GIT-CONES FOR QUIVERS

BASED ON THE TALK BY DAIVA PUCINSKAITE

The talk is based on the paper *Notes on GIT-fans for quivers* by Calin Chindris.

Assumption.

Throughout the talk we assume that Q is a quiver without oriented cycles and $\beta \in \mathbb{N}^{Q_0}$.

Notation.

If $\alpha \in \mathbb{N}^{Q_0}$, then we write $\alpha \hookrightarrow \beta$ if for each $W \in \operatorname{rep}_Q(\beta)$ there exists a subrepresentation U of W such that $\mathbf{dim}\, U = \alpha$.

Definition.

We call

$$C(Q, \beta) := \{\sigma \in \mathbb{R}^{Q_0} \mid \sigma(\beta) \text{ and } \forall\ \alpha \hookrightarrow \beta : \sigma(\alpha) \leq 0\}$$

the cone of effective weights for β.

Remark.

If $\operatorname{rep}(Q, \beta)^{\mathrm{ss}}_{\sigma} \neq \varnothing$ for $\sigma \in \mathbb{Z}^{Q_0}$, then $\sigma \in C(Q, \beta)$.

Definition.

If $\sigma_1, \sigma_2 \in C(Q, \beta)$, then we call σ_1 and σ_2 GIT-equivalent and write $\sigma_1 \sim \sigma_2$, if $\operatorname{rep}_Q(\beta)^{\mathrm{ss}}_{\sigma_1} = \operatorname{rep}_Q(\beta)^{\mathrm{ss}}_{\sigma_2}$.

Notation.

For $\sigma \in C(Q, \beta)$ we denote by $\langle \sigma \rangle$ the equivalence class of σ under the above relation.

Notation.

If $\sigma \in C(Q, \beta)$, then

$$C(\sigma) := \{\sigma' \in C(Q, \beta) \mid \operatorname{rep}_Q(\beta)^{\mathrm{ss}}_{\sigma} \subset \operatorname{rep}_Q(\beta)^{\mathrm{ss}}_{\sigma'}\}.$$

Remark.

If $\sigma_1, \sigma_2 \in C(Q, \beta)$, then $\sigma_1 \sim \sigma_2$ if and only if $\sigma_1 \in C(\sigma_2)$ and $\sigma_2 \in C(\sigma_1)$.

Theorem.

If $\sigma \in \mathbb{R}^{Q_0}$, then $C(\sigma)$ is a rational polyhedral cone.

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Proof.

Let

$$D_\sigma := \{\mathbf{dim}\, U \mid U \subset W \text{ and } W \in \mathrm{rep}_Q(\beta)^{\mathrm{ss}}_\sigma\}.$$

Then

$$C(\sigma) := \{\sigma' \in \mathbb{R}^{Q_0} \mid \sigma'(\beta) = 0 \text{ and } \forall\, \alpha \in D_\sigma : \sigma'(\alpha) \leq 0\}.$$

Remark.

The set $\{C(\sigma) \mid \sigma \in C(Q,\beta)\}$ is finite. Indeed, if $C(\sigma_1) \neq C(\sigma_2)$ for $\sigma_1, \sigma_2 \in \mathbb{R}^{Q_0}$, then there exists $\alpha \in \mathbb{N}^{Q_0}$ such that $\alpha \leq \beta$ and either $\sigma_1(\alpha) \leq 0$ and $\sigma_2(\alpha) > 0$ or $\sigma_1(\alpha) > 0$ and $\sigma_2(\alpha) \leq 0$. Since the set of such dimension vectors if finite, the claim follows.

Definition.

For $W \in \mathrm{rep}_Q(\beta)$ we call

$$\Omega(W) := \{\sigma \in C(Q,\beta) \mid W \in \mathrm{rep}_Q(\beta)^{\mathrm{ss}}_\sigma\}$$

the orbit cone of W.

Remark.

If $W \in \mathrm{rep}_Q(\beta)$, then $\Omega(W)$ is a rational polyhedral cone.

Remark.

The set $\{\Omega(W) \mid W \in \mathrm{rep}_Q(\beta)\}$ is finite.

Definition.

If $\sigma \in C(Q,\beta)$, then we call $W \in \mathrm{rep}_Q(\beta)^{\mathrm{ss}}_\sigma$ σ-polystable, if $\mathrm{GL}(\beta)W$ is closed in $\mathrm{rep}_Q(\beta)^{\mathrm{ss}}_\sigma$.

Proposition.

If $\sigma \in C(Q,\beta)$, then

$$C(\sigma) = \bigcap_{\substack{W \in \mathrm{rep}_Q(\beta)^{\mathrm{ss}}_\sigma \\ W \text{ is } \sigma\text{-polystable}}} \Omega(W).$$

Theorem.

If $\sigma \in C(Q,\beta)$, then $\langle \sigma \rangle = \mathrm{relint}(\sigma)$.