# REPRESENTATION DIMENSION

BASED ON THE TALK BY DAIVA PUCINSKAITE

Notation.

If $M$ is a module over an artin algebra $\Lambda$, then by $I(M)$ we denote the injective envelope of $M$.

Notation.

For an artin algebra $\Lambda$ we define

$$\mathscr{A}(\Lambda) := \{\Gamma \mid \Gamma \text{ is an artin algebra, } \operatorname{dom.dim} \Gamma \geq 2, \\ \Lambda \text{ is Morita equivalent to } \operatorname{End}_\Gamma(I(\Gamma))^{\mathrm{op}}\}.$$

Remark.

If $\Lambda$ is an artin algebra, then

$$\mathscr{A}(\Lambda) = \{\operatorname{End}_\Lambda(N)^{\mathrm{op}} \mid N \text{ is a generator-cogenerator of } \operatorname{mod} \Lambda\}.$$

Definition.

If $\Lambda$ is an artin algebra, then the representation dimension $\operatorname{rep.dim} \Lambda$ of $\Lambda$ is defined by:

$$\operatorname{rep.dim} \Lambda := \begin{cases} 1 & \Lambda \text{ is semisimple,} \\ \min\{\operatorname{gl.dim} \Gamma \mid \Gamma \in \mathscr{A}(\Lambda)\} & \text{otherwise.} \end{cases}$$

Lemma.

If $\Lambda$ is an artin algebra which is not semisimple, then $\operatorname{rep.dim} \Lambda \geq 2$.

Proof.

Assume that $\operatorname{rep.dim} \Lambda \leq 1$ and choose $\Gamma \in \mathscr{A}(\Lambda)$ such that $\operatorname{gl.dim} \Gamma \leq 1$. If $0 \to \Gamma \to I_0 \xrightarrow{f} I_1$ is a minimal injective resolution of $\Gamma$, then $f$ is surjective, since $\operatorname{gl.dim} \Gamma \leq 1$. Moreover, $I_1$ is projective, since $\operatorname{dom.dim} \Gamma \geq 2$. Consequently, $f$ splits and $\Gamma$ is selfinjective. In particular, $I(\Gamma) = \Gamma$. Additionaly, $\Gamma$ is hereditary and selfinjective artin algebra, hence semisimple. But $\Lambda$ is Morita equivalent to $\operatorname{End}_\Gamma(\Gamma)^{\mathrm{op}} \simeq \Gamma$, thus semisimple, contradiction.

Proposition.

Let $\Lambda$ be an artin algebra.

(1) $\operatorname{rep.dim} \Lambda = 1$ if and only if $\Lambda$ is semisimple.
(2) $\operatorname{rep.dim} \Lambda = 2$ if and only if $\Lambda$ is representation-finite but not semisimple.

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PROOF.

(1) Follows from the definition and the above lemma.

(2) It follows from the theorem of Auslander stating the following:

- If $M$ is an additive generator of $\operatorname{mod}\Lambda$ for a representation-finite artin algebra $\Lambda$ and $\Gamma := \operatorname{End}_\Lambda(M)^{\mathrm{op}}$, then $\operatorname{gl.dim}\Gamma \leq 2$ and $\operatorname{dom.dim}\Gamma \geq 2$.
- If $\Gamma$ is an artin algebra such that $\operatorname{gl.dim}\Gamma \leq 2$ and $\operatorname{dom.dim}\Gamma \geq 2$, then $\operatorname{End}_\Gamma(I(\Gamma))^{\mathrm{op}}$ is representation-finite.

LEMMA.

Let $V$ be a module over an artin algebra $\Lambda$ and $\Gamma := \operatorname{End}_\Lambda(V)$. If for each $\Lambda$-module $M$ there exists an exact sequence $0 \to V_1 \to V_2 \to M \to 0$ with $V_1, V_2 \in \operatorname{add} V$, such that the sequence

$$0 \to \operatorname{Hom}_\Lambda(-, V_1) \to \operatorname{Hom}_\Lambda(-, V_2) \to \operatorname{Hom}_\Lambda(-, M) \to 0$$

is exact, then $\operatorname{gl.dim}\Gamma \leq 3$.

PROOF.

Let $\mathscr{V}$ be the category of contravariant coherent functors from $\operatorname{add} V$ to the category $Ab$ of abelian groups, i.e. the category of the functors of the form $\operatorname{Coker}\operatorname{Hom}_\Lambda(-, f)$, where $f$ is a morphism in $\operatorname{add} V$. It is known that $\mathscr{V}$ is equivalent to $\operatorname{mod}\Gamma$ and the projective objects in $\mathscr{V}$ are representable functors, i.e. $F \in \mathscr{V}$ is projective in $\mathscr{V}$ if and only if there exists $V' \in \operatorname{add} V$ such that $F \simeq \operatorname{Hom}_\Lambda(-, V')$. Thus in order to show that $\operatorname{gl.dim}\Gamma \leq 3$, it is enough to prove that for each $F \in \mathscr{V}$ there exists an exact sequence

$$0 \to V_1 \to V_2 \to V_3 \to V_4$$

with $V_1, V_2, V_3, V_4 \in \operatorname{add} V$, such that the sequence

$$0 \to \operatorname{Hom}_\Lambda(-, V_1) \to \operatorname{Hom}_\Lambda(-, V_2) \to \operatorname{Hom}_\Lambda(-, V_3) \\ \to \operatorname{Hom}_\Lambda(-, V_4) \to F \to 0$$

is exact. However, if $F \in \mathscr{V}$, then there exists $f : V_2 \to V_3$ with $V_2, V_3 \in \operatorname{add} V$ such that $F \simeq \operatorname{Coker}(-, f)$, hence it is enough to apply the condition from the lemma for $\operatorname{Ker} f$.

PROPOSITION.

Let $\Lambda$ be an artin algebra and $n \in \mathbb{N}$. If $\operatorname{rep.dim}\Lambda/\mathfrak{r}_\Lambda^{n-1} \leq 2$, then $\operatorname{rep.dim}\Lambda/\mathfrak{r}_\Lambda^n \leq 3$.

PROOF.

Without loss of generality we may assume that $\mathfrak{r}_\Lambda^n = 0$, i.e. $\Lambda/\mathfrak{r}_\Lambda^n \simeq \Lambda$. Since $\operatorname{rep.dim}\Lambda/\mathfrak{r}_\Lambda^{n-1} \leq 2$, there exists an additive generator $N$ of the full subcategory of $\operatorname{mod}\Lambda$ formed by $M \in \operatorname{mod}\Lambda$ such that $\mathfrak{r}_\Lambda^{n-1}M = 0$. Let $V := N \oplus \Lambda \oplus D(\Lambda)$ and $\Gamma := \operatorname{End}_\Lambda(V)^{\mathrm{op}}$. We show that $\operatorname{gl.dim}\Gamma \leq 3$ using the previous lemma, i.e. we prove that for each

$\Lambda$-module $M$ there exists an exact sequence $0 \to V_1 \to V_2 \to M \to 0$ with $V_1, V_2 \in \operatorname{add} V$, such that the sequence

$$0 \to \operatorname{Hom}_\Lambda(-, V_1) \to \operatorname{Hom}_\Lambda(-, V_2) \to \operatorname{Hom}_\Lambda(-, M) \to 0$$

is exact

Take $M \in \operatorname{mod}\Lambda$. Obviously, we may assume that $M$ is indecomposable. If $M \in \operatorname{add} V$, then the claim is obvious, thus assume that $M \notin \operatorname{add} V$. Let $M' := \{m \in M \mid \mathfrak{r}_\Lambda^{n-1} m = 0\}$ and let $g : P \to M/M'$ be the projective cover of $M/M'$. There exists $h : P \to M$ such that $g = ph$, where $p : M \to M/M'$ is the canonical projection. Let $f := [h, i] : P \oplus M' \to M$, where $i : M' \to M$ is the canonical embedding. Observe that $f$ is surjective. Moreover, if $K := \operatorname{Ker} f$, then

$$K \simeq \{p \in P \mid h(p) \in M'\} \subset \mathfrak{r}_\Lambda P \in \operatorname{add} V,$$

thus it remains to prove that $\operatorname{Hom}_\Lambda(X, f)$ is surjective for each $X \in \operatorname{add} V$. We may again assume that $X$ is indecomposable.

Obviously, $\operatorname{Hom}_\Lambda(X, f)$ is surjective if $X$ is projective. Moreover, if $\mathfrak{r}_\Lambda^{n-1} X = 0$, then $\operatorname{Hom}_\Lambda(X, i)$ is an isomorphism, hence $\operatorname{Hom}_\Lambda(X, f)$ is surjective. Thus assume that $X$ is injective and $\mathfrak{r}_\Lambda^{n-1} X \neq 0$. Let $S$ be the socle of $X$. Then $S$ is simple and $\mathfrak{r}_\Lambda^{n-1}(X/S) = 0$. In particular, $\operatorname{Hom}_\Lambda(X/S, i)$ is an isomorphism. Let $q : X \to X/S$ be the canonical projection. Since $M$ is indecomposable and not injective, $\varphi$ cannot be injective for $\varphi \in \operatorname{Hom}_\Lambda(X, M)$. Since $S$ is simple, this implies that $\operatorname{Hom}_\Lambda(q, M)$ is an isomorphism. Similarly one shows that $\operatorname{Hom}_\Lambda(q, M')$ is an isomorphism, using that $\mathfrak{r}_\Lambda^{n-1} X \neq 0$. Finally,

$$\operatorname{Hom}_\Lambda(X, i) \operatorname{Hom}_\Lambda(q, M') = \operatorname{Hom}_\Lambda(q, M) \operatorname{Hom}_\Lambda(X/S, i),$$

hence $\operatorname{Hom}_\Lambda(X, i)$ is also an isomorphism, thus the claim follows.

Corollary.

If $\mathfrak{r}_\Lambda^2 = 0$ for an artin algebra $\Lambda$, then $\operatorname{rep.dim} \Lambda \leq 3$.

Proposition.

If $\operatorname{gl.dim} \Lambda \leq 1$ for an artin algebra $\Lambda$, then $\operatorname{rep.dim} \Lambda \leq 3$.

Proof.

Let $V := \Lambda \oplus D(\Lambda)$ and $\Gamma := \operatorname{End}_\Lambda(V)^{\mathrm{op}}$. We again prove that $\operatorname{gl.dim} \Gamma \leq 3$ using the above lemma. Let $M$ be an indecomposable $\Lambda$-module. We may assume that $M$ is not injective. This implies that $M$ has no nonzero injective submodules. Let $f : P \to M$ be the projective cover of $M$. Since $\operatorname{gl.dim} \Lambda \leq 1$, $\operatorname{Ker} f$ is projective. Moreover, $\operatorname{Hom}_\Lambda(Q, f)$ is surjective for each projective $\Lambda$-module $Q$. Finally, if $I$ is injective, then $\operatorname{Hom}_\Lambda(I, M) = 0$, since $M$ has no nonzero injective submodules and the image of a map from an injective module is injective over hereditary algebras, and this finished the proof.