Matematyka Dyskretna

Wykład VIII

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Twierdzenie 2.14 (reguła włączania i wyłączania)

Jeśli $X_1, \ldots, X_n$ są zbiorami, to

$$|X_1 \cup \cdots \cup X_n| = \sum_{\varnothing \neq I \subseteq [1,n]} (-1)^{|I|-1} |X_I| = \sum_{k=1}^{n} (-1)^{k-1} \cdot \sum_{I \in C_{n,k}} |X_I|,$$

gdzie $X_{\{i_1,\ldots,i_k\}} := X_{i_1} \cap \cdots \cap X_{i_k}$.

Dowód

Indukcja na n.

1° $n \leq 2$.

1.1 $n = 1$.

$|X_1 \cup \cdots \cup X_n| = |X_1|$.

$\sum_{\varnothing \neq I \subseteq [1,1]} (-1)^{|I|-1} |X_I| = (-1)^{|\{1\}|-1} \cdot |X_{\{1\}}| = |X_1|$.

1.2 $n = 2$.

$|X_1 \cup \cdots \cup X_n| = |X_1 \cup X_2|$.

$\sum_{\varnothing \neq I \subseteq [1,2]} (-1)^{|I|-1} |X_I| = |X_1| + |X_2| - |X_1 \cap X_2|$.

Twierdzenie 2.14 (reguła włączania i wyłączania)

Jeśli $X_1, \ldots, X_n$ są zbiorami, to

$$|X_1 \cup \cdots \cup X_n| = \sum_{\varnothing \neq I \subseteq [1,n]} (-1)^{|I|-1} |X_I| = \sum_{k=1}^{n} (-1)^{k-1} \cdot \sum_{I \in C_{n,k}} |X_I|,$$

gdzie $X_{\{i_1,\ldots,i_k\}} := X_{i_1} \cap \cdots \cap X_{i_k}$.

2° $n > 2$.

Niech $Y_i := X_i \cap X_n$, $i \in [1, n-1]$.

$$\begin{aligned}
|X_1 \cup \cdots \cup X_n| &= |(X_1 \cup \cdots \cup X_{n-1}) \cup X_n| \\
&= |X_1 \cup \cdots \cup X_{n-1}| + |X_n| - |(X_1 \cup \cdots \cup X_{n-1}) \cap X_n| && [\text{(Z.I) dla } 2] \\
&= |X_1 \cup \cdots \cup X_{n-1}| + |X_n| - |(X_1 \cap X_n) \cup \cdots \cup (X_{n-1} \cap X_n)| \\
&= |X_1 \cup \cdots \cup X_{n-1}| + |X_n| - |Y_1 \cup \cdots \cup Y_{n-1}| \\
&= \textstyle\sum_{\varnothing \neq I \subseteq [1,n-1]} (-1)^{|I|-1} \cdot |X_I| + |X_n| - \sum_{\varnothing \neq I \subseteq [1,n-1]} (-1)^{|I|-1} \cdot |Y_I| && [\text{(Z.I) dla } n-1] \\
&= \textstyle\sum_{\varnothing \neq I \subseteq [1,n-1]} (-1)^{|I|-1} \cdot |X_I| + |X_n| - \sum_{\varnothing \neq I \subseteq [1,n-1]} (-1)^{|I|-1} \cdot |X_{I \cup \{n\}}| \\
&= \textstyle\sum_{\substack{\varnothing \neq I \subseteq [1,n] \\ n \notin I}} (-1)^{|I|-1} \cdot |X_I| + \sum_{I = \{n\}} (-1)^{|I|-1} \cdot |X_I| + \sum_{\substack{\{n\} \neq I \subseteq [1,n] \\ n \in I}} (-1)^{|I|-1} \cdot |X_I| \\
&= \textstyle\sum_{\varnothing \neq I \subseteq [1,n]} (-1)^{|I|-1} \cdot |X_I|. \quad \square
\end{aligned}$$

Przykład

Jeśli $n := p_1^{m_1} \cdots p_l^{m_l}$, $m_1, \dots, m_l > 0$, $p_1, \dots, p_l \in \mathbb{P}$, $p_i \neq p_j$ dla $i \neq j$, to

$$\varphi(n) = n \cdot \left(1 - \tfrac{1}{p_1}\right) \cdots \left(1 - \tfrac{1}{p_l}\right).$$

Dowód

Dla $i \in [1, k]$ definiujemy X_i wzorem

$$X_i := \{m \in [0, n-1] : p_i \mid m\}.$$

Zauważmy, ze

$$|X_{i_1} \cap \cdots \cap X_{i_k}| = \frac{n}{p_{i_1} \cdots p_{i_k}}.$$

Stąd

$$\begin{aligned}
\varphi(n) &= |[0, n-1] \setminus (X_1 \cup \cdots \cup X_l)| \\
&= n - |X_1 \cup \cdots \cup X_l| \\
&= n - \textstyle\sum_{k=1}^{l} \sum_{1 \leq i_1 < \cdots < i_k \leq l} (-1)^{k-1} \cdot |X_{i_1} \cap \cdots \cap X_{i_k}| \\
&= n + \textstyle\sum_{k=1}^{l} \sum_{1 \leq i_1 < \cdots < i_k \leq l} (-1)^{k} \cdot |X_{i_1} \cap \cdots \cap X_{i_k}| \\
&= n + \textstyle\sum_{k=1}^{l} \sum_{1 \leq i_1 < \cdots < i_k \leq l} (-1)^{k} \cdot \frac{n}{p_{i_1} \cdots p_{i_k}} \\
&= n \cdot \left(1 + \textstyle\sum_{k=1}^{l} \sum_{1 \leq i_1 < \cdots < i_k \leq l} \left(-\frac{1}{p_{i_1}}\right) \cdots \left(-\frac{1}{p_{i_k}}\right)\right) \\
&= n \cdot \left(1 - \tfrac{1}{p_1}\right) \cdots \left(1 - \tfrac{1}{p_l}\right). \quad \square
\end{aligned}$$

Oznaczenie (permutacje bez punktów stałych)

Dla $n \in \mathbb{N}$ definiujemy P_n i P'_n wzorami $P_n := P_{[1,n]}$ i

$$P'_n := \{a \in P_n : a(i) \neq i \text{ dla wszystkich } i \in [1,n]\}.$$

Lemat 2.15

Jeśli $n \in \mathbb{N}$, to $|P'_n| = n! \cdot \sum_{k \in [0,n]} \frac{(-1)^k}{k!}$.

Przypomnienie

(2.1): $|P_X| = (|X|)!$.

(2.3): $|C_{n,k}| = \binom{n}{k}$.

Dowód

Dla $i \in [1,n]$ niech $X_i := \{a \in P_n : a(i) = i\}$.

Zauważmy, że $|X_{i_1} \cap \cdots \cap X_{i_k}| = P_{[1,n] \setminus \{i_1,\ldots,i_k\}} \overset{(2.1)}{=} (n-k)!$.
Stąd

$$\begin{aligned}
|P'_n| &= |P_n \setminus (X_1 \cup \cdots \cup X_n)| = |P_n| - |X_1 \cup \cdots \cup X_n| \\
&= |P_n| - \sum_{k \in [1,n]} (-1)^{k-1} \cdot \sum_{\{i_1,\ldots,i_k\} \in C_{n,k}} |X_{i_1} \cap \cdots \cap X_{i_k}| \\
&= n! + \sum_{k \in [1,n]} (-1)^k \cdot \sum_{I \in C_{n,k}} (n-k)! \\
&= n! + \sum_{k \in [1,n]} (-1)^k \cdot \binom{n}{k} \cdot (n-k)! \qquad [(2.3)] \\
&= (-1)^0 \cdot \frac{n!}{0!} + \sum_{k \in [1,n]} (-1)^k \cdot \frac{n!}{k!} = n! \cdot \sum_{k \in [0,n]} \frac{(-1)^k}{k!}. \quad \square
\end{aligned}$$

Oznaczenie

Dla $x \in \mathbb{R}$ definiujemy $[x] \in \mathbb{Z}$ wzorem

$$[x] := \begin{cases} \lfloor x \rfloor & \text{jeśli } x - \lfloor x \rfloor < \frac{1}{2}, \\ \lfloor x \rfloor + 1 & \text{jeśli } x - \lfloor x \rfloor \geq \frac{1}{2}. \end{cases}$$

Uwaga

Jeśli $x \in \mathbb{R}$, $k \in \mathbb{Z}$ i $|x - k| < \frac{1}{2}$, to $[x] = k$.

Wniosek 2.16

(1) Jeśli $n \in \mathbb{N}_+$, to $|P_n'| = [\frac{n!}{e}]$.

(2) $\lim_{n\to\infty} \frac{|P_n'|}{|P_n|} = \frac{1}{e}$.

Dowód

Wiadomo, że $\left| \frac{1}{e} - \sum_{k\in[0,n]} \frac{(-1)^k}{k!} \right| < \frac{1}{(n+1)!}$. $(*)$

W szczególności $\lim_{n\to\infty} \sum_{k\in[0,n]} \frac{(-1)^k}{k!} = \frac{1}{e}$. $(**)$

Stąd $\lim_{n\to\infty} \frac{|P_n'|}{|P_n|} \overset{(2.15)+(2.1)}{=} \lim_{n\to\infty} \sum_{k\in[0,n]} \frac{(-1)^k}{k!} \overset{(**)}{=} \frac{1}{e}$.

Ponadto,

$$\left| \frac{n!}{e} - |P_n'| \right| \overset{(2.15)}{=} \left| \frac{n!}{e} - n! \cdot \sum_{k\in[0,n]} \frac{(-1)^k}{k!} \right| = n! \cdot \left| \frac{1}{e} - \sum_{k\in[0,n]} \frac{(-1)^k}{k!} \right| \overset{(*)}{<} \frac{n!}{(n+1)!} = \frac{1}{n+1} \leq \frac{1}{2}. \square$$