THE DIMENSION OF A TRIANGULATED CATEGORY & REPRESENTATION DIMENSION OF EXTERIOR ALGEBRAS

BASED ON THE TALKS BY ANGELA HOLTMANN

The talks were based on the paper *Representation dimension of exterior algebras* by Raphaël Rouquier.

Definition.

The weak representation dimension $\operatorname{w.rep.dim} \mathscr{A}$ of an abelian category $\mathscr{A}$ is the smallest $l \in \mathbb{N}$, $l \geq 2$, such that there exists $M \in \mathscr{A}$ such that for each $L \in \mathscr{A}$ there exists a complex

$$C : \cdots \to 0 \to C_m \to \cdots \to C_n \to 0 \to \cdots$$

such that $C_m, \ldots, C_n \in \operatorname{add} M$, L is a direct summand of $H^0(C)$, $H^d(C) = 0$ for all $d \in \mathbb{Z}$, $d \neq 0$, and $n - m \leq l - 2$.

Remark.

Auslander's lemma implies that $\operatorname{w.rep.dim} A \leq \operatorname{rep.dim} A$ for an algebra A.

Remark.

The representation dimension is not invariant under derived equivalence.

Notation.

For a triangulated category $\mathscr{T}$, $M \in \mathscr{T}$, and $l \in \mathbb{N}_+$, we define $\langle M \rangle_l$ inductively as follows: $\langle M \rangle_1 := \operatorname{add}\{M[n] \mid n \in \mathbb{Z}\}$ and $\langle M \rangle_{l+1}$ is the additive closure of the category generated by $L \in \mathscr{T}$ such that there exists a distinguished triangle $M' \to L \to M'' \to M'[1]$ with $M' \in \langle M \rangle_1$ and $M'' \in \langle M \rangle_l$.

Definition.

The dimension $\dim \mathscr{T}$ of a triangulated category $\mathscr{T}$ is the smallest $d \in \mathbb{N}$ such that there exists $M \in \mathscr{T}$ with $\mathscr{T} = \langle M \rangle_{d+1}$.

Remark.

If $F : \mathscr{T} \to \mathscr{S}$ is a triangle functor between triangulated categories such that $\mathscr{S} = \operatorname{add}(F\mathscr{T})$, then $\dim \mathscr{S} \leq \dim \mathscr{T}$. In particular, if F is a derived equivalence, then $\dim \mathscr{T} = \dim \mathscr{S}$.

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Proposition.

If A is an algebra, then

$$\dim \mathscr{D}^b(A)/\operatorname{perf} A \leq \min\{\mathrm{LL}(A)-1, \mathrm{w.rep.dim}\, A-2\}.$$

Proof.

The proof is based on the fact proved by Rickard that for each $M \in \mathscr{D}^b(A)$ there exists $L \in \operatorname{mod} A$ and $n \in \mathbb{Z}$ such that $M \simeq L[n]$ in $\mathscr{D}^b(A)/\operatorname{perf} A$.

Proposition.

If A is an algebra, then

$$\dim \mathscr{D}^b(A) \leq \min\{\mathrm{gl.dim}\, A, \mathrm{rep.dim}\, A\}.$$

Proof.

We first prove that $\dim \mathscr{D}^b(A) \leq \mathrm{gl.dim}\, A$. Let

$$0 \to P_r \to \cdots \to P_0 \to A \to 0$$

be the minimal projective resolution of A over its enveloping algebra. It follows that $r = \mathrm{gl.dim}\, A$. Since $P \otimes_A C \in \langle A \rangle_1$ for each projective module P over the enveloping algebra of A and $C \in \mathscr{D}^b(A)$, it follows that $C \in \langle A \rangle_{r+1}$.

For the second inequality let M be a generator-cogenerator of M such that $\mathrm{rep.dim}\, A = \mathrm{gl.dim}\, M$. We have an essentially surjective triangle functor $\mathscr{K}^b(\operatorname{add} M) \to \mathscr{D}^b(A)$, hence $\dim \mathscr{D}^b(A) \leq \dim \mathscr{K}^b(\operatorname{add} M)$. Moreover, $\operatorname{add} M \simeq \operatorname{proj} \operatorname{End}_A(M)^{\mathrm{op}}$ and

$$\mathscr{K}^b(\operatorname{proj} \operatorname{End}_A(M)^{\mathrm{op}}) \simeq \mathscr{D}^b(\operatorname{End}_A(M)^{\mathrm{op}}).$$

Consequently,

$$\begin{aligned} \dim \mathscr{D}^b(A) &\leq \dim \mathscr{K}^b(\operatorname{add} M) = \dim \mathscr{K}^b(\operatorname{proj} \operatorname{End}_A(M)^{\mathrm{op}}) \\ &= \dim \mathscr{D}^b(\operatorname{End}_A(M)^{\mathrm{op}}) \leq \mathrm{gl.dim}\, \operatorname{End}_A(M)^{\mathrm{op}} = \mathrm{rep.dim}\, A. \end{aligned}$$

Proposition.

If A is selfinjective, then $\mathrm{rep.dim}\, A \leq \mathrm{LL}(A)$.

Theorem.

If $n \in \mathbb{N}_+$, then $\mathrm{rep.dim}\, \Lambda(k^n) = n+1$.

Remark.

The proof of the theorem makes use of differential modules. For an algebra A by a differential A-module we mean a pair (M, d) consisting of an A-module M and $d \in \operatorname{End}_A(M)$ such that $d^2 = 0$. The cohomology of a differential A-module (M, d) is by definition $\operatorname{Ker} d/\operatorname{Im} d$. The class of exact sequences of differential A-modules which split as sequences of A-modules establish a structure of an exact category in the category of differential A-modules. It is a Frobenius category, the corresponding stable category is called the homotopy category of differential A-modules. A morphism of differential A-modules is called a quasi-isomorphism if it induces an isomorphism of the cohomologies. The localization of the homotopy category with respect to the quasi-isomorphisms is denoted $\mathscr{D}\mathrm{diff}(A)$. We have an obvious forgetful functor $\mathscr{D}(A) \to \mathscr{D}\mathrm{diff}(A)$ which induces an isomorphism $\prod_{n\in\mathbb{N}} \mathrm{Ext}^n_A(X,Y) \simeq \mathrm{Hom}_{\mathscr{D}\mathrm{diff}}(X,Y)$.