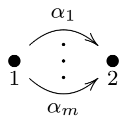
LOCALIZATION IN KRONECKER MODULI SPACES AND APPLICATIONS

BASED ON THE TALK BY THORSTEN WEIST

Assumption.

Throughout the talk $m \geq 3$ will be a fixed integer. We also denote by Δ the m-Kronecker quiver



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Finally, by μ we denote the slope function $\mu : \mathbb{Z}^2 \to \mathbb{R}$, $(d, e) \mapsto \frac{d}{d+e}$.

Definition.

A representation X is called stable, if $\mu(Y) < \mu(X)$ for all proper nonzero subrepresentations Y of X.

Remark.

A representation X of dimension vector (d, e) is stable if and only if

$$\dim_k \Big(\sum_{k=1}^{m} X_{\alpha_k}(U) \Big) > \frac{e}{d} \dim_k U$$

for each proper nonzero subspace U of X_1.

Notation.

If $(d, e) = 1$ for $d, e \in \mathbb{N}$, then we denote by $K^m_{d,e}$ the moduli space of stable representations of dimension vector (d, e).

Notation.

Let $T = (\mathbb{C}^*)^m$. If $(d, e) = 1$, $d, e \in \mathbb{N}$, then the action of T on the representation space by multiplication induces an action on $K^m_{d,e}$.

Notation.

We define the quiver $\hat{\Delta}$ by $\hat{\Delta}_0 := \Delta_0 \times \mathbb{Z}^m$ and

$$\hat{\Delta}_1 := \{(\alpha_i, \chi) : (1, \chi) \to (2, \chi + \mathbf{e}_i) \mid i \in [1, m],\ \chi \in \mathbb{Z}^m\}.$$

$\mathbb{Z}^m$ acts $\hat{\Delta}$ by $\mu(i, \chi) = (i, \chi + \mu)$ and this induces an action on the dimension vectors.

Remark.

If $X \in (K^m_{d,e})^T$, then there exist $\mathbb{Z}^m$-gradings in X_1 and X_2 such that

Date: 07.12.2007.

$X_{\alpha_i}((X_1)_\chi) \subset (X_2)_{\chi+\mathbf{e}_i}$. Consequently, X determines the representation of $\hat{\Delta}$ of dimension vector $(\dim_k(X_1)_\chi, \dim_k(X_2)_\chi)_{\chi\in\mathbb{Z}^m}$.

Theorem (Reineke).

If $(d,e) = 1$ for $d, e \in \mathbb{N}$, then $(K^m_{d,e})^T$ is isomorphic to the disjoint union $\bigcup K_{\mathbf{d}}(\hat{\Delta})$, where $\mathbf{d}$ ranges all equivalence classes of dimension vectors such that $\sum_{\chi\in\mathbb{Z}^m} d_{1,\chi} = d$ and $\sum_{\chi\in\mathbb{Z}^m} d_{1,\chi} = e$. In particular,

$$\chi(K^m_{d,e}) = \sum \chi K_{\mathbf{d}}(\hat{\Delta}).$$

Theorem.

If $(d,e) = 1$ for $d, e \in \mathbb{N}$, then there exists unique $d_s \in [0,d]$, $e_s \in [0,e]$, and $C_{d,e} > 0$, such that $(d_s, e_s) = 1$ and

$$\chi(K^m_{d_e+nd, e_s+ne}) \geq \exp(C_{d,e} nd)$$

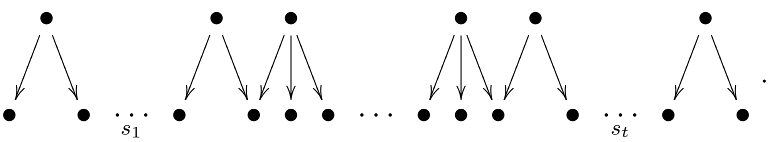
for $n \gg 0$.

Assumption.

We assume that $m = 3$. Moreover, since $K^m_{d,e} \simeq K^m_{e,d}$ and $K^m_{d,e} \simeq K^m_{e,me-d}$, we may assume that $d < e < 2e$.

Definition.

If $s = (s_1, \ldots, s_t) \in \mathbb{N}^t$, then we denote by $\Delta(s)$ the quiver with



We will often identify s with $\Delta(s)$. We say that s is compatible with (d,e) if $s_1 + \cdots + s_t + t - 1 = d$ and $s_1 + \cdots + s_t + 2t - 2 = e$. s is called stable if there exists a stable representation of $\Delta(s)$ of dimension vector (1). If s is stable, then there exists $l \in \mathbb{N}_+$ such that s is of simple type l, i.e. $s_i \in \{l-1, l\}$ for all $i \in [1,t]$. Finally, $\hat{s} = (s_1 - 1, s_2, \ldots, s_t)$.

Definition.

If $l \in \mathbb{N}_+$, then we define function η^l_n and θ^l_n between the set of quivers of simple type l, by

$$\eta^l_n(l-1) = (l-1, l^{n-1}), \qquad \eta^l_n(l) = (l-1, l^n),$$
$$\theta^l_n(l-1) = ((l-1)^{n+1}, l), \qquad \theta^l_n(l) = ((l-1)^n, l).$$

One can show, that if s is stable of simple type l, then $\theta^l_n(\hat{s})$ and $\eta^l_n(\hat{s})$ are stable of simple type l.

Lemma.

Let $d, e \in \mathbb{N}$, $(d,e) = 1$.

(1) There exists (up to coloring) a unique stable quiver $s_{d,e}$ of simple type (d,e).

(2) If d_s is minimal such that $d \mid 1 + d_s e$ and $e_s = (1 + d_s e)/d$, then $s_{d_s+nd,e_s+ne} = (s_{d_s,e_s}, (\hat{s}_{d,e})^n)$.

(3) There exists a unique sequence $(l_1, \ldots, l_n)$ such that

$$\hat{s}_{d,e} = \eta^{l_1}_{l_2} \circ \cdots \circ \eta^{l_1}_{l_{n-1}} \circ \theta^{l_1}_{l_n}(l_1)$$

or

$$\hat{s}_{d,e} = \eta^{l_1}_{l_2} \circ \cdots \circ \eta^{l_1}_{l_{n-1}} \circ \eta^{l_1}_{l_n}(l_1).$$