Quasi-tilted one-point extensions of wild hereditary algebras

based on the talk by Otto Kerner (Düsseldorf)

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Throughout k denotes a fixed algebraically closed filed. Let H be a connected wild hereditary algebra and M a nonzero regular module. The Auslander–Reiten quiver $\Gamma(H)$ of H consists of the preprojective component $\mathscr{P}(H)$, the components of the form $\mathbb{Z}\mathbb{A}_\infty$ and the preinjective component $\mathscr{Q}(H)$. Let $A := H[M]$. We know that $\mathscr{P}(A) = \mathscr{P}(H)$. Let $T_r = \tau_H^{-r} H \oplus P_\omega$, where P_ω is the new projective A-module. Then T_r is a tilting module such that

$$\operatorname{End}(T_r) = \begin{pmatrix} \operatorname{End}(\tau_H^{-r} H) & \operatorname{Hom}(\tau_H^{-r} H, P_\omega) \\ 0 & \operatorname{End}(P_\omega) \end{pmatrix} = \begin{pmatrix} H & \tau_H^r M \\ 0 & k \end{pmatrix},$$

since $\operatorname{Hom}(\tau_H^{-r} H, P_\omega) \simeq \operatorname{Hom}(\tau_H^{-r} H, M) \simeq \operatorname{Hom}(H, \tau_H^r M) \simeq \tau_H^r M$. One can show that $\mathbf{dim}\, \tau_H^r M \gg 0$ for $|r| \ll 0$.

Assume that $H[M]$ is tilted of type $\tilde{A}$, where $\tilde{A}$ is a wild hereditary algebra. Then the number of simple $\tilde{A}$-modules is at least 3.

Let X be a simple regular $\tilde{A}$-module with $\operatorname{Ext}(X, X) = 0$. Let $X^\perp = \{M \mid \operatorname{Hom}(X, M) = \operatorname{Ext}(X, M) = 0\} \subset \operatorname{mod} \tilde{A}$. By the Bongartz construction there exists a hereditary algebra C such that $X^\perp$ is equivalent to $\operatorname{mod} C$. Let P be a minimal projective generator of $X^\perp$. Then $T = P \oplus X$ is a tilting module and $\operatorname{End}(T)$ is a connected tilted algebra. We have

$$\operatorname{End} T = \begin{pmatrix} C & \operatorname{Hom}(P, Z) \\ 0 & k \end{pmatrix},$$

where $0 \to \tau_A X \to Z \to X \to 0$ is an Auslander–Reiten sequence. We can show that $\operatorname{End}(T)$ is wild. We can also show that Z is an elementary $\operatorname{End}(T)$-module, hence also simple regular.

Let P' be a preprojective tilting C-module. Then $P' \oplus X$ is a tilting $\tilde{A}$-module such that

$$\operatorname{End}(P' \oplus X) = \begin{pmatrix} \operatorname{End}(P') & \operatorname{Hom}(P', Z) \\ 0 & k \end{pmatrix}.$$

Here $\mathrm{Hom}(P', Z)$ is an $\mathrm{End}(P')$-regular module and $\mathrm{End}(P')$ is concealed. Moreover, $\mathrm{End}(P')$ is hereditary if and only if P' is a directing (slice) module.

One can dually show that if $H[M]$ is a tilted algebra, then $H[M] = \mathrm{End}(T)$, where $T = X \oplus P'$, with X simple regular, P' a preprojective C-module ($\mathrm{mod}\, C \simeq X^{\perp}$) such that $\mathrm{End}(P') = H$ and $M = \mathrm{Hom}(P', Z)$, where Z is as above. Moreover, $X \oplus \tau_C^{-r} P'$ is a tilting module with the property $\mathrm{End}(X \oplus \tau_C^{-r} P') = H[\tau_H^r M]$. For r big enough $H[\tau_H^r M]$ is a tilted algebra with a regular connecting component.

Let $\mathscr{H}$ be the category of coherent sheaves over a weighted projective line $\mathbb{X}$. The Auslander–Reiten quiver of $\mathscr{H}$ consists of a family $\mathrm{Vect}\,\mathbb{X}$ of $\mathbb{Z}\mathbb{A}_\infty$ components and a tubular family. If T is a tilting object from $\mathrm{Vect}\,\mathbb{X}$ then $\mathrm{End}(T)$ is concealed canonical. Assume that $H[M]$ is quasi-tilted or concealed canonical of type $\mathscr{H}$.

Let X be a simple object in $\mathrm{Vect}\,\mathbb{X}$ with $\mathrm{Ext}(X, X) = 0$. We may define $Z \in X^{\perp}$ similarly as above. Let $\tilde{T}$ be any tilting object in $\mathrm{Vect}\,\mathbb{X}$. For $m \gg 0$ we have $\mathrm{Ext}(X, \tau^{-m} T_i) \neq 0$ for any indecomposable direct summand T_i of $\tilde{T}$ and $\mathrm{Hom}(X, \tau^{-m}\tilde{T}) = 0$. We can construct the universal sequence $0 \to \tau^{-m}\tilde{T} \to M \to X^r \to 0$. Then M is a projective generator of $X^{\perp}$ and we can replay the above considerations.

If $H[M]$ is quasi-tilted of canonical type then $H[M] = \mathrm{End}(T)$ where T is a tilting object in $\mathscr{H}^{\mathrm{op}}$, $T = X \oplus P$ with X quasi-simple in some $\mathbb{Z}\mathbb{A}_\infty$ component and P belongs to a preprojective component of $X^{\perp} \simeq \mathrm{mod}\, C$, $M = \mathrm{Hom}(P, Z)$. Further, $H[\tau_H^r M]$ is concealed canonical of type $\mathscr{H}$ for $r \gg 0$.

Theorem. *Let $H[M]$ be a concealed canonical algebra and let N be a natural number. Then there exists a natural number r such that in the Auslander–Reiten quiver of $H[\tau_H^s M]$, $s \geq r$, all indecomposable modules in tubes have dimension at least N, they are sincere or almost sincere.* □