FINITENESS OF REPRESENTATION DIMENSION

BASED ON THE TALK BY ANDRE BEINEKE

The talk was based on the paper *Finiteness of representation dimension* by Osamu Iyama.

Definition.
An ideal I in an artin algebra Λ is called a heredity ideal if $I^2 = I$, $I \cdot \operatorname{rad} \Lambda \cdot I = 0$ and I is a projective Λ-module.

Definition.
A chain

$$0 = I_r \subset I_{r-1} \subset \cdots \subset I_1 \subset I_0 = I$$

of ideals of an artin algebra Λ is called a heredity chain if I_{i-1}/I_i is a heredity ideal in Λ/I_i for each $i \in [1, r]$.

Definition.
An artin algebra Λ is called quasi-hereditary if it possesses a heredity chain of ideals.

Notation.
If X is a module over an artin algebra Λ, then

$$X' := \{f(x) \mid f \in \operatorname{rad}_\Lambda(X, X),\ x \in X\}.$$

Notation.
For a module M over an artin algebra Λ we define a chain

$$M_0 \supset M_1 \supset M_2 \supset \cdots$$

of submodules of M as follows: $M_0 := M$, $M_i := M'_{i-1}$ for $i \in \mathbb{N}_+$.

Notation.
For modules M and N over an artin algebra Λ we put

$$\eta_M N := \sum_{f \in \operatorname{Hom}_\Lambda(M,N)} \operatorname{Im} f.$$

Remark.
If M and N are modules over an artin algebra Λ, then there exists $n \in \mathbb{N}$ and $f : M^n \to N$ such that $\eta_M N = \operatorname{Im} f$.

Lemma.
Let M be a Λ-module over an artin algebra Λ and $m, n \in \mathbb{N}$. If $m \leq n$, then $\eta_{M_n} X \in \operatorname{add} M_n$ for all $X \in \operatorname{add} M_m$.

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Proof.

We prove the claim by induction on $n-m$. Without loss of generality we may assume that X is indecomposable. If $n=m$, then the claim is obvious, thus assume that $n>m$. Let $i:\eta_{M_n}X\to X$ be the canonical injection, and fix epimorphisms $f:M_n^u\to\eta_{M_n}X$ and $g:M_m^v\to M_n^u$. If $ifg\notin\operatorname{rad}_\Lambda(M_m^v,X)$, then i is an isomorphism and f splits, hence the claim follows. Now assume that $ifg\in\operatorname{rad}_\Lambda(M_m^v,X)$. This implies that $X\in\operatorname{add}M_{m+1}$, thus the claim follows by inductive hypothesis.

Notation.

For modules M and X over an artin algebra Λ we put

$$\langle X\rangle_M := \{f\in\operatorname{End}_\Lambda(M)\mid f\text{ factors through }\operatorname{add}X\}.$$

Lemma.

Let X,Y,Z be modules over an artin algebra Λ and $M:=X\oplus Y\oplus Z$. If $\eta_Y Z=Z$, $\operatorname{rad}_\Lambda(Y,Y)\subset\langle Z\rangle_M$ and $\eta_Y X\in\operatorname{add}Y$, then $\langle Y\oplus Z\rangle_M/\langle Z\rangle_M$ is a heredity ideal in $\operatorname{End}_\Lambda(M)/\langle Z\rangle_M$.

Theorem.

Let M be a module over an artin algebra Λ, $r:=\min\{i\in\mathbb{N}_+\mid M_r=0\}$, and $N:=\bigoplus_{i\in[0,r-1]}M_i$. If $I_j:=\langle\bigoplus_{i=j}^{r-1}M_i\rangle_N$ for $j\in[0,r]$, then

$$0=I_r\subset I_{r-1}\subset\cdots\subset I_1\subset I_0=\operatorname{End}_\Lambda(N)$$

is a heredity chain in $\operatorname{End}_\Lambda(N)$.

Proof.

For each $j\in[1,r]$ we use the previous lemma with $X:=\bigoplus_{i\in[0,j-1]}M_i$, $Y:=M_i$, and $Z:=\bigoplus_{i\in[j+1,r-1]}M_i$.

Theorem.

If M is a module over an artin algebra Λ, then there exists a Λ-module N such that $\operatorname{End}_\Lambda(M\oplus N)$ is quasi-hereditary.

Corollary.

If Λ is an artin algebra, then $\operatorname{rep.dim}\Lambda<\infty$.