SOME ASPECTS OF DJAMENT'S PROOF OF THE ARTINIAN CONJECTURE ON LEVEL 3

BASED ON THE TALK BY ALEXANDER ZIMMERMANN

The talk is based on the papers *Catégories de foncteurs en grassmanniennes et filtration de Krull* and *Le foncteur* $V \mapsto \mathbb{F}_2[V]^{\otimes 3}$ *entre* $\mathbb{F}_2$-*espaces vectoriels est noethérien* by Aurélien Djament.

Notation.
Let $k = \mathbb{F}_2$. By $\mathscr{F} = \mathscr{F}(k)$ we denote the category of functors $\operatorname{mod} k \to \operatorname{Mod} k$.

Lemma.
$\mathscr{F}$ is an abelian category with enough projective and injective.

Remark.
For $V \in \operatorname{mod} k$, let P_V be the functor $k[\operatorname{Hom}_k(V, -)]$, where $k[X]$ denotes the vector space with basis X for a set X. Then $\operatorname{Hom}_{\mathscr{F}}(P_V, F) \simeq F(V)$ for each $F \in \mathscr{F}$. It follows that P_V is a set of projective generators of $\mathscr{F}$.

Remark.
For $F \in \mathscr{F}$, let $(DF)(V) = (F(V^*))^*$, where $(-)^*$ denotes the k-dual. It is known $\operatorname{Hom}_{\mathscr{F}}(F, DG) = \operatorname{Hom}_{\mathscr{F}}(G, DF)$, hence $I_V = DP_V$ is an injective object for each $V \in \operatorname{mod} k$.

Definition.
We call a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of $n \in \mathbb{N}$ 2-regular, if $\lambda_1 > \cdots > \lambda_l > 0$.

Notation.
For a 2-regular partition λ we denote by W_λ the subfunctor of $\Lambda^{\lambda^1} \otimes \cdots \otimes \Lambda^{\lambda_l}$ such that $W_\lambda(V)$ is the subspace of $\Lambda^{\lambda_1}(V) \otimes \cdots \otimes \Lambda^{\lambda_l}(V)$ generated by

$$(u_1 \wedge \cdots \wedge u_{\lambda_1}) \otimes \cdots \otimes (u_1 \wedge \cdots \wedge u_{\lambda_l}),\ u_1, \ldots, u_{\lambda_1} \in V,$$

for $V \in \operatorname{mod} k$.

Theorem.
If λ is 2-regular partition, then W_λ has a unique quotient S_λ. Moreover, S_λ's are the simple objects in $\mathscr{F}$.

Date: 30.11.2007.

Conjecture (Aritinian Conjecture).

I_V is artinian for $V \in \operatorname{mod} k$. Equivalently, P_V is noetherian for $V \in \operatorname{mod} k$.

Remark.

Kuhn proved in 1994 this conjecture for vector spaces of dimension 1. Powell proved in 1998 this conjecture for two-dimensional vector spaces. Finally, Djament showed that this conjecture for vector spaces of dimension 3. In the talk we describe the tools used in his proof.

Notation.

Let $\mathscr{E}_{\mathrm{gr}}$ be the category with objects (V, W) such that $V \in \operatorname{mod} k$ and W is a subspace of V. If $(V, W), (V', W') \in \mathscr{E}_{\mathrm{gr}}$, then by a morphism we mean a linear map $f : V \to V'$ such that $f(W) = W'$. By $\mathscr{E}_{\mathrm{gr},n}$ ($\mathscr{E}_{\mathrm{gr},\leq n}$) we denote the full subcategory with the objects (V, W) such that $\dim_k W = n$ ($\dim_k W \leq n$, respectively). Moreover, $\mathscr{F}_{\mathrm{gr}}$ ($\mathscr{F}_{\mathrm{gr},n}$, $\mathscr{F}_{\mathrm{gr},\leq n}$) denotes the category of functors $\mathscr{E}_{\mathrm{gr}} \to \operatorname{Mod} k$ ($\mathscr{E}_{\mathrm{gr},n} \to \operatorname{Mod} k$, $\mathscr{E}_{\mathrm{gr},\leq n} \to \operatorname{Mod} k$, respectively).

Fix $n \in \mathbb{N}$. We have restriction functors $R_n : \mathscr{F}_{\mathrm{gr}} \to \mathscr{F}_{\mathrm{gr},n}$ and $R_{\leq n} : \mathscr{F}_{\mathrm{gr}} \to \mathscr{F}_{\mathrm{gr},\leq n}$. The Grassmann integral $\omega : \mathscr{F}_{\mathrm{gr}} \to \mathscr{F}$ is defined by $\omega(F)(V) = \bigoplus_{W \subset V} F(V, W)$. On the other hand, we have the functor $\iota : \mathscr{F} \to \mathscr{F}_{\mathrm{gr}}$ given by $\iota(F)(V, W) = F(V)$. It follows that (ω, ι) is an adjoint pair. Finally, let $P_n : \mathscr{F}_{\mathrm{gr},n} \to \mathscr{F}_{\mathrm{gr}}$ be defined by

$$(\mathscr{P}_n F)(V, W) = \begin{cases} F(V, W) & (V, W) \in \mathscr{E}_{\mathrm{gr},n}, \\ 0 & \text{otherwise}, \end{cases}$$

and $\omega_n = \omega \mathscr{P}_n$.

Definition.

For $L \in \operatorname{mod} k$ we define $\Delta_L : \mathscr{F}_{\mathrm{gr}} \to \mathscr{F}_{\mathrm{gr}}$ by $\Delta_L(F)(V, W) = F(V \oplus L, W)$. The assignment $L \mapsto \Delta_L$ is functorial and $\Delta_k = \operatorname{Id} \oplus \Delta$, since the exact sequence $0 \to k \to 0$ induces the sequence $\Delta_0 \to \Delta_k \to \Delta_0$. We call Δ the difference. $F \in \mathscr{F}$ is called polynomial of degree d if $\Delta^{d+1}(F) = 0$.

Proposition.

$(\Delta_k, - \otimes_k P_k)$ are adjoint. Moreover, the finite composition length objects in $\mathscr{F}_{\mathrm{gr},\leq n}$ and $\mathscr{F}_{\mathrm{gr},n}$ are polynomial objects with image in $\operatorname{mod} k$.

Definition.

We call the subcategory $\mathscr{B}$ of an abelian category $\mathscr{A}$ closed under short exact sequence if for each every short exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ such that $A_i, A_j \in \mathscr{B}$ for $i \neq j$, $A_1, A_2, A_3 \in \mathscr{B}$.

Notation.

We denote by $\mathscr{F}^{\omega-\mathrm{cons}(n)}$ the smallest subcategory of $\mathscr{F}$ closed under short exact sequences and containing $\omega_{\leq n}(X)$ for all finite object X in $\mathscr{F}_{\mathrm{gr},\leq n}$.

Definition.

We call the subcategory $\mathscr{B}$ of an abelian category $\mathscr{A}$ thick if for each every short exact sequence $0 \to A \to B \to C \to 0$, $B \in \mathscr{B}$ if and only if $A, B \in \mathscr{B}$.

Proposition.

If $\mathscr{F}^{\omega-\mathrm{cons}(i)}$ is thick for each $i \in [1, n]$, then $P_V \otimes F$ is noetherian for each finite object F and V with $\dim V \leq n$.

Definition.

If $F \in \mathscr{F}$, then $- \otimes_k F : \mathscr{F} \to \mathscr{F}$ commutes with limits, hence possesses a left adjoint which we denote $(- : F)$ all call the division by F. It follows that $((- : F) : G) \simeq ((- : G) : F)$.

Theorem.

Let $L(i)$ be the injective hull of $S_{i,i-1,\ldots,1}$. If $F \in \mathscr{F}^{\omega-\mathrm{cons}(i-1)}$ for each F such that $(F : L(i)) = 0$ and $\mathscr{F}^{\omega-\mathrm{cons}(j)}$ is thick for each $j \in [1, i-1]$, then $\mathscr{F}^{\omega-\mathrm{cons}(j)}$ is thick for each $j \in [1, i]$.