STABILITY IN ABELIAN CATEGORIES

BASED ON THE TALK BY NILS MAHRT

The talk was based on the paper *Stability for an abelian category* by Alexei Rudakov.

§1. General stability

Assumption.

Throughout the talk $\mathscr{A}$ we be an abelian category.

Definition.

By a total preorder in $\mathscr{A}$ we mean a relations $\leq$ on the non-zero objects of $\mathscr{A}$ such that for all non-zero $A, B, C \in \mathscr{A}$ the following hold:

(1) $A \leq B$ and $B \leq A$ if $A \simeq B$,
(2) if $A \leq B$ and $B \leq C$, then $A \leq C$,
(3) either $A \leq B$ or $B \leq A$.

Assumption.

For the rest of this sections talk we assume that $\leq$ is a fixed total preorder in $\mathscr{A}$.

Notation.

If $A, B \in \mathscr{A}$ are non-zero, then we write $A \approx B$ if $A \leq B$ and $B \leq A$. We also write $A < B$ if $A \leq B$ but $A \not\approx B$.

Remark.

If $A, B \in \mathscr{A}$ are non-zero, then either $A < B$ or $A \approx B$ or $A > B$.

Definition.

A total preorder $\leq$ is called a stability structure if it has a seesaw property, i.e. for each $\circ \in \{<, \simeq, >\}$ and each short exact sequence

$$0 \to A \to B \to C \to 0$$

with $A, B, C \in \mathscr{A}$ non-zero,

$$A \circ B \iff A \circ C \iff B \circ C.$$

Assumption.

For the rest of this section we assume that $\leq$ is a stability structure.

Lemma.

For each $\circ \in \{<, \simeq, >\}$, each short exact sequence

$$0 \to A \to B \to C \to 0$$

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with $A, B, C \in \mathscr{A}$ non-zero, and each $D \in \mathscr{A}$ non-zero, if $A \circ D$ and $C \circ D$, then $B \circ D$.

Definition.

We call $B \in \mathscr{A}$ stable if $B \neq 0$ and for each proper non-zero subobject A of B, $A < B$.

Definition.

We call $B \in \mathscr{A}$ stable if $B \neq 0$ and for each non-zero subobject A of B, $A \leq B$.

Theorem.

If $\varphi : A \to B$ is non-zero for semistable A and B with $A \geq B$, then the following hold:

(1) $A \approx B$,
(2) if B is stable, then φ is an epimorphism,
(3) if A is stable, then φ is a monomorphism,
(4) if A and B are stable, then φ is an isomorphism.

Proof.

Since A and B are semistable, $A \leq \operatorname{Im} \varphi \leq B$, hence (1) follows. Moreover, if B is stable, then $\operatorname{Im} \varphi = B$ since $\operatorname{Im} \varphi \neq 0$ and $\operatorname{Im} \varphi \approx B$, which implies (2). Additionally, if $\operatorname{Ker} \varphi \neq 0$, then $\operatorname{Ker} \varphi \simeq A$, hence A cannot be stable. This implies (3). Finally, (4) follows immediately from (2) and (3).

§2. Slope stability

Assumption.

Throughout this section we assume that $c, r : \mathscr{A} \to \mathbb{R}$ are functions which are additive on exact sequences and $r(A) > 0$ for all non-zero $A \in \mathscr{A}$.

Notation.

For a non-zero $A \in \mathscr{A}$ let $\mu(A) := \frac{c(A)}{r(A)}$.

Definition.

We define the total preorder on $\mathscr{A}$ by

$$A \leq B \iff \mu(A) \leq \mu(B)$$

for non-zero $A, B \in \mathscr{A}$.

Remark.

If $A, B \in \mathscr{A}$ are non-zero, then

$$A \leq B \iff c(A)r(B) < c(B)r(A).$$

Lemma.

$\leq$ defines a stability structure.

Proof.

We have to show that $\leq$ has a seesaw property. Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence with non-zero $A, B, C \in \mathscr{A}$. Observe that $c(B) = c(A) + c(C)$ and $r(B) = r(A) + r(C)$. Now the claim follows immediately from the above remark.

Definition.

Let $\theta : K_0(\mathscr{A}) \to \mathbb{R}$ be a group homomorhism. We call a non-zero $M \in \mathscr{A}$ θ-stable if $\theta(M) = 0$ and $\theta(N) > 0$ for each non-zero proper subobject of M.

Proposition.

Let $M \in \mathscr{A}$ be non-zero and $\theta := -c + \frac{c(M)}{r(M)} r$. Then M is θ-stable if and only if M is stable with respect to the stability structure determined by (c, r).

§3. Filtrations

Lemma.

Let $B, D \in \mathscr{A}$ be non-zero and assume that B has a filtration

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_m = B$$

with factors $G_i := F_i / F_{i-1}$, $i \in [1, m]$. If $\circ \in \{<, \approx, >\}$ and $G_i \circ D$ for all $i \in [1, m]$, then $B \circ D$.

Lemma.

Let $B \in \mathscr{A}$ be non-zero and assume that B has a filtration

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_m = B$$

with factors $G_i := F_i / F_{i-1}$, $i \in [1, m]$. Additionally, let $G_{i,j} := G_i / G_{j-1}$ for $i \in [1, m]$ and $j \in [1, i]$. If $G_m < \cdots < G_1$, then for $G_{i,j} < G_{p,q}$ if $i \geq p$, $j \geq q$, and $(i, j) \neq (p, q)$.

Definition.

We call a non-zero object in $\mathscr{A}$ quasi-noetherian if any chain

$$A_1 \subset A_2 \subset A_3 \subset \cdots$$

of non-zero subobjects of B such that $A_n \leq A_{n+1}$ for all $n \in \mathbb{N}$ stabilizes.

Definition.

We call a non-zero object in $\mathscr{A}$ weakly noetherian it is quasi-noetherian and any chain

$$A_1 \subset A_2 \subset A_3 \subset \cdots$$

of non-zero subobjects of B such that $A_n \geq A_{n+1}$ for all $n \in \mathbb{N}$ stabilizes.

Definition.

We call a non-zero object in $\mathscr{A}$ weakly artinian if any chain

$$A_1 \supset A_2 \supset A_3 \supset \cdots$$

of non-zero subobjects of B such that $A_n \leq A_{n+1}$ for all $n \in \mathbb{N}$ stabilizes.

Proposition.

If $B \in \mathscr{A}$ is non-zero, quasi-noetherian and weakly artinian, then there exists a uniquely determined non-zero subobject $B^\#$ of B such that

(1) if A is a non-zero subobject of B, then $A \leq B^\#$,
(2) if A is a non-zero subobject of B and $A \approx B^\#$, then $A \subset B^\#$.

Moreover, $B^\#$ is semistable and B is semistable if and only if $B = B^\#$.

Theorem.

Assume that every non-zero object in $\mathscr{A}$ is weakly artinian and weakly noetherian. Then for any non-zero $B \in \mathscr{A}$ there exists a unique filtration

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_m = B$$

with factors $G_i := F_i/F_{i-1}$, $i \in [1, m]$, such that G_i is semistable for each $i \in [1, m]$

Proof.

Let $F_0 := 0$. Fix $n \in \mathbb{N}$ and assume that we have already defined F_n. If $F_n \neq B$, then let $\pi : B \to B/F_n$ be the canonical projection. Let $F_{n+1} := \pi^{-1}((B/F_n)^\#)$. Observe that $G_{n+1} \simeq (B/F_n)^\#$ is semistable. If $n > 0$, then we have a short exact sequence

$$0 \to G_n \to F_{n+1}/F_{n-1} \to G_{n+1} \to 0$$

which implies that $G_n > G_{n+1}$. Moreover, since $G_n \subsetneq F_{n+1}/F_{n-1}$ and $G_n = (B/F_{n-1})^\#$, hence $G_n > F_{n+1}/F_{n-1}$. Consequently, $G_n > G_{n+1}$, and $F_n > F_{n+1}$. Since B it weakly noetherian, it implieas that $F_m = B$ for some $m \in \mathbb{N}_+$ and finishes the proof of existence.