D}^b(R_{\mathfrak{m}})}(Q_{\mathfrak{m}}, X_1) = 0$$

for all $X_1, X_2 \in \operatorname{mod} R_{\mathfrak{m}}$ and $\mathfrak{m} \in \Omega$.

Let Ω be a nonempty open subset of $\operatorname{MaxSpec}(R)$ such that $R_{\mathfrak{m}} \otimes_R (P_0 / \operatorname{Im} \partial_P^1)$ is a free $R_{\mathfrak{m}}$-module for each $\mathfrak{m} \in \Omega$. If $X_1, X_2 \in \operatorname{mod} R_{\mathfrak{m}}$ for $\mathfrak{m} \in \Omega$, then

$$\operatorname{Hom}_{\mathscr{D}^b(R_{\mathfrak{m}})}(Q_{\mathfrak{m}}, X_2[1]) \simeq \operatorname{Hom}_{\mathscr{K}(R_{\mathfrak{m}})}(Q_{\mathfrak{m}}, X_2[1]) = 0$$

and the claim follows.