AUSLANDER ALGEBRAS

BASED ON THE TALK BY LINGLING YAO

The talk was based on Section VI.5 of *Representation Theory of Artin Algebras* by Auslander, Reiten and Smalø.

DEFINITION.
Let Σ be an Artin algebra. The dominant dimension $\operatorname{dom.dim}_\Sigma A$ of a Σ-module A is the supremum of the set of $t \in \mathbb{N}$ such that $I_0, \ldots, I_{t-1}$ are projective for a minimal injective resolution

$$0 \to A \to I_0 \to I_1 \to \cdots$$

of A.

DEFINITION.
An artin algebra Γ is called an Auslander algebra if $\operatorname{gl.dim} \Gamma \leq 2$ and $\operatorname{dom.dim}_\Gamma \Gamma \geq 2$.

DEFINITION.
Let Λ be an artin algebra and let $\mathscr{C}$ be the full subcategory of $\operatorname{mod} \Lambda$. A Λ-module M is called an additive generator of $\mathscr{C}$ if $\operatorname{add} M = \mathscr{C}$.

REMARK.
There exists an additive generator of $\operatorname{mod} \Lambda$ for an artin algebra Λ if and only if Λ is of finite representation type.

NOTATION.
If M is a module over an artin algebra Λ, then $\Gamma_M := \operatorname{End}_\Lambda(M)^{\mathrm{op}}$.

REMARK.
If M is a module over an artin algebra Λ, then $\operatorname{Hom}_\Lambda(M, -)$ induces an equivalence between $\operatorname{add} M$ and the category of projective Γ_M-modules.

REMARK.
If M and N are modules over an artin algebra Λ such that $\operatorname{add} M = \operatorname{add} N$, then Γ_M and Γ_N are Morita equivalent.

LEMMA.
Let M be an additive generator of $\operatorname{mod} \Lambda$ for an artin algebra Λ.

(1) If $P_1 \xrightarrow{g} P_0 \to X \to 0$ is a projective presentation of $X \in \operatorname{mod} \Gamma_M$, then there exists an exact sequence $0 \to A_2 \to A_1 \xrightarrow{h} A_0$ of Λ-modules such that $g = \operatorname{Hom}_\Lambda(M, h)$.
(2) $\operatorname{gl.dim} \Gamma_M \leq 2$.

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PROPOSITION.

Let M be an additive generator of $\operatorname{mod}\Lambda$ for an artin algebra Λ.

(1) If Λ is semisimple, then Λ and Γ_M are Morita equivalent.
(2) If Λ is not semisimple, then $\operatorname{gl.dim}\Gamma_M = 2$.

LEMMA.

Let M be an additive generator of $\operatorname{mod}\Lambda$ for an artin algebra Λ.

(1) If I is an injective Λ-module, then $\operatorname{Hom}_\Lambda(M, I)$ is an injective Γ_M-module.
(2) If $0 \to A \to I_0 \to I_1$ is a minimal injective resolution of a Λ-module A, then
$$0 \to \operatorname{Hom}_\Lambda(M, A) \to \operatorname{Hom}_\Lambda(M, I_0) \to \operatorname{Hom}_\Lambda(M, I_1)$$
is a minimal injective resolution of $\operatorname{Hom}_\Lambda(M, A)$.
(3) A Γ_M-module N is projective-injective if and only if there exists an injective Λ-module I such that $N \simeq \operatorname{Hom}_\Lambda(M, I)$.
(4) $\operatorname{Hom}_\Lambda(M, -)$ induces an equivalence between the category of injective Λ-modules and the category of projective-injective Γ_M-modules.

PROPOSITION.

Let M be an additive generator of $\operatorname{mod}\Lambda$ for an artin algebra Λ which is not semisimple.

(1) $\operatorname{gl.dim}\Gamma_M = 2 = \operatorname{dom.dim}_{\Gamma_M}\Gamma_M$.
(2) $\Lambda \simeq \operatorname{End}_{\Gamma_M}(D(M))^{\mathrm{op}}$.

REMARK.

If M is an additive generator of $\operatorname{mod}\Lambda$ for an artin algebra Λ, then $D(M)$ and $\operatorname{Hom}_\Lambda(M, D(\Lambda))$ are isomorphic Γ_M-modules.

LEMMA.

Let Σ be an artin algebra.

(1) If $\operatorname{pd}_\Sigma A = n < \infty$ for a Σ-module A, then $\operatorname{Ext}^n_\Sigma(A, \Sigma) \neq 0$.
(2) Suppose $\operatorname{gl.dim}\Sigma = n < \infty$.
   (a) $\operatorname{id}_\Sigma \Sigma = n$.
   (b) If
   $$0 \to \Sigma \to I_0 \to \cdots \to I_n \to 0$$
   is a minimal injective resolution of Σ, then $\bigoplus_{i\in[0,n]} I_i$ is an additive generator for the category of injective Λ-modules.

PROPOSITION.

Let Q be an additive generator for the category of projective-injective Σ-modules for an artin algebra Σ such that $\operatorname{gl.dim}\Sigma = 2 = \operatorname{dom.dim}\Sigma$, and $\Lambda := \operatorname{End}_\Sigma(Q)^{\mathrm{op}}$.

(1) Λ is of finite representation type but not semisimple.

(2) $\Sigma \simeq \mathrm{End}_\Lambda(D(Q))^{\mathrm{op}}$.

THEOREM.

Let

$$\mathscr{X} := \{(\Lambda, M) \mid \Lambda \text{ is representation finite, but not semisimple and } M \text{ is an additive generator of } \operatorname{mod}\Lambda\},$$

and

$$\mathscr{Y} := \{(\Gamma, Q) \mid \Gamma \text{ is not semisimple Auslander algebra and } Q \text{ is an additive generator of the category of projective-injective } \Gamma\text{-modules}\}.$$

Then

$$\mathscr{X} \ni (\Lambda, M) \mapsto (\mathrm{End}_\Lambda(M)^{\mathrm{op}}, D(M)) \in \mathscr{Y}$$

and

$$\mathscr{Y} \ni (\Gamma, Q) \mapsto (\mathrm{End}_\Gamma(Q)^{\mathrm{op}}, D(Q)) \in \mathscr{X}$$

are mutually inverse bijections (modulo isomorphism).