COMPLEXITY AND THE DIMENSION OF A TRIANGULATED CATEGORY, II

BASED ON THE TALK BY DAIVA PUCINSKAITE

The talk was based on the paper *Complexity and the dimension of a triangulated category* by Petter Andreas Bergh and Steffen Oppermann.

Assumption.

Throughout the talk Λ is an Artin algebra over a commutative artinian ring k such that there exists a commutative noetherian graded k-algebra H of finite type with the following properties:

(1) for each $M \in \operatorname{mod} \Lambda$ there exists a graded ring homomorphism $\varphi_M : H \to \operatorname{Ext}^*_\Lambda(M, M)$,
(2) if $M, N \in \operatorname{mod} \Lambda$, then the actions of H on $\operatorname{Ext}^*_\Lambda(M, N)$ via φ_M and φ_N coincide and $\operatorname{Ext}^*_\Lambda(M, N)$ is a finitely generated H-module with respect to this action.

Notation.

For $X, Y \in \operatorname{mod} \Lambda$ we put

$$A(X, Y) := \{\eta \in H \mid \eta\theta = 0 \text{ for all } \theta \in \operatorname{Ext}^*_\Lambda(X, Y)\}.$$

Lemma.

If $X, Y \in \operatorname{mod} \Lambda$, then $\gamma(\operatorname{Ext}^*_\Lambda(X, Y)) = \gamma(H/A(X, Y))$.

Proof.

Let $\theta_1, \ldots, \theta_t$ be generators of $\operatorname{Ext}^*_\Lambda(X, Y)$ over H. Then $A_H(X, Y)$ is the kernel of the map $H \to \bigoplus_{i\in[1,t]} \operatorname{Ext}^*_\Lambda(X, Y)$ given by $\eta \mapsto (\eta\theta_1, \ldots, \eta\theta_t)$ and this shows that $\gamma(\operatorname{Ext}^*_\Lambda(X, Y)) \geq \gamma(H/A(X, Y))$. The other inequality follows since $\operatorname{Ext}^*_\Lambda(X, Y)$ is a finitely generated graded $H/A(X, Y)$-module.

Lemma.

If $M \in \operatorname{mod} \Lambda$, then $\sqrt{A(M, \Lambda/\operatorname{rad} \Lambda)} = \sqrt{A(\Lambda/\operatorname{rad} \Lambda, M)}$.

Proof.

Let $\eta \in \sqrt{A(M, \Lambda/\operatorname{rad} \Lambda)}$. By easy induction on $\ell(X)$ one shows that $\eta \in \sqrt{A(M, X)}$ for all $X \in \operatorname{mod} \Lambda$. In particular, $\eta \in \sqrt{A(M, M)}$, hence $\varphi_M(\eta^n) = 0$ for some $n \in \mathbb{N}_+$. This immediately implies that $\eta \in \sqrt{A(\Lambda/\operatorname{rad} \Lambda, M)}$. The other inclusion follows similarly.

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Proposition.

If $M \in \operatorname{mod}\Lambda$, then

$$\gamma(\operatorname{Ext}^*_\Lambda(M, \Lambda/\operatorname{rad}\Lambda)) = \gamma(\operatorname{Ext}^*_\Lambda(\Lambda/\operatorname{rad}\Lambda, M)).$$

Proof.

Benson showed that $\dim(H/A) = \gamma(H/A)$ for any graded ideal A of H. Consequently, we have the following sequence of equalities

$$\begin{aligned}
\gamma(\operatorname{Ext}^*_\Lambda(M, \Lambda/\operatorname{rad}\Lambda)) &= \gamma(H/A(M, \Lambda/\operatorname{rad}\Lambda)) \\
&= \dim(H/A(M, \Lambda/\operatorname{rad}\Lambda)) \\
&= \dim(H/\sqrt{A(M, \Lambda/\operatorname{rad}\Lambda)}) \\
&= \dim(H/\sqrt{A(\Lambda/\operatorname{rad}\Lambda, M)}) \\
&= \dim(H/A(\Lambda/\operatorname{rad}\Lambda, M)) \\
&= \gamma(H/A(\Lambda/\operatorname{rad}\Lambda, M)) \\
&= \gamma(\operatorname{Ext}^*_\Lambda(\Lambda/\operatorname{rad}\Lambda, M)).
\end{aligned}$$

Corollary.

Λ is Gorenstein, i.e. $\operatorname{id}_\Lambda \Lambda < \infty$ and $\operatorname{pd}_\Lambda D(\Lambda) < \infty$.

Proof.

Since $\gamma(\operatorname{Ext}^*_\Lambda(\Lambda/\operatorname{rad}\Lambda, \Lambda)) = \gamma(\operatorname{Ext}^*_\Lambda(\Lambda, \Lambda/\operatorname{rad}\Lambda)) = 0$, the first part follows. The second part follows similarly.

Notation.

We put

$$\operatorname{MCM}(\Lambda) := \{M \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^m_\Lambda(M, \Lambda) = 0 \text{ for each } m \in \mathbb{N}_+\}.$$

Remark.

Since Λ is Gorenstein, $\mathscr{D}^b(\Lambda)/\mathscr{D}^{\mathrm{perf}}(\Lambda)$ is equivalent to $\underline{\operatorname{MCM}}(\Lambda)$.

Lemma.

If $X \in \operatorname{mod}\Lambda$ and $n := \operatorname{id}_\Lambda \Lambda$, then $\Omega^n X \in \operatorname{MCM}(\Lambda)$.

Proof.

This follows since $\operatorname{Ext}^m_\Lambda(\Omega^n X, \Lambda) \simeq \operatorname{Ext}^{m+n}_\Lambda(X, \Lambda)$ for all $m \in \mathbb{N}_+$.

Lemma.

If $n := \operatorname{id}_\Lambda \Lambda$, then $\operatorname{Ext}^m_\Lambda(X, Y) \simeq \operatorname{Ext}^{m+1}_\Lambda(X, \Omega Y)$ for all $m > n$.

Proof.

It is enough to apply $\operatorname{Hom}_\Lambda(X, -)$ to the exact sequence $0 \to \Omega Y \to P \to Y \to 0$ with P projective.

Corollary.

If $X, Y \in \operatorname{mod}\Lambda$, then

$$\operatorname{cx}_{\mathscr{D}^b(\Lambda)}(X, Y) = \operatorname{cx}_{\mathscr{D}^b(\Lambda)}(\Omega^p X, \Omega^q Y)$$

for all $p, q \in \mathbb{N}$.

LEMMA.

If $X, Y \in \mathrm{MCM}(\Lambda)$, then $\mathrm{Hom}_{\underline{\mathrm{MCM}}(\Lambda)}(X, Y[m]) \simeq \mathrm{Ext}^m_\Lambda(X, Y)$ for all $m \in \mathbb{N}_+$.

PROOF.

It follows by induction since we have an exact sequence $0 \to Z \to P \to Z[1] \to 0$ with P projective for each $Z \in \mathrm{MCM}(\Lambda)$.

COROLLARY.

If $X, Y \in \mathrm{MCM}(\Lambda)$, then

$$\mathrm{cx}_{\underline{\mathrm{MCM}}(\Lambda)}(X, Y) = \mathrm{cx}_{\mathscr{D}^b(\Lambda)}(X, Y).$$

LEMMA.

Let $n := \mathrm{id}_\Lambda \Lambda$ and $C := \Omega^n(\Lambda/\operatorname{rad}\Lambda)$. If $\mathrm{cx}_{\underline{\mathrm{MCM}}(\Lambda)}(K, C) = 0$ for $K \in \mathrm{MCM}(\Lambda)$, then K is a projective Λ-module.

PROOF.

Observe that

$$\begin{aligned} \mathrm{cx}_\Lambda(K) &= \mathrm{cx}_{\mathscr{D}^b(\Lambda)}(K, \Lambda/\operatorname{rad}\Lambda) \\ &= \mathrm{cx}_{\mathscr{D}^b(\Lambda)}(K, C) = \mathrm{cx}_{\underline{\mathrm{MCM}}(\Lambda)}(K, C) = 0, \end{aligned}$$

hence $\mathrm{pd}_\Lambda K < \infty$. The claim follows since the only modules of finite projective dimension in $\mathrm{MCM}(\Lambda)$ are the projective ones.

THEOREM.

$\mathscr{D}^b(\Lambda)/\mathscr{D}^{\mathrm{perf}}(\Lambda)$ satisfies the condition (Fgc) and

$$\dim(\mathscr{D}^b(\Lambda)/\mathscr{D}^{\mathrm{perf}}(\Lambda)) \geq \mathrm{cx}_\Lambda(\Lambda/\operatorname{rad}\Lambda) - 1.$$

PROOF.

Recall that $\mathscr{D}^b(\Lambda)/\mathscr{D}^{\mathrm{perf}}(\Lambda)$ is equivalent to $\underline{\mathrm{MCM}}(\Lambda)$, hence the first claim follows. Let $n := \mathrm{id}_\Lambda \Lambda$ and $C := \Omega^n(\Lambda/\operatorname{rad}\Lambda)$. We show that C is a periodicity generator. Indeed, if $\mathrm{cx}_{\underline{\mathrm{MCM}}(\Lambda)}(M, C) = 1$ for $M \in \mathrm{MCM}(\Lambda)$, then we construct a triangle $M \to M[n] \to K \to M[1]$ for some $n \in \mathbb{N}_+$ and $K \in \mathrm{MCM}(\Lambda)$ such that $\mathrm{cx}_{\underline{\mathrm{MCM}}(\Lambda)}(K, C) = 0$. According to the previous lemma $K = 0$ in $\underline{\mathrm{MCM}}(\Lambda)$, hence $M \simeq M[n]$. Consequently,

$$\dim(\mathscr{D}^b(\Lambda)/\mathscr{D}^{\mathrm{perf}}(\Lambda)) \geq \sup\{\mathrm{cx}_{\underline{\mathrm{MCM}}(\Lambda)}(M, C) \mid M \in \mathrm{MCM}(\Lambda)\} - 1.$$

Observe that

$$\begin{aligned} \mathrm{cx}_{\underline{\mathrm{MCM}}(\Lambda)}(C, C) &= \mathrm{cx}_{\mathscr{D}^b(\Lambda)}(C, C) \\ &= \mathrm{cx}_{\mathscr{D}^b(\Lambda)}(\Lambda/\operatorname{rad}\Lambda, \Lambda/\operatorname{rad}\Lambda) = \mathrm{cx}_\Lambda(\Lambda/\operatorname{rad}\Lambda), \end{aligned}$$

hence the claim follows.