THE GABRIEL–ROITER FILTATIONS OF THE MODULES IN A HOMOGENEOUS TUBE

CLAUS MICHAEL RINGEL

Assumption.

Throughout the talk Λ we be an artin algebra.

Notation.

For a Λ-module M we denote by $|M|$ its length.

Definition.

Let M be Λ-module. A filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n$$

is called an M-filtration, if $M_i/M_{i-1} \simeq M$ for each $i \in [1, n]$.

Notation.

For a Λ-module M we denote by $\mathscr{F}(M)$ the category of modules with M-filtrations.

Definition.

A monomorphism $f : X \to Y$ with $X \neq 0$ is called mono-irreducible if f does not split and for each monomorphism $g : X \to Z$ and $h : Z \to Y$ such that $f = hg$ either h is an isomorphism or g splits. We also call each map $0 \to Y$ mono-irreducible.

Definition.

A filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n$$

is called homogeneous if the inclusion $M_{i-1} \subset M_i$ is homogeneous for each $i \in [1, n]$.

Notation.

$\mathfrak{P}_{\text{fin}}(\mathbb{N}_+) := \{I \subset \mathbb{N}_+ \mid I \text{ is finite}\}$.

Definition.

Let $I, J \in \mathfrak{P}_{\text{fin}}(\mathbb{N}_+)$. Then $I < J$ if $\min((I \setminus J) \cup (J \setminus I)) \in J$.

Remark.

Let $I, J \in \mathfrak{P}_{\text{fin}}(\mathbb{N}_+)$. Then $I < J$ if and only if

$$\sum_{i \in I} \frac{1}{2^i} < \sum_{j \in J} \frac{1}{2^j}.$$

Date: 02.11.2007.

Definition.

By a Gabriel–Roiter measure we mean a function $\mu : \operatorname{mod}\Lambda \to \mathfrak{P}_{\mathrm{fin}}(\mathbb{N})$ defined for a Λ-module M by the condition that $\mu(M)$ is the maximum of the sets $\{l_1, \ldots, l_t\}$ such that there exists a filtration

$$M_1 \subset \cdots \subset M_t \subset M$$

with M_i indecomposable and $|M_i| = l_i$ for each $i \in [1, t]$.

Definition.

If $\mu(M) = \{1, \ldots, l_t\}$ for a Λ-module M, then any filtration

$$M_1 \subset \cdots \subset M_t \subset M$$

such that M_i is indecomposable and $|M_i| = l_i$ for each $i \in [1, t]$, is called a Gabriel–Roiter filtration.

Definition.

Let $I, J \in \mathfrak{P}_{\mathrm{fin}}(\mathbb{N}_+)$. We say J starts with I if $I = J \cap \{1, \ldots, \max I\}$.

Definition.

Let U be an indecomposable submodule of a Λ-module Y. We say that U is a "solid" submodule of Y provided $\mu(Y)$ starts with $\mu(U)$.

Theorem.

Let M and Y be indecomposable modules such that Y possesses a homogeneous M-filtration consisting of indecomposable modules. If U is a "solid" submodule of Y, then

(1) there exists a submodule U' of M such that $U' \simeq U$, provided $|U| \leq |M|$,
(2) any Gabriel–Roiter filtration of U contains a submodule isomorphic to M, provided $|U| \geq |M|$.

Proof.

Assume that $|U| \leq |M|$. Let

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = Y$$

be a homogeneous M-filtration of Y consisting of indecomposable modules. Without loss of generality we may assume that $U \not\subset M_{n-1}$. Since Y is indecomposable and $|U| \leq |M|$, $U + M_{n-1} \neq Y$. Consequently, $U + M_{n-1} = V \oplus M_{n-1}$ for some Λ-module V, since the inclusion $M_{n-1} \hookrightarrow M_n$ is mono-irreducible. Observe that V is isomorphic to a submodule of M. Since U is a "solid" submodule of Y, it is also a "solid" submodule of $U + M_{n-1} = V \oplus M_{n-1}$, and by the strong Gabriel–Roiter property, either U is a isomorphic to a "solid" submodule of V or U is isomorphic to a "solid" submodule M_{n-1}. However, the latter possibility cannot hold and we are done.

Theorem.

Let M and Y be indecomposable modules such that $\mathrm{End}_\Lambda(M)$ is a division ring, and Y possesses a unique M-filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = Y,$$

this filtration is M-homogeneous and consists of indecomposable modules. If U is a "solid" submodule of Y and $|U| \geq |M|$, then $U = M_i$ for some $i \in [1, n]$. In particular,

$$\mu(Y) = \mu(Y) \cup \{|M_2|, \ldots, |M_n|\}.$$