

RATNER'S PROPERTY AND MIXING FOR SPECIAL FLOWS OVER TWO-DIMENSIONAL ROTATIONS (EXTENDED VERSION)

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ABSTRACT. We consider special flows over two-dimensional rotations by (α, β) on \mathbb{T}^2 and under piecewise C^2 roof functions f satisfying von Neumann's condition

$$\int_{\mathbb{T}^2} f_x(x, y) dx dy \neq 0 \neq \int_{\mathbb{T}^2} f_y(x, y) dx dy.$$

Such flows are shown to be always weakly mixing and never partially rigid. For an uncountable set of (α, β) with both α and β of unbounded partial quotients the strong mixing property is proved to hold. It is also proved that while specifying to a subclass of roof functions and to ergodic rotations for which α and β are of bounded partial quotients the corresponding special flows enjoy so called weak Ratner's property. As a consequence, such flows turn out to be mildly mixing.

1. INTRODUCTION

Mixing properties, especially strong and mild mixing, of special flows over one- and multi-dimensional irrational rotations under some regular roof functions have been intensively studied during last few years, e.g. [4]–[6], [8], [9], [16], [18]–[24]. Such special flows appear often while studying smooth flows (or at least ergodic components of smooth flows) on some compact manifolds; indeed, a choice of a natural transversal may lead to a special representation over a rotation, see e.g. [2], [8], [15], [20].

It is already in 1932 when von Neumann [25] considered special flows over irrational rotations on $\mathbb{T} = [0, 1)$ under roof functions f which were piecewise C^1 . He proved weak mixing of such flows whenever the condition

$$(1) \quad \int_{\mathbb{T}} f'(x) dx \neq 0$$

was satisfied. Linear functions $f(x) = ax + b$ for $0 \leq x < 1$ (with $a \neq 0$ and $b \in \mathbb{R}$ so that $f > 0$) are the simplest examples of roof functions satisfying von Neumann's condition (1). Piecewise C^1 -functions are of bounded variation, hence, as shown by Kochergin [18] in 1972, the corresponding special flows are not mixing. A natural question whether a special flow over an irrational rotation by $\alpha \in [0, 1)$ under f piecewise C^1 and satisfying (1) can enjoy a stronger property than weak mixing was answered positively in [8]; indeed, such flows turn out to be mildly mixing whenever α has bounded partial quotients. As a matter of fact, the mild mixing property has been proved in [8] in two independent steps: first, the absence of partial rigidity (which does not require any Diophantine condition on α) has

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been proved and then so called Ratner's property has been established for α with bounded partial quotients.

In the present paper we consider special flows over an ergodic two-dimensional rotation $T(x, y) = (x + \alpha, y + \beta)$. Our roof functions $f : \mathbb{T}^2 \rightarrow \mathbb{R}_+$ will be piecewise C^2 (discontinuities of f are contained in finitely many horizontal and vertical lines, see Definition 3) and will satisfy a two-dimensional analog of (1)

$$(2) \quad \int_{\mathbb{T}^2} f_x(x, y) dx dy \neq 0 \text{ or } \int_{\mathbb{T}^2} f_y(x, y) dx dy \neq 0.$$

In what follows (2) will be referred to as the *weak von-Neumann's condition*. We will observe that this condition implies the weak mixing property of the corresponding special flows T^f (Theorem 3.2) as well as the absence of partial rigidity (Theorem 4.1). As in [8], our aim will be to prove that such flows are mildly mixing. If we want the strategy from [8] of showing the mild mixing property (under some Diophantine assumptions on (α, β)) to work we need to prove an analog of Ratner's property for such flows. This is done only partially, namely, in a restricted class of roof functions satisfying (2) and both α and β are assumed to have bounded partial quotients, see Theorem 7.4 in which so called weak Ratner's property is proved to hold. The class of roof functions includes all positive linear functions $f(x, y) = ax + by + c$ with $a/b \in \mathbb{R} \setminus \mathbb{Q}$. Then, the mild mixing property follows (Theorem 8.2). Proving (even the weak) Ratner's property of such flows is of independent interest, as it has some other ergodic consequences (Theorem 5.9, see also [28]). Recall that the original notion, introduced by Ratner in [26] and called there \mathcal{H}_p -property, is as follows:

Ratner's property. Let (X, d) be a σ -compact metric space, μ a probability Borel measure on (X, d) and $(S_t)_{t \in \mathbb{R}}$ a μ -preserving flow. The flow $(S_t)_{t \in \mathbb{R}}$ is called \mathcal{H}_p -flow, $p \neq 0$, if for every $\varepsilon > 0$ and $N \in \mathbb{N}$ there exist $\kappa = \kappa(\varepsilon) > 0$, $\delta = \delta(\varepsilon, N) > 0$ and a Borel subset $Z = Z(\varepsilon, N) \subset X$ with $\mu(Z) > 1 - \varepsilon$ such that if $x, x' \in Z$, x' is not in the orbit of x and $d(x, x') < \delta$, then there are $M = M(x, x') \geq N$, $L = L(x, x') \geq N$ with $L/M \geq \kappa$ such that if we denote

$$K^\pm = \{n \in \mathbb{Z} \cap [M, M + L] : d(S_{np}(x), S_{(n \pm 1)p}(x')) < \varepsilon\}$$

then either $\#K^+/L > 1 - \varepsilon$ or $\#K^-/L > 1 - \varepsilon$.

Ratner's property, originally proved by M. Ratner [26] for horocycle flows, in the framework of special flows over irrational rotations first appeared in [8]. In fact, already in [8] the original definition of Ratner has been modified and $\pm p$ was replaced by a finite subset of $\mathbb{R} \setminus \{0\}$. In the present paper we need a further weakening of the definition: we introduce a compact set $P \subset \mathbb{R} \setminus \{0\}$ so that the orbits of two close different points are close up to a shift of time belonging to P on sufficiently long pieces of orbits. We call this property *weak Ratner's property* (see Definition 4).

Unlike the one-dimensional rotation case, special flows over two-dimensional rotations even under smooth functions can be mixing, see [5], [6]. In Section 9 we show that special flows with piecewise C^2 roof functions and satisfying the following *strong von Neumann's condition*

$$(3) \quad \int_{\mathbb{T}^2} f_x(x, y) dx dy \neq 0 \text{ and } \int_{\mathbb{T}^2} f_y(x, y) dx dy \neq 0$$

are mixing for uncountably many $(\alpha, \beta) \in \mathbb{T}^2$ (Theorem 9.3). The main tool to prove mixing property we use is a Fayad's criterion from [5]. In particular, in the linear case $f(x, y) = ax + by + c$ mixing is possible for a special choice of α, β – a phenomenon which can not happen in the one-dimensional case.

1.1. Plan of the paper. The plan on the paper is as follows: Section 2 introduces terminology and notation that will be used throughout the remainder of the paper. In Section 3 we will show weak mixing of the special flow T^f (Theorem 3.2) assuming that the roof function $f : \mathbb{T}^2 \rightarrow \mathbb{R}_+$ is piecewise C^2 and satisfies (2). In Section 4 we will establish the absence of partial rigidity under the same assumption (Theorem 4.1). The proofs of these results are proved in spirit to the one-dimensional case in [8].

Next part of the paper deals with mild mixing. We use a criterion from [8]: If a flow is not partially rigid and it is a finite extension of each of its non-trivial factors (finite fibers factor property) then it is mildly mixing. The absence of partial rigidity being already established, in order to deal with the second assumption the notion of weak Ratner's property is introduced in Section 5. Then, in Theorem 5.9, it is proved that weak Ratner's property implies finite fibers factor property.

In Section 6 we present techniques (Lemma 6.3 and Proposition 6.4) that help us in proving the weak Ratner property to hold for special flows built over rotations. In Section 7 we introduce a class of piecewise C^2 von Neumann roof functions on \mathbb{T}^2 and we consider the corresponding special flows over ergodic rotations whose both coordinates have bounded partial quotients. Using techniques from Section 6 for this class of special flows, we prove weak Ratner's property (see Theorem 7.4), which finally establishes mild mixing. Moreover, in Section 8 we provide an example from this class which is mildly mixing but is not mixing.

Section 9 deals with mixing property for special flows with piecewise C^2 roof functions satisfying strong von Neumann's condition (3) and it uses methods different from earlier sections. We first notice that Fayad's criterion [5] (alternating uniform stretch of the Birkhoff sums in the vertical and horizontal directions) of mixing of special flows for C^2 roof functions can be extended to piecewise C^2 case. Then we prove mixing over an uncountable family of rotations by (α, β) on \mathbb{T}^2 (both α and β have unbounded partial quotients).

We will discuss some other consequences of the results proved in the paper as well as some open problems in Section 10.

Our special thanks go to A. Katok who was the first to conjecture that already linearity over two dimensional rotations may be sufficient for strong mixing property of the corresponding special flows. Such mixing flows are apparently the simplest examples of mixing flows in the framework of special flows under regular roof functions and over multi-dimensional rotations.

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2. NOTATION

Let T be an ergodic automorphism of a standard probability Borel space (X, \mathcal{B}, μ) , this is for every T -invariant set $A \in \mathcal{B}$, either A or its complement $X \setminus A$ has measure zero. Assume $f : X \rightarrow \mathbb{R}$ is a strictly positive integrable function and let $\mathcal{B}(\mathbb{R})$

and $\lambda_{\mathbb{R}}$ denote Borel σ -algebra and Lebesgue measure on \mathbb{R} respectively. Then by $\mathcal{T}^f = (T_t^f)_{t \in \mathbb{R}}$ we will mean the corresponding special flow under f (see e.g. [3], Chapter 11) acting on $(X^f, \mathcal{B}^f, \mu^f)$, where $X^f = \{(x, s) \in X \times \mathbb{R} : 0 \leq s < f(x)\}$ and \mathcal{B}^f (μ^f) is the restriction of $\mathcal{B} \otimes \mathcal{B}(\mathbb{R})$ ($\mu \otimes \lambda_{\mathbb{R}}$) to X^f . Under the action of the flow \mathcal{T}^f each point in X^f moves vertically at unit speed, and we identify the point $(x, f(x))$ with $(Tx, 0)$. Given $m \in \mathbb{Z}$ we put

$$f^{(m)}(x) = \begin{cases} f(x) + f(Tx) + \dots + f(T^{m-1}x) & \text{if } m > 0 \\ 0 & \text{if } m = 0 \\ -(f(T^m x) + \dots + f(T^{-1}x)) & \text{if } m < 0. \end{cases}$$

Then for every $(x, s) \in X^f$ we have

$$T_t^f(x, s) = (T^n x, s + t - f^{(n)}(x)),$$

where $n \in \mathbb{Z}$ is unique such that $f^{(n)}(x) \leq s + t < f^{(n+1)}(x)$.

If X is equipped with a metric d whose Borel σ -algebra is equal to \mathcal{B} then we will consider on X^f the metric d^f defined by

$$(4) \quad d^f((x_1, s_1), (x_2, s_2)) = d(x_1, x_2) + |s_1 - s_2| \text{ for } (x_1, s_1), (x_2, s_2) \in X^f.$$

Definition 1. A measure-preserving flow $(S_t)_{t \in \mathbb{R}}$ on a standard probability Borel space (X, \mathcal{B}, μ) is *mixing* if

$$\lim_{t \rightarrow \infty} \mu(S_t A \cap B) = \mu(A)\mu(B) \text{ for all } A, B \in \mathcal{B}.$$

If for all $A, B \in \mathcal{B}$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |\mu(S_t A \cap B) - \mu(A)\mu(B)| dt = 0$$

then $(S_t)_{t \in \mathbb{R}}$ is *weakly mixing*.

Of course, mixing implies weak mixing, and the following conditions are equivalent (see [3]):

- (i) $(S_t)_{t \in \mathbb{R}}$ is weakly mixing;
- (ii) the Cartesian product flow $(S_t \times S'_t)_{t \in \mathbb{R}}$ is ergodic provided that $(S'_t)_{t \in \mathbb{R}}$ is an ergodic flow on a standard probability Borel space;
- (iii) if $F : X \rightarrow \mathbb{C}$ is an eigenfunction corresponding to an eigenvalue $\theta \in \mathbb{R}$, i.e. $F(S_t x) = e^{it\theta} F(x)$ then $\theta = 0$ and F is constant.

Definition 2. A measure-preserving flow $(S_t)_{t \in \mathbb{R}}$ on a standard probability Borel space is *mildly mixing* if its Cartesian product with an arbitrary ergodic (finite or infinite conservative) measure-preserving transformation remains ergodic.

Recall that a measure-preserving flow $(S'_t)_{t \in \mathbb{R}}$ on a standard probability Borel space (X', \mathcal{B}', μ') is a *factor* of the flow $(S_t)_{t \in \mathbb{R}}$ if there exists a measurable map $\psi : X \rightarrow X'$ such that the image of μ via ψ is μ' and $\psi \circ S_t = S'_t \circ \psi$ for every $t \in \mathbb{R}$. Then the flow is $(S_t)_{t \in \mathbb{R}}$ called an *extension* of $(S'_t)_{t \in \mathbb{R}}$. If additionally, ψ is finite-to-one almost everywhere then $(S_t)_{t \in \mathbb{R}}$ a *finite extension* of $(S'_t)_{t \in \mathbb{R}}$.

A measure-preserving flow $(S_t)_{t \in \mathbb{R}}$ on a standard probability Borel space (X, \mathcal{B}, μ) is *rigid* if there exists a sequence (t_n) , $t_n \rightarrow \infty$ such that $\mu(S_{t_n} B \Delta B) \rightarrow 0$ as $n \rightarrow \infty$ for every $B \in \mathcal{B}$.

It is also proved in [11] that a probability measure-preserving flow $(S_t)_{t \in \mathbb{R}}$ on (X, \mathcal{B}, μ) is mildly mixing iff $(S_t)_{t \in \mathbb{R}}$ has no non-trivial rigid factor, i.e.

$$\liminf_{t \rightarrow \infty} \mu(S_t B \Delta B) > 0 \text{ for every } B \in \mathcal{B} \text{ with } 0 < \mu(B) < 1.$$

It follows that the mixing property of a flow implies its mild mixing which in turn implies the weak mixing property.

Assume that T is an ergodic automorphism and $f : X \rightarrow \mathbb{R}_+$ is in $L^1(X, \mathcal{B}, \mu)$. It is well-known (see e.g. [14]) that the special flow T^f is weakly mixing if and only if for every $s \in \mathbb{R} \setminus \{0\}$ the equation

$$(5) \quad \psi(Tx)/\psi(x) = e^{2\pi i s f(x)}$$

has no measurable solution $\psi : X \rightarrow \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. Assume moreover that T is *rigid*, i.e. for some increasing sequence (q_n) , $\mu(T^{q_n} A \cap A) \rightarrow \mu(A)$ for each $A \in \mathcal{B}$. We will make use of the following simple criterion of weak mixing of special flows over rigid systems.

Proposition 2.1. *Under the above assumptions suppose additionally that there exists $C > 0$ such that*

$$\left| \int_X e^{2\pi i s f^{(n)}(x)} d\mu(x) \right| \leq C/|s|$$

for every $s \neq 0$ and for all n large enough. Then (5) has no measurable solution for $s \neq 0$ and therefore the special flow T^f is weakly mixing.

Proof. Suppose that for some $s \neq 0$ and a measurable $\psi : X \rightarrow \mathbb{S}^1$

$$\psi(Tx)/\psi(x) = e^{2\pi i s f(x)}.$$

Then for all $k \in \mathbb{Z} \setminus \{0\}$ and all n large enough we have

$$\left| \int_X \psi^k(T^{q_n} x) \overline{\psi^k(x)} d\mu(x) \right| = \left| \int_X e^{2\pi i k s f^{(q_n)}(x)} d\mu(x) \right| \leq C/|ks|$$

and since clearly $\psi^k \circ T^{q_n} \cdot \overline{\psi^k} \rightarrow 1$ in measure, when $n \rightarrow \infty$, we obtain a contradiction. \square

We denote by \mathbb{T}^d the torus $\mathbb{R}^d/\mathbb{Z}^d$ which we will constantly identify with the d -cube $[0, 1)^d$. Let $\lambda_{\mathbb{T}^d}$ stand for Lebesgue measure on \mathbb{T}^d .

A homeomorphism T of a compact topological space X is called *uniquely ergodic* if it admits a unique T -invariant probability Borel measure μ . Then the measure-preserving automorphism T of (X, μ) is ergodic and for every continuous function $f : X \rightarrow \mathbb{C}$

$$(6) \quad \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) \rightarrow \int_X f d\mu \text{ uniformly in } x \in X.$$

Recall that if $T : \mathbb{T}^d \rightarrow \mathbb{T}^d$ is the rotation by a vector $(\alpha_1, \dots, \alpha_d) \in \mathbb{T}^d$ such that $\alpha_1, \dots, \alpha_d, 1$ are independent over \mathbb{Q} then T is uniquely ergodic. Moreover, using standard arguments this gives (6) for every Riemann integrable function $f : \mathbb{T}^d \rightarrow \mathbb{C}$ with $\mu = \lambda_{\mathbb{T}^d}$.

For a real number t denote by $\{t\}$ its fractional part and by $\|t\|$ its distance to the nearest integer number. For an irrational $\alpha \in \mathbb{T}$ denote by (q_n) its sequence of denominators (see e.g. [17]), that is we have

$$(7) \quad \frac{1}{2q_n q_{n+1}} < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}},$$

where

$$\begin{aligned} q_0 &= 1, & q_1 &= a_1, & q_{n+1} &= a_{n+1}q_n + q_{n-1} \\ p_0 &= 0, & p_1 &= 1, & p_{n+1} &= a_{n+1}p_n + p_{n-1}. \end{aligned}$$

Let $[0; a_1, a_2, \dots]$ stand for the continued fraction expansion of α . The rational numbers p_n/q_n are called *the convergents* of the continued fraction. The number α is said to have *bounded partial quotients* if the sequence (a_n) is bounded. Then there exists a natural number C such that $\|n\alpha\| \geq 1/(C|n|)$ for every non-zero integer n . It follows that $q_{s+1} \leq Cq_s$ holds for each natural s .

Definition 3. A function $f : \mathbb{T}^2 \rightarrow \mathbb{R}$ is called a *piecewise C^r -function* if there exist $0 \leq a_1 < \dots < a_N < 1$ and $0 \leq b_1 < \dots < b_M < 1$ such that $f : (a_j, a_{j+1}) \times (b_k, b_{k+1}) \rightarrow \mathbb{R}$ is of class C^r and it has a C^r -extension to $[a_j, a_{j+1}] \times [b_k, b_{k+1}]$ for every $1 \leq j \leq N$ and $1 \leq k \leq M$, where $a_{N+1} = a_1$ and $b_{M+1} = b_1$ and the intervals $[a_N, a_1]$ and $[b_M, b_1]$ are meant mod 1.

Remark 2.2. Modifying f on a set of measure zero, if necessary, we can always assume that f is of class C^r on every set $[a_j, a_{j+1}] \times [b_k, b_{k+1}]$.

3. WEAK MIXING

In this section we will show weak mixing assuming that the roof function $f : \mathbb{T}^2 \rightarrow \mathbb{R}_+$ is piecewise C^2 and satisfies the von Neumann condition (2) (in the following section we will establish the absence of partial rigidity under the same assumption). We recall that all rotations on tori are rigid.

Lemma 3.1 (see [13]). *Let $h : \mathbb{T} \rightarrow \mathbb{R}$ be a piecewise absolutely continuous map with N discontinuities. Suppose that $h' : \mathbb{T} \rightarrow \mathbb{R}$ is of bounded variation and $|h'(x)| \geq \theta > 0$ for all $x \in \mathbb{T}$. Then*

$$\left| \int_{\mathbb{T}} e^{2\pi i h(x)} dx \right| \leq \frac{N}{\pi\theta} + \frac{\text{Var } h'}{2\pi\theta^2}.$$

Proof. Suppose that $0 \leq a_1 < \dots < a_N < 1$ are all discontinuities of h (we set $a_{N+1} = a_1$). Using integration by parts we obtain

$$\begin{aligned} \int_{a_j}^{a_{j+1}} e^{2\pi i h(x)} dx &= \int_{a_j}^{a_{j+1}} \frac{1}{2\pi i h'(x)} d e^{2\pi i h(x)} \\ &= \left[\frac{e^{2\pi i h(x)}}{2\pi i h'(x)} \right]_{a_j^+}^{a_{j+1}^-} - \int_{a_j}^{a_{j+1}} e^{2\pi i h(x)} d \frac{1}{2\pi i h'(x)}. \end{aligned}$$

Moreover,

$$\left| \int_{a_j}^{a_{j+1}} e^{2\pi i h(x)} d \frac{1}{2\pi i h'(x)} \right| \leq \frac{1}{2\pi} \text{Var}_{[a_j, a_{j+1}]} \frac{1}{h'} \leq \frac{1}{2\pi\theta^2} \text{Var}_{[a_j, a_{j+1}]} h'$$

and

$$\left| \left[\frac{e^{2\pi i h(x)}}{2\pi i h'(x)} \right]_{a_j^+}^{a_{j+1}^-} \right| \leq \frac{1}{\pi\theta}.$$

It follows that

$$\left| \int_{\mathbb{T}} e^{2\pi i h(x)} dx \right| \leq \sum_{j=1}^N \left(\frac{1}{\pi\theta} + \frac{1}{2\pi\theta^2} \text{Var}_{[a_j, a_{j+1}]} h' \right) = \frac{N}{\pi\theta} + \frac{\text{Var } h'}{2\pi\theta^2}.$$

□

Theorem 3.2. *Let $T : \mathbb{T}^2 \rightarrow \mathbb{T}^2$, $T(x, y) = (x + \alpha, y + \beta)$ be an ergodic rotation. Suppose that $f : \mathbb{T}^2 \rightarrow \mathbb{R}_+$ is a piecewise C^2 -function satisfying (2). Then the special flow T^f is weakly mixing.*

Proof. Suppose that $\int_{\mathbb{T}^2} f_x(x, y) dx dy \neq 0$. The proof of the symmetric case runs similarly. By Proposition 2.1, it suffices to show that there exist $C > 0$ and $n_0 \in \mathbb{N}$ such that for every $s \neq 0$ and $n \geq n_0$ we have $\left| \int_{\mathbb{T}^2} e^{2\pi i s f^{(n)}(x, y)} dx dy \right| \leq C/|s|$. Since $f_x : \mathbb{T}^2 \rightarrow \mathbb{R}$ is Riemann integrable and T is uniquely ergodic, $(f^{(n)})_x/n = (f_x)^{(n)}/n$ tends uniformly to $\int_{\mathbb{T}^2} f_x(x, y) dx dy \neq 0$. Therefore there exist $\theta > 0$ and $n_0 \in \mathbb{N}$ such that $|(f^{(n)})_x(x, y)| \geq \theta n$ for all $(x, y) \in \mathbb{T}^2$ and $n \geq n_0$. Fix $n \geq n_0$ and $y \in \mathbb{T}$. Since $\mathbb{T} \ni x \mapsto f^{(n)}(x, y) \in \mathbb{R}$ is a piecewise C^2 -function with at most nN discontinuities, by Lemma 3.1 applied to $f^{(n)}(\cdot, y)$,

$$\begin{aligned} \left| \int_{\mathbb{T}} e^{2\pi i s f^{(n)}(x, y)} dx \right| &\leq \frac{nN}{\pi|s|\theta n} + \frac{\text{Var } s(f^{(n)})'(\cdot, y)}{2\pi s^2 \theta^2 n^2} \\ &\leq \frac{N}{\pi|s|\theta} + \frac{\sum_{k=0}^{n-1} \|f''(\cdot, y + k\beta)\|_{C^0}}{2\pi s \theta^2 n^2} \\ &\leq \frac{N}{\pi|s|\theta} + \frac{\|f_{xx}\|_{C^0}}{2\pi|s|\theta^2 n}, \end{aligned}$$

so also

$$\left| \int_{\mathbb{T}^2} e^{2\pi i s f^{(n)}(x, y)} dx dy \right| \leq \int_{\mathbb{T}} \left| \int_{\mathbb{T}} e^{2\pi i s f^{(n)}(x, y)} dx \right| dy \leq \frac{N}{\pi|s|\theta} + \frac{\|f_{xx}\|_{C^0}}{2\pi|s|\theta^2 n},$$

which completes the proof. □

4. ABSENCE OF PARTIAL RIGIDITY

Let us recall that a flow $(S_t)_{t \in \mathbb{R}}$ acting on a standard probability Borel space (X, \mathcal{B}, μ) is called *partially rigid* if there exist $\kappa > 0$ and $\mathbb{R} \ni r_t \rightarrow \infty$ such that $\liminf_{t \rightarrow \infty} \mu(A \cap S_{r_t} A) \geq \kappa \mu(A)$ for each $A \in \mathcal{B}$.

Theorem 4.1. *Let $T : \mathbb{T}^2 \rightarrow \mathbb{T}^2$, $T(x, y) = (x + \alpha, y + \beta)$ be an ergodic rotation. Suppose that $f : \mathbb{T}^2 \rightarrow \mathbb{R}_+$ is a piecewise C^1 -function satisfying (2). Then the special flow T^f is not partially rigid.*

To prove Theorem 4.1 we will need the following.

Lemma 4.2. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of piecewise C^1 -functions $f_n : \mathbb{T} \rightarrow \mathbb{R}_+$ for which there exist $0 < c < C$, $0 < \theta < \Theta$, $m_0 \in \mathbb{N}$, $N \in \mathbb{N}$ and finite sets $D(f_n) \subset \mathbb{T}$ containing all discontinuity points of f_n such that*

$$(8) \quad f_{n-1}(x) + c \leq f_n(x) \leq f_{n-1}(x) + C \text{ for all } n \in \mathbb{N}, x \in \mathbb{T} \text{ (} f_0 \equiv 0 \text{),}$$

$$(9) \quad D(f_n) \subset D(f_{n+1}) \text{ and } \#D(f_n) \leq Nn,$$

$$(10) \quad \theta n \leq |f'_n(x)| \leq \Theta n \text{ for all } n \geq n_0 \text{ and } x \in \mathbb{T} \setminus D(f_n).$$

Then for every $t \geq 2Cn_0$ and $0 < \varepsilon < c/4$ we have

$$\lambda_{\mathbb{T}}(\{x \in \mathbb{T} : \exists j \in \mathbb{N} |f_j(x) - t| < \varepsilon\}) < \frac{16C}{\theta c^2} (Nc + \Theta)\varepsilon.$$

Proof. Fix $t \geq 2Cn_0$ and $0 < \varepsilon < c/4$. Notice that, by (8), $jc \leq f_j \leq jC$ for all $j \geq 0$. Let J stand for the set of all natural j such that $|f_j(x) - t| < \varepsilon$ for some $x \in \mathbb{T}$. Then for such j and x we have $t + \varepsilon > f_j(x) \geq cj$ and $t - \varepsilon < f_j(x) \leq Cj$, whence

$$(11) \quad t/(2C) \leq (t - \varepsilon)/C < j < (t + \varepsilon)/c \leq 2t/c$$

for any $j \in J$; in particular, J is finite and $j \in J$ implies

$$n_0 \leq \frac{t}{2C} < j.$$

Let $\bar{j} = \max J$. Set $k := \#D(f_{\bar{j}}) \leq N\bar{j} < 2Nt/c$. The elements of $D(f_{\bar{j}})$ partition \mathbb{T} into subintervals I_1, \dots, I_k . Notice that for every $j \in J$ the function f_j is of class C^1 and strictly monotone (because of (9) and (10)) on the interval I_i , $i = 1, \dots, k$.

Fix $1 \leq i \leq k$. For every $j \in J$ let $I_{i,j} = \overline{\{x \in I_i : |f_j(x) - t| < \varepsilon\}}$. Since f_j is monotone on I_i , $I_{i,j}$ is an interval although it can be empty. If $I_{i,j} = [z_1, z_2]$ is not empty then, by (10) and (11),

$$(12) \quad \theta j |I_{i,j}| \leq |(f_j)_-(z_2) - (f_j)_+(z_1)| \leq 2\varepsilon \leq \frac{4C\varepsilon j}{t}.$$

Now suppose that $x \in I_{i,j}$ and $y \in I_{i,j'}$ with $j \neq j'$. Since x, y are in the same interval of continuity of f_j , by (10) and (8), it follows that

$$(13) \quad \begin{aligned} \Theta \bar{j} |y - x| &\geq \Theta j |y - x| \geq |f_j(y) - f_j(x)| \\ &\geq |f_j(y) - f_{j'}(y)| - |f_{j'}(y) - t| - |f_j(x) - t| \geq c - 2\varepsilon \geq \frac{c}{2}. \end{aligned}$$

In particular, there is no overlap between $I_{i,j}$ and $I_{i,j'}$.

Let $K_i = \{j \in J : I_{i,j} \neq \emptyset\}$ and suppose that $s = \#K_i \geq 1$. Then there exist $s - 1$ pairwise disjoint subintervals $H_l \subset I_i$, $l = 1, \dots, s - 1$ that are disjoint from intervals $I_{i,j}$, $j \in K_i$ and fill up the space between those intervals. In view of (13) and (11) we have $|H_l| \geq c/(2\bar{j}\Theta) \geq c^2/(4t\Theta)$ for $l = 1, \dots, s - 1$. Therefore, by (12) and (13), we obtain

$$\begin{aligned} \sum_{j \in K_i} |I_{i,j}| &\leq s \frac{4C\varepsilon}{t\theta} = \frac{4C\varepsilon}{t\theta} + \frac{16C\varepsilon\Theta}{c^2\theta} (s - 1) \frac{c^2}{4t\Theta} \\ &\leq \frac{4C\varepsilon}{t\theta} + \frac{16C\varepsilon\Theta}{c^2\theta} \sum_{l=1}^{s-1} |H_l| \leq \frac{4C\varepsilon}{t\theta} + \frac{16C\varepsilon\Theta}{c^2\theta} |I_i|. \end{aligned}$$

Since

$$B := \{x \in \mathbb{T} : \exists_{j \in \mathbb{N}} |f_j(x) - t| < \varepsilon\} \subset \bigcup_{i=1}^k \bigcup_{j \in K_i} I_{i,j},$$

it follows that

$$\begin{aligned} \lambda_{\mathbb{T}}(B) &\leq \sum_{i=1}^k \sum_{j \in K_i} |I_{i,j}| \leq \sum_{i=1}^k \left(\frac{4C\varepsilon}{t\theta} + \frac{16C\varepsilon\Theta}{c^2\theta} |I_i| \right) \\ &= \frac{4C\varepsilon k}{t\theta} + \frac{16C\varepsilon\Theta}{c^2\theta} \sum_{i=1}^k |I_i| = \frac{4C\varepsilon k}{t\theta} + \frac{16C\varepsilon\Theta}{c^2\theta} \\ &\leq \frac{8C\varepsilon N}{c\theta} + \frac{16C\varepsilon\Theta}{c^2\theta} \leq \frac{16C\varepsilon}{c^2\theta} (Nc + \Theta). \end{aligned}$$

□

Proof of Theorem 4.1. Suppose that $\int_{\mathbb{T}^2} f_x(x, y) dx dy \neq 0$. The proof of the symmetric case runs similarly. Let c, C be positive numbers such that $0 < c \leq f(x, y) \leq C$ for every $(x, y) \in \mathbb{T}^2$. Assume, contrary to our claim, that T^f is partially rigid. By Lemma 7.1 in [8], there exist $(t_n)_{n \in \mathbb{N}}$, $t_n \rightarrow +\infty$ and $0 < u \leq 1$ such that for every $0 < \varepsilon < c$ we have

$$(14) \quad \liminf_{n \rightarrow \infty} \lambda_{\mathbb{T}^2} \left(\{(x, y) \in \mathbb{T}^2 : \exists_{j \in \mathbb{N}} |f^{(j)}(x, y) - t_n| < \varepsilon\} \right) \geq u.$$

Let $0 \leq a_1 < \dots < a_N < 1$ and $0 \leq b_1 < \dots < b_M < 1$ be points determining the lines of points of discontinuity for f . Since $f_x : \mathbb{T}^2 \rightarrow \mathbb{R}$ is Riemann integrable, by the unique ergodicity of T , there exist $0 < \theta < \Theta$ and $m_0 \in \mathbb{N}$ such that $m\theta \leq |(f_x)^{(m)}(x, y)| \leq m\Theta$ for all $(x, y) \in \mathbb{T}^2$ and $m \geq m_0$.

Take $0 < \varepsilon < \frac{c^2\theta}{32C(Nc+\Theta)}u$. Fix $y \in \mathbb{T}$. For every $m \in \mathbb{N}$ let us consider the map $\mathbb{T} \ni x \mapsto f^{(m)}(x, y) \in \mathbb{R}_+$ and set $D(f^{(m)}(\cdot, y)) = \{a_k - j\alpha : 1 \leq k \leq N, 0 \leq j < m\}$. Then $f^{(m)}(\cdot, y)$ is piecewise C^1 and its discontinuity points are contained in $D(f^{(m)}(\cdot, y))$. Moreover, $D(f^{(m)}(\cdot, y)) \subset D(f^{(m+1)}(\cdot, y))$, $\#D(f^{(m)}(\cdot, y)) \leq Nm$ and

$$f^{(m)}(x, y) = f^{(m-1)}(x, y) + f \circ T^{m-1}(x, y) \in f^{(m-1)}(x, y) + [c, C].$$

Now an application of Lemma 4.2 to the sequence $(f^{(m)}(\cdot, y))_{m \in \mathbb{N}}$ gives

$$\lambda_{\mathbb{T}} \left(\{x \in \mathbb{T} : \exists_{j \in \mathbb{N}} |f^{(j)}(x, y) - t_n| < \varepsilon\} \right) < \frac{16C}{\theta c^2} (Nc + \Theta)\varepsilon < u/2$$

whenever $t_n > 2Cm_0$. By Fubini's Theorem,

$$\begin{aligned} &\lambda_{\mathbb{T}^2} \left(\{(x, y) \in \mathbb{T}^2 : \exists_{j \in \mathbb{N}} |f^{(j)}(x, y) - t_n| < \varepsilon\} \right) \\ &= \int_{\mathbb{T}} \lambda_{\mathbb{T}} \left(\{x \in \mathbb{T} : \exists_{j \in \mathbb{N}} |f^{(j)}(x, y) - t_n| < \varepsilon\} \right) dy < u/2 \end{aligned}$$

whenever $t_n > 2Cm_0$, contrary to (14). □

5. WEAK RATNER'S PROPERTY

In this section we introduce and discuss consequences of weak Ratner's property. Weak Ratner's property will be one more weakening of the classical Ratner condition from [26]. The present idea has already been used in case P is finite in [8] and [9].

Definition 4. Let (X, d) be a σ -compact metric space, \mathcal{B} be the σ -algebra of Borel subsets of X , μ a probability Borel measure on (X, d) . Assume that $(S_t)_{t \in \mathbb{R}}$ is a flow on (X, \mathcal{B}, μ) . Let $P \subset \mathbb{R} \setminus \{0\}$ be a compact subset and $t_0 \in \mathbb{R} \setminus \{0\}$. The flow $(S_t)_{t \in \mathbb{R}}$ is said to have *the property* $R(t_0, P)$ if for every $\varepsilon > 0$ and $N \in \mathbb{N}$ there exist $\kappa = \kappa(\varepsilon) > 0$, $\delta = \delta(\varepsilon, N) > 0$ and a subset $Z = Z(\varepsilon, N) \in \mathcal{B}$ with $\mu(Z) > 1 - \varepsilon$ such that if $x, x' \in Z$, x' is not in the orbit of x and $d(x, x') < \delta$, then there are $M = M(x, x') \geq N$, $L = L(x, x') \geq N$ such that $L/M \geq \kappa$ and there exists $\rho = \rho(x, x') \in P$ such that

$$\frac{\#\{n \in \mathbb{Z} \cap [M, M+L] : d(S_{nt_0}(x), S_{nt_0+\rho}(x')) < \varepsilon\}}{L} > 1 - \varepsilon.$$

Moreover, we say that $(S_t)_{t \in \mathbb{R}}$ has *the property* $R(P)$ if the set of $s \in \mathbb{R}$ such that the flow $(S_t)_{t \in \mathbb{R}}$ has the $R(s, P)$ -property is uncountable. Flows with the latter property are said to have *weak Ratner's property*.

Remark 5.1. Note that the original Ratner notion of \mathcal{H}_p -flow, introduced in [26], is equivalent to requiring that a flow has $R(p, \{-p, p\})$ -property.

The notion we introduce is different from the concept of Ratner's property presented by Witte in [30]. The main difference is that Witte admits compact subsets in the centralizer of the flow $(S_t)_{t \in \mathbb{R}}$ as the set of displacements. In our approach this set is included in the flow. It should be emphasized that Witte has used his notion to prove certain rigidity phenomena of some translations on homogeneous space but not to study the structure of joinings which is one of our aims.

The following result is a simple consequence of Birkhoff's Ergodic Theorem.

Lemma 5.2. *Let $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ be an ergodic automorphism and $A \in \mathcal{B}$. For every $\varepsilon > 0$, $\delta > 0$ and $\kappa > 0$ there exist $N = N(\varepsilon, \delta, \kappa) \in \mathbb{N}$ and $X(\varepsilon, \delta, \kappa) \in \mathcal{B}$ with $\mu(X(\varepsilon, \delta, \kappa)) > 1 - \delta$ such that for every $M, L \in \mathbb{N}$ with $L \geq N$ and $L/M \geq \kappa$ we have*

$$\left| \frac{1}{L} \sum_{n=M}^{M+L} \chi_A(T^n x) - \mu(A) \right| < \varepsilon \text{ for all } x \in X(\varepsilon, \delta, \kappa). \quad \square$$

Remark 5.3. If the set $P \subset \mathbb{R} \setminus \{0\}$ is finite then using Luzin's theorem and Lemma 5.2 one can easily show that the $R(s, P)$ -property does not depend on the choice of the metric d on X compatible with \mathcal{B} . We have been unable to decide whether for P infinite (and compact) the $R(s, P)$ -property depends on the choice of the metric; it is very likely that it does. This is why we are forced to put one more assumption on d , see (15) below (see also Remark 5.5 below).

We will constantly assume that $(S_t)_{t \in \mathbb{R}}$ satisfies the following "almost continuity" condition

$$(15) \quad \begin{aligned} & \text{for every } \varepsilon > 0 \text{ there exists } X(\varepsilon) \in \mathcal{B} \text{ with } \mu(X(\varepsilon)) > 1 - \varepsilon \text{ such that} \\ & \text{for every } \varepsilon' > 0 \text{ there exists } \varepsilon_1 > 0 \text{ such that} \\ & d(S_t x, S_{t'} x) < \varepsilon' \text{ for all } x \in X(\varepsilon) \text{ and } t, t' \in [-\varepsilon_1, \varepsilon_1]. \end{aligned}$$

Notice that if $(S_t)_{t \in \mathbb{R}}$ is a special flow acting on a space Y^f equipped with a metric of the form (4) then (15) holds.

We intend to prove a version of famous Ratner's theorem which describes the structure of ergodic joinings between a system satisfying weak Ratner's property and an arbitrary one, see Theorem 5.9.

Assume that $\mathcal{S} = (S_t)_{t \in \mathbb{R}}$ and $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ are ergodic flows acting on (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) respectively. By a *joining* one means any $(S_t \times T_t)_{t \in \mathbb{R}}$ -invariant probability measure ρ on $(X \times Y, \mathcal{B} \otimes \mathcal{C})$ with the marginals μ and ν respectively. We then write $\rho \in J(\mathcal{S}, \mathcal{T})$. The set of ergodic joinings is denoted by $J^e(\mathcal{S}, \mathcal{T})$.

An essential step of the proof of Theorem 5.9 will be based on the following result.

Lemma 5.4. *Let $(S_t)_{t \in \mathbb{R}}$ and $(T_t)_{t \in \mathbb{R}}$ be ergodic flows acting on (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) respectively and let $\rho \in J(\mathcal{S}, \mathcal{T}) \cap J^e(S_1, T_1)$. Assume that $(S_t)_{t \in \mathbb{R}}$ and (X, d) satisfy (15). Let $P \subset \mathbb{R}$ be a non-empty compact set. Suppose that $A \in \mathcal{B}$ with $\mu(\partial A) = 0$ and $B \in \mathcal{C}$. Then for every $\varepsilon, \delta, \kappa > 0$ there exist $N = N(\varepsilon, \delta, \kappa) \in \mathbb{N}$ and $\Theta(\varepsilon, \delta, \kappa) \in \mathcal{B} \otimes \mathcal{C}$ with $\rho(\Theta(\varepsilon, \delta, \kappa)) > 1 - \delta$ such that for every $M, L \in \mathbb{N}$ with $L \geq N$ and $L/M \geq \kappa$ we have*

$$\left| \frac{1}{L} \sum_{j=M}^{M+L} \chi_{S_{-p}A \times B}(S_j x, T_j y) - \rho(S_{-p}A \times B) \right| < \varepsilon$$

for all $(x, y) \in \Theta(\varepsilon, \delta, \kappa)$ and $p \in P$.

Remark 5.5. If in Lemma 5.4 we take $\rho = \mu \otimes \nu$, $B = Y$ and $\kappa = 1$ then for every $\varepsilon, \delta > 0$ there exist $N(\varepsilon, \delta) \in \mathbb{N}$ and $\Theta(\varepsilon, \delta) \in \mathcal{B}$ with $\mu(\Theta(\varepsilon, \delta)) > 1 - \delta$ such that for every $L \geq N(\varepsilon, \delta)$ we have

$$(16) \quad \sup_{p \in P} \left| \frac{1}{L} \sum_{j=0}^L \chi_A(S_{j+p}x) - \mu(A) \right| < \varepsilon \text{ for all } x \in \Theta(\varepsilon, \delta).$$

As it was pointed to us by E. Lesigne, if we let (S_t) be an arbitrary flow, and $A \in \mathcal{B}$ be also arbitrary then (16) fails to be true for $P = [0, 1]$. This is one more reason to justify our additional assumption (15) on (S_t) and d .

Proof of Lemma 5.4. Fix $\varepsilon, \delta, \kappa > 0$. Let $V_\varepsilon(A) = \{z \in X : d(z, A) < \varepsilon\}$. Since $\mu(\partial A) = 0$, there exists $\varepsilon' > 0$ such that $\mu(V_{\varepsilon'}(A)) - \mu(A) < \varepsilon/4$,

$$\mu(A) - \mu((V_{\varepsilon'}(A^c))^c) = \mu(V_{\varepsilon'}(A^c)) - \mu(A^c) < \varepsilon/4.$$

By (15), there exists $\varepsilon_1 > 0$ such that $d(S_t x, S_{t'} x) < \varepsilon'$ for all $x \in X(\varepsilon/4)$ and $t, t' \in [-\varepsilon_1, \varepsilon_1]$. It follows that

$$\begin{aligned} & \mu \left(\bigcup_{t \in [-\varepsilon_1, \varepsilon_1]} S_{-t} A \right) \\ & \leq \mu \left(\bigcup_{t \in [-\varepsilon_1, \varepsilon_1]} S_{-t} A \cap X(\varepsilon/4) \right) + \mu \left(\bigcup_{t \in [-\varepsilon_1, \varepsilon_1]} S_{-t} A \cap X(\varepsilon/4)^c \right) \\ & \leq \mu(V_{\varepsilon'}(A)) + \mu(X(\varepsilon/4)^c) < \mu(A) + \varepsilon/2. \end{aligned}$$

Similarly $\mu\left(\bigcup_{t \in [-\varepsilon_1, \varepsilon_1]} S_{-t}A^c\right) \leq \mu(A^c) + \varepsilon/2$, and hence

$$\mu\left(\bigcap_{t \in [-\varepsilon_1, \varepsilon_1]} S_{-t}A\right) = 1 - \mu\left(\bigcup_{t \in [-\varepsilon_1, \varepsilon_1]} S_{-t}A^c\right) \geq 1 - (\mu(A^c) + \varepsilon/2) = \mu(A) - \varepsilon/2.$$

For every $\varepsilon > 0$ and $p \in \mathbb{R}$ set

$$I(\varepsilon, p) = \bigcap_{t \in [-\varepsilon, \varepsilon]} (S_{-t-p}A \times B) \text{ and } U(\varepsilon, p) = \bigcup_{t \in [-\varepsilon, \varepsilon]} (S_{-t-p}A \times B).$$

It follows that for every $p \in \mathbb{R}$ we have

$$\begin{aligned} & \rho(U(\varepsilon_1, p)) - \rho(S_{-p}A \times B) \\ &= \rho\left(\bigcup_{t \in [-\varepsilon_1, \varepsilon_1]} (S_{-t-p}A \times B) \setminus S_{-p}A \times B\right) \\ &= \rho\left(\left(\bigcup_{t \in [-\varepsilon_1, \varepsilon_1]} S_{-t-p}A \setminus S_{-p}A\right) \times B\right) \leq \mu\left(\bigcup_{t \in [-\varepsilon_1, \varepsilon_1]} S_{-t-p}A \setminus S_{-p}A\right) \\ &= \mu\left(\left(\bigcup_{t \in [-\varepsilon_1, \varepsilon_1]} S_{-t}A\right) \setminus A\right) < \varepsilon/2 \end{aligned}$$

and similarly

$$\rho(S_{-p}A \times B) - \rho(I(\varepsilon_1, p)) < \varepsilon/2.$$

Let $Q \subset P$ be a finite set such that $P \subset Q + [-\varepsilon_1/2, \varepsilon_1/2]$. By Lemma 5.2 applied to $T_1 \times S_1 : (X \times Y, \rho) \rightarrow (X \times Y, \rho)$ and sets $U(\varepsilon_1/2, q)$, $I(\varepsilon_1/2, q)$ for $q \in Q$, there exist $N \in \mathbb{N}$ and $\Theta \subset \mathcal{B} \otimes \mathcal{C}$ with $\rho(\Theta) > 1 - \delta$ such that for every $M, L \in \mathbb{N}$ with $L \geq N$ and $L/M \geq \kappa$ we have

$$\left| \frac{1}{L} \sum_{j=M}^{M+L} \chi_{U(\varepsilon_1/2, q)}(S_j x, T_j y) - \rho(U(\varepsilon_1/2, q)) \right| < \varepsilon/2$$

and

$$\left| \frac{1}{L} \sum_{j=M}^{M+L} \chi_{I(\varepsilon_1/2, q)}(S_j x, T_j y) - \rho(I(\varepsilon_1/2, q)) \right| < \varepsilon/2$$

for all $(x, y) \in \Theta$ and $q \in Q$. Take $p \in P$ and choose $q \in Q$ such that $p \in q + [-\varepsilon_1/2, \varepsilon_1/2]$. Then

$$I(\varepsilon_1, p) \subset I(\varepsilon_1/2, q) \subset S_{-p}A \times B \subset U(\varepsilon_1/2, q) \subset U(\varepsilon_1, p).$$

Thus

$$\begin{aligned} \frac{1}{L} \sum_{j=M}^{M+L} \chi_{S_{-p}A \times B}(S_j x, T_j y) &\leq \frac{1}{L} \sum_{j=M}^{M+L} \chi_{U(\varepsilon_1/2, q)}(S_j x, T_j y) \\ &< \rho(U(\varepsilon_1/2, q)) + \varepsilon/2 \leq \rho(U(\varepsilon_1, p)) + \varepsilon/2 < \rho(S_{-p}A \times B) + \varepsilon \end{aligned}$$

and

$$\begin{aligned} \frac{1}{L} \sum_{j=M}^{M+L} \chi_{S_{-p}A \times B}(S_j x, T_j y) &\geq \frac{1}{L} \sum_{j=M}^{M+L} \chi_{I(\varepsilon_1/2, q)}(S_j x, T_j y) \\ &> \rho(I(\varepsilon_1/2, q)) - \varepsilon/2 \geq \rho(I(\varepsilon_1, p)) - \varepsilon/2 > \rho(S_{-p}A \times B) - \varepsilon, \end{aligned}$$

which completes the proof. \square

Lemma 5.6. *For every $A \in \mathcal{B}$ there exists a set $\Upsilon \subset (0, +\infty)$ such that $(0, +\infty) \setminus \Upsilon$ is countable and $\mu(\partial V_\epsilon(A)) = 0$ for all $\epsilon \in \Upsilon$.*

Proof. Note that $\partial V_\epsilon(A) \subset \{x \in X : d(x, A) = \epsilon\}$ and $\{\{x \in X : d(x, A) = \epsilon\} : \epsilon > 0\}$ is a family of closed pairwise disjoint sets. Since μ is finite, the set of all $\epsilon > 0$ such that $\mu(\{x \in X : d(x, A) = \epsilon\}) > 0$ is countable. It follows that $\mu(\partial V_\epsilon(A)) > 0$ for at most countably many $\epsilon > 0$. \square

Remark 5.7. Since (X, d) is a Polish space, by the regularity of μ and Lemma 5.6, we can find $\{A_i : i \in \mathbb{N}\}$ a dense family in (\mathcal{B}, μ) such that $\mu(\partial A_i) = 0$ for all $i \in \mathbb{N}$.

Lemma 5.8 (see the proof of Theorem 3 in [26]). *Let $(S_t)_{t \in \mathbb{R}}$ and $(T_t)_{t \in \mathbb{R}}$ be ergodic flows acting on (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) respectively and let $\rho \in J^e(\mathcal{S}, \mathcal{T})$. Suppose that there exists $U \in \mathcal{B} \otimes \mathcal{C}$ with $\rho(U) > 0$ and $\delta > 0$ such that if $(x, y) \in U$, $(x', y) \in U$ then either x and x' are in the same orbit or $d(x, x') \geq \delta$. Then ρ is a finite extension of ν . \square*

Theorem 5.9. *Let (X, d) be a σ -compact metric space, \mathcal{B} be the σ -algebra of Borel subsets of X , μ a probability Borel measure on (X, d) . Let $(S_t)_{t \in \mathbb{R}}$ be a weakly mixing flow on the space (X, \mathcal{B}, μ) that satisfies the $R(P)$ -property where $P \subset \mathbb{R} \setminus \{0\}$ is a nonempty compact set. Assume that $(S_t)_{t \in \mathbb{R}}$ and (X, d) satisfy (15).*

Let $(T_t)_{t \in \mathbb{R}}$ be an ergodic flow on (Y, \mathcal{C}, ν) and let ρ be an ergodic joining of $(S_t)_{t \in \mathbb{R}}$ and $(T_t)_{t \in \mathbb{R}}$. Then either $\rho = \mu \otimes \nu$, or ρ is a finite extension of ν .

Proof. Suppose that $\rho \in J^e(\mathcal{S}, \mathcal{T})$ and $\rho \neq \mu \otimes \nu$. Since the flow $(S_t \times T_t)_{t \in \mathbb{R}}$ is ergodic on $(X \times Y, \rho)$, we can find $t_0 \neq 0$ such that the automorphism $S_{t_0} \times T_{t_0} : (X \times Y, \rho) \rightarrow (X \times Y, \rho)$ is ergodic and the flow $(S_t)_{t \in \mathbb{R}}$ has the $R(t_0, P)$ -property. To simplify notation we assume that $t_0 = 1$.

By Remark 5.7, there exist two families $\{A_i : i \in \mathbb{N}\}$ and $\{B_i : i \in \mathbb{N}\}$ dense in (\mathcal{B}, μ) and (\mathcal{C}, ν) respectively such that $\mu(\partial A_i) = 0$ for all $i \in \mathbb{N}$. Let us consider the map

$$\mathbb{R} \ni t \mapsto \varrho(t) := \sum_{i,j=1}^{\infty} \frac{1}{2^{i+j}} |\rho(S_{-t}A_i \times B_j) - \rho(A_i \times B_j)| \in \mathbb{R}.$$

Since

$$|\varrho(t) - \varrho(t')| \leq \sum_{i,j=1}^{\infty} \frac{1}{2^{i+j}} |\rho(S_{-t}A_i \times B_j) - \rho(S_{-t'}A_i \times B_j)| \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \mu(S_{-t}A_i \Delta S_{-t'}A_i)$$

and $\mathbb{R} \ni t \mapsto S_t \in \text{Aut}(X, \mathcal{B}, \mu)$ is a continuous representation, the function ϱ is continuous. Notice that $\varrho(t) > 0$ for $t \neq 0$. Indeed, if $\varrho(t) = 0$ then $\rho(S_{-t}A_i \times B_j) = \rho(A_i \times B_j)$ for all $i, j \in \mathbb{N}$, and hence $\rho(S_{-t}A \times B) = \rho(A \times B)$ for all $A \in \mathcal{B}, B \in \mathcal{C}$. By the ergodicity of S_t , we obtain $\rho = \mu \otimes \nu$.

Since $P \subset \mathbb{R} \setminus \{0\}$ is compact, there exists $\varepsilon > 0$ such that $\rho(p) \geq \varepsilon$ for $p \in P$. Let M be a natural number such that $\sum_{i,j>M} 1/2^{i+j} < \varepsilon/2$. Since

$$\sum_{i,j=1}^M \frac{1}{2^{i+j}} |\rho(S_{-p}A_i \times B_j) - \rho(A_i \times B_j)| \geq \varepsilon/2 \text{ for all } p \in P,$$

we have

$$(17) \quad \forall p \in P \exists 1 \leq i, j \leq M |\rho(S_{-p}A_i \times B_j) - \rho(A_i \times B_j)| \geq \varepsilon > 0.$$

Since $\mu(\partial(A_i)) = 0$, by Lemma 5.6, we can choose $0 < \varepsilon_1 < \varepsilon/8$ such that

$$\mu(V_{\varepsilon_1}(A_i) \setminus A_i) < \varepsilon/2 \text{ and } \mu(\partial V_{\varepsilon_1}(A_i)) = 0$$

for $1 \leq i \leq M$. It follows that

$$(18) \quad \begin{aligned} |\rho(A_i \times B_j) - \rho(V_{\varepsilon_1}(A_i) \times B_j)| &< \varepsilon/2, \\ |\rho(S_{-t}A_i \times B_j) - \rho(S_{-t}V_{\varepsilon_1}(A_i) \times B_j)| &< \varepsilon/2 \end{aligned}$$

for all $1 \leq i, j \leq M$ and $t \in \mathbb{R}$.

Let $\kappa := \kappa(\varepsilon_1) (> 0)$. By Lemma 5.2 applied to the sets $V_{\varepsilon_1}(A_i) \times B_j$ and the automorphism $S_1 \times T_1$, and Lemma 5.4 applied to the pairs of sets A_i, B_j , $i, j = 1, \dots, M$, there exist a measurable set $U \subset X \times Y$ with $\rho(U) > 3/4$ and $N \in \mathbb{N}$ such that if $(x, y) \in U$, $p \in P$, $1 \leq i, j \leq M$, $l \geq N$ and $l/m \geq \kappa$ then

$$(19) \quad \left| \frac{1}{l} \sum_{k=m}^{m+l} \chi_{V_{\varepsilon_1}(A_i) \times B_j}(S_k x, T_k y) - \rho(V_{\varepsilon_1}(A_i) \times B_j) \right| < \frac{\varepsilon}{8},$$

$$(20) \quad \left| \frac{1}{l} \sum_{k=m}^{m+l} \chi_{S_{-p}A_i \times B_j}(S_k x, T_k y) - \rho(S_{-p}A_i \times B_j) \right| < \frac{\varepsilon}{8}$$

and similar inequalities hold for $A_i \times B_j$ for (19) and $S_{-p}V_{\varepsilon_1}(A_i) \times B_j$ for (20).

Next, by the property $R(1, P)$, we obtain relevant $\delta = \delta(\varepsilon_1, N) > 0$ and $Z = Z(\varepsilon_1, N) \in \mathcal{B}$, $\mu(Z) > 1 - \varepsilon_1$.

Now assume that $(x, y) \in U$, $(x', y) \in U$, $x, x' \in Z$ and x' is not in the orbit of x . We claim that $d(x, x') \geq \delta$. Suppose that, on the contrary, $d(x, x') < \delta$. Then, by the property $R(1, P)$, there exist $M = M(x, x')$, $L = L(x, x') \geq N$ with $L/M \geq \kappa$ and $p = p(x, x') \in P$ such that $(\#K_p)/L > 1 - \varepsilon_1$, where

$$K_p = \{n \in \mathbb{Z} \cap [M, M+L] : d(S_n(x), S_{n+p}(x')) < \varepsilon_1\}.$$

From (17), there exist $1 \leq i, j \leq M$ such that

$$(21) \quad |\rho(S_{-p}A_i \times B_j) - \rho(A_i \times B_j)| \geq \varepsilon > 0.$$

If $k \in K_p$ and $S_{k+p}x' \in A_i$, then $S_k x \in V_{\varepsilon_1}(A_i)$. Hence

$$(22) \quad \begin{aligned} &\frac{1}{L} \sum_{k=M}^{M+L} \chi_{S_{-p}A_i \times B_j}(S_k x', T_k y) \\ &\leq \frac{\#(\mathbb{Z} \cap [M, M+L] \setminus K_p)}{L} + \frac{1}{L} \sum_{k \in K_p} \chi_{A_i \times B_j}(S_{k+p}x', T_k y) \\ &\leq \varepsilon/8 + \frac{1}{L} \sum_{k=M}^{M+L} \chi_{V_{\varepsilon_1}(A_i) \times B_j}(S_k x, T_k y). \end{aligned}$$

Now from (20), (22), (19) and (18) it follows that

$$\begin{aligned} \rho(S_{-p}A_i \times B_j) &\leq \frac{1}{L} \sum_{k=M}^{M+L} \chi_{S_{-p}A_i \times B_j}(S_k x', T_k y) + \varepsilon/8 \\ &\leq \varepsilon/4 + \frac{1}{L} \sum_{k=M}^{M+L} \chi_{V_{\varepsilon_1}(A_i) \times B_j}(S_k x, T_k y) \\ &< \varepsilon/2 + \rho(V_{\varepsilon_1}(A_i) \times B_j) < \varepsilon + \rho(A_i \times B_j). \end{aligned}$$

Applying similar arguments we get

$$\rho(A_i \times B_j) < \varepsilon + \rho(S_{-p}A_i \times B_j).$$

Consequently,

$$|\rho(A_i \times B_j) - \rho(S_{-p}A_i \times B_j)| < \varepsilon,$$

contrary to (21).

In summary, we have found a measurable set $U_1 = U \cap (Z(\varepsilon_1, N) \times Y)$ and $\delta(\varepsilon_1, N) > 0$ such that $\rho(U_1) > 3/4 - \varepsilon_1 > 1/2$ and if $(x, y) \in U_1$, $(x', y) \in U_1$ then either x and x' are in the same orbit or $d(x, x') \geq \delta(\varepsilon_1, N)$. Now an application of Lemma 5.8 completes the proof. \square

6. WEAK RATNER'S PROPERTY FOR SPECIAL FLOWS

In this section we present techniques that will help us to prove the weak Ratner property for special flows built over isometries. The following is a general version of Lemma 5.2 in [8]. We omit its proof since it is showed as in [8].

Proposition 6.1. *Let (X, d) be a compact metric space, \mathcal{B} the σ -algebra of Borel subsets of X and let μ be a probability Borel measure on (X, d) . Assume that $T : (X, \mu) \rightarrow (X, \mu)$ is an ergodic isometry and $f : X \rightarrow \mathbb{R}$ is a bounded positive measurable function which is bounded away from zero. Let $P \subset \mathbb{R} \setminus \{0\}$ be a nonempty compact subset. Assume that for every $\varepsilon > 0$ and $N \in \mathbb{N}$ there exist $\kappa = \kappa(\varepsilon) > 0$, $0 < \delta = \delta(\varepsilon, N) < \varepsilon$ and $Z = Z(\varepsilon, N) \in \mathcal{B}$, $\mu(Z) > 1 - \varepsilon$ such that if $x, y \in Z$, $0 < d(x, y) < \delta$, then there are natural numbers $M = M(x, y) \geq N$, $L = L(x, y) \geq N$ such that $L/M \geq \kappa$ and there exists $p = p(x, y) \in P$ such that*

$$\frac{1}{L} \# \left\{ M \leq n < M + L : |f^{(n)}(x) - f^{(n)}(y) - p| < \varepsilon \right\} > 1 - \varepsilon.$$

Suppose that $\gamma \in \mathbb{R}$ is a positive number such that the γ -time automorphism $T_\gamma^f : X^f \rightarrow X^f$ is ergodic. Then the special flow T^f has the $R(\gamma, P)$ -property. \square

Definition 5. Let $0 < a < b$. A sequence $(x_n)_{n \geq 0}$ taking values in $[-R, R] \cap \mathbb{Z}$ ($R > 0$) is called a -sparse if there exists an increasing sequence $(k_m)_{m \geq 0}$, $k_0 = 0$, of natural numbers such that

- (i) $x_n \neq 0$ with $n \geq 1$ if and only if $n = k_m$ for some $m \geq 1$;
- (ii) $k_{m+1} - k_m \geq a$ for all $m \geq 1$;

If additionally

- (iii) $k_{m+1} - k_m \leq b$ for all $m \geq 0$.

then $(x_n)_{n \geq 0}$ is (a, b) -sparse.

Remark 6.2. If $(x_n)_{n \geq 0}$ is a -sparse then $\left| \sum_{k=0}^{n-1} x_k \right| \leq R(1 + n/a)$.

Let $T : X \rightarrow X$ be an isometry of a metric space (X, d) . Let $f : X \rightarrow \mathbb{R}$ be a Borel function and let $H = \{h_1, \dots, h_s\}$, $s \geq 3$, a collection of real numbers. Assume that

$$(23) \quad h_1, \dots, h_{s-1} \text{ are linearly independent over } \mathbb{Q} \text{ and } h_{s-1} = h_s.$$

Let $N_j : X \times X \rightarrow \mathbb{Z}$, $j = 1, \dots, s+1$, and $b : X \times X \rightarrow \mathbb{R}$ be Borel functions such that for some constants $R, B > 0$

$$(24) \quad |N_j(x, y)| \leq R, \quad j = 1, \dots, s+1 \quad \text{and} \quad |b(x, y)| \leq B \quad \text{whenever} \quad d(x, y) \leq 1.$$

Moreover, suppose that there exist positive constants $C_0, C_1 < C_2$ such that for any pair of distinct $x, y \in X$ with $d(x, y) \leq 1/2$ we have

$$(25) \quad |f^{(n)}(y) - f^{(n)}(x) - \left(\sum_{j=1}^s N_j^{(n)}(x, y) h_j \right)| \\ \leq |b(x, y) N_{s+1}(x, y) - b(T^n x, T^n y) N_{s+1}(T^n x, T^n y)| + C_0 n d(x, y),$$

$$(26) \quad \forall_{1 \leq j \leq s+1} (N_j(T^n x, T^n y))_{n \geq 0} \text{ is } C_1/d(x, y)\text{-sparse,}$$

$$(27) \quad \exists_{1 \leq j \leq s-2} (N_j(T^n x, T^n y))_{n \geq 0} \text{ is } (C_1/d(x, y), C_2/d(x, y))\text{-sparse,}$$

where $N_j^{(n)}(x, y) := \sum_{k=0}^{n-1} N_j(T^k x, T^k y)$.

Lemma 6.3. *Under the above assumptions there exist $0 < p_0 \leq p_1$ such that for every $\varepsilon > 0$ and $N \in \mathbb{N}$ there exist $\kappa = \kappa(\varepsilon) > 0$, $0 < \delta = \delta(\varepsilon, N) < \varepsilon$ such that if $x, y \in X$, $0 < d(x, y) < \delta$, then there are natural numbers $M = M(x, y) \geq N$, $L = L(x, y) \geq N$ such that $L/M \geq \kappa$ and there exists $p = p(x, y)$ with $p_0 \leq |p| \leq p_1$ such that*

$$\frac{1}{L} \# \left\{ M \leq n < M + L : |f^{(n)}(x) - f^{(n)}(y) - p| < \varepsilon \right\} > 1 - \varepsilon.$$

Proof. Let

$$H' := \left\{ \sum_{j=1}^s r_j h_j : r_j \in [-R, R] \cap \mathbb{Z}, j = 1, \dots, s \right\} \setminus \{0\}$$

and $h := \min\{|w| : w \in H'\} > 0$. Fix $0 < \varepsilon < \min(1/2, C_0 C_1/s, h/(4s))$ and $N \geq 2$. Without loss of generality we can assume that $0 < C_1 \leq 1 \leq C_0, C_2$. Set

$$(28) \quad \delta := \varepsilon^3 C_1 / (2C_0 N) \quad \text{and} \quad \kappa := \varepsilon / (6s C_0 C_2).$$

Fix two distinct $x, y \in X$ with

$$d := d(x, y) < \delta$$

and set $L = \lceil \varepsilon / (C_0 d) \rceil$. Note that

$$(29) \quad N/\varepsilon^2 \leq \varepsilon / (2C_0 d) \leq L \leq \varepsilon / (C_0 d) < C_1 / (sd).$$

The number M will be chosen between C_1/d and $3sC_2/d$, and we will precise its value later. Then

$$L/M \geq \frac{\varepsilon / (2C_0 d)}{3sC_2/d} = \frac{\varepsilon}{6sC_0 C_2} = \kappa \quad \text{and} \quad M \geq C_1/d \geq N.$$

By assumptions, there exists an increasing sequence $(k_m)_{m \geq 0}$, $k_0 = 0$ of natural numbers such that

$$(30) \quad N_j(T^n x, T^n y) = 0 \text{ for } k_m < n < k_{m+1} \text{ and for all } m \geq 0 \text{ and } j = 1, \dots, s;$$

$$(31) \quad \text{for each } m \geq 1 \text{ there exists } 1 \leq j \leq s \text{ with } N_j(T^{k_m} x, T^{k_m} y) \neq 0;$$

$$(32) \quad k_{m+1} - k_m \leq C_2/d \text{ for } m \geq 0;$$

$$(33) \quad k_{m+s} - k_m \geq C_1/d \text{ for } m \geq 1.$$

Since $k_{m+s} - k_m \geq C_1/d > sL$,

$$(34) \quad \text{for every } m \geq 1 \text{ there exists } m \leq m' < m + s \text{ such that } k_{m'+1} - k_{m'} > L.$$

We use (34) for $m = s + 1$ and obtain m_1 , and apply again (34) for $m_1 + 1$ to have $s < m_1 < m_2 \leq m_1 + s$ such that $k_{m_i+1} - k_{m_i} > L$ for $i = 1, 2$. It follows that the set

$$\{(m_1, m_2) \in \mathbb{N}^2 : s < m_1 \leq 2s, m_1 < m_2 \leq m_1 + s, k_{m_i+1} - k_{m_i} > L, i = 1, 2\}$$

is not empty. Pick a pair (m_1, m_2) from this set with the smallest $m_2 - m_1$. Then

$$(35) \quad k_{m+1} - k_m \leq L \text{ for all } m_1 < m < m_2$$

and

$$(36) \quad \begin{aligned} &\text{for each } 1 \leq j \leq s \text{ there exists at most one natural number } m \\ &\text{for which } N_j(T^{k_m} x, T^{k_m} y) \neq 0 \text{ and } m_1 < m \leq m_2. \end{aligned}$$

Indeed, suppose contrary to our claim that there exist $1 \leq j \leq s$ and $m_1 < m'_1 < m'_2 \leq m_2$ such that $N_j(T^{k_{m'_i}} x, T^{k_{m'_i}} y) \neq 0$ for $i = 1, 2$. Then $m'_2 - m'_1 < m_2 - m_1 \leq s$. Since $(N_j(T^n x, T^n y))_{n \geq 0}$ is C_1/d -sparse, $k_{m'_2} - k_{m'_1} \geq C_1/d > sL$. Therefore, there exists $m'_1 \leq m' < m'_2$ such that $k_{m'+1} - k_{m'} > L$, contrary to the definition of (m_1, m_2) .

Take $M_1 \in \{k_{m_1+1} - L, k_{m_1+1} - L + 1\}$ and $M_2 \in \{k_{m_2} + 1, k_{m_2} + 2\}$ so that $N_{s+1}(T^{M_i} x, T^{M_i} y) = 0$ for $i = 1, 2$. In view of (35) and (29)

$$(37) \quad M_2 - M_1 \leq L + 2 + \sum_{m_1 < m < m_2} (k_{m+1} - k_m) \leq (s+1)L \leq \frac{2s}{C_0} \frac{\varepsilon}{d}.$$

By (30) and (36), for each $j = 1, \dots, s$

$$N_j^{(M_2)}(x, y) - N_j^{(M_1)}(x, y) = \sum_{m_1 < m \leq m_2} N_j(T^{k_m} x, T^{k_m} y) \in [-R, R] \cap \mathbb{Z}.$$

Moreover, in view of (31), there exists $1 \leq j_0 \leq s-2$ such that $N_{j_0}(T^{k_{m_2}} x, T^{k_{m_2}} y) \neq 0$, hence

$$N_{j_0}^{(M_2)}(x, y) - N_{j_0}^{(M_1)}(x, y) = N_{j_0}(T^{k_{m_2}} x, T^{k_{m_2}} y) \neq 0.$$

It follows that

$$\sum_{j=1}^s (N_j^{(M_2)}(x, y) - N_j^{(M_1)}(x, y)) h_j \in H'.$$

Therefore

$$\left| \sum_{j=1}^s N_j^{(M_2 - M_1)}(T^{M_1} x, T^{M_1} y) h_j \right| \geq h.$$

As $N_{s+1}(T^{M_1}x, T^{M_1}y) = N_{s+1}(T^{M_2}x, T^{M_2}y) = 0$, in view of (25) and (37),

$$\begin{aligned} & |f^{(M_2-M_1)}(T^{M_1}y) - f^{(M_2-M_1)}(T^{M_1}x) - \sum_{j=1}^s N_j^{(M_2-M_1)}(T^{M_1}x, T^{M_1}y)h_j| \\ & \leq C_0(M_2 - M_1)d < 2s\varepsilon < h/2. \end{aligned}$$

It follows that

$$\begin{aligned} & \left| f^{(M_2)}(y) - f^{(M_2)}(x) - (f^{(M_1)}(y) - f^{(M_1)}(x)) \right| \\ & = \left| f^{(M_2-M_1)}(T^{M_1}y) - f^{(M_2-M_1)}(T^{M_1}x) \right| > h/2. \end{aligned}$$

Consequently, either for $M = M_1$ or $M = M_2$ we have

$$|f^{(M)}(y) - f^{(M)}(x)| > h/4 =: p_0 > 0,$$

and let $M = M_i$. Since $k_{m_i} < M < M + L - 1 \leq k_{m_i+1}$, by (30), for all $1 \leq j \leq s$

$$N_j(T^n x, T^n y) = 0 \text{ for all } M \leq n < M + L - 1,$$

hence

$$(38) \quad N_j^{(n-M)}(T^M x, T^M y) = \sum_{M \leq k < n} N_j(T^k x, T^k y) = 0 \text{ for all } M \leq n < M + L.$$

Since $s < m_1 \leq 2s$ and $m_2 \leq m_1 + s \leq 3s$, in view of (32) and (33),

$$C_1/d \leq k_{m_1} \text{ and } k_{m_2} \leq 3sC_2/d,$$

hence

$$(39) \quad C_1/d \leq M \leq 3sC_2/d + 2.$$

As $N_{s+1}(T^M x, T^M y) = 0$, by (25),

$$|f^{(M)}(y) - f^{(M)}(x)| \leq \sum_{j=1}^s \left| N_j^{(M)}(x, y) \right| |h_j| + |b(x, y)N_{s+1}(x, y)| + C_0Md.$$

Since $(N_j(T^n x, T^n y))_{n \geq 0}$ is C_1/d -sparse, by Remark 6.2 and (39),

$$\left| N_j^{(M)}(x, y) \right| \leq R(1 + Md/C_1) \leq R(3sC_2/C_1 + 2),$$

hence

$$\left| f^{(M)}(y) - f^{(M)}(x) \right| \leq R \left((3sC_2/C_1 + 2) \sum_{j=1}^s |h_j| + B + 3sC_0C_2 + 2 \right) =: p_1.$$

Let $p := f^{(M)}(y) - f^{(M)}(x)$. Then, in view of (38), (25) and (29), for each $M \leq n < M + L$ we have

$$\begin{aligned} |f^{(n)}(y) - f^{(n)}(x) - p| &= \left| f^{(n)}(y) - f^{(n)}(x) - (f^{(M)}(y) - f^{(M)}(x)) \right| \\ &= \left| f^{(n-M)}(T^M y) - f^{(n-M)}(T^M x) - \sum_{j=1}^s N_j^{(n-M)}(T^M x, T^M y)h_j \right| \\ &\leq C_0(n - M)d + B |N_{s+1}(T^M x, T^M y)| + B |N_{s+1}(T^n x, T^n y)| \\ &< C_0Ld + B |N_{s+1}(T^n x, T^n y)| \leq \varepsilon + B |N_{s+1}(T^n x, T^n y)|. \end{aligned}$$

Since $(N_{s+1}(T^n x, T^n y))_{n \geq 0}$ is C_1/d -sparse,

$$\#\{M \leq n < M + L : N_{s+1}(T^n x, T^n y) \neq 0\} \leq dL/C_1 + 1.$$

It follows that

$$\#\{M \leq n < M + L : |f^{(n)}(y) - f^{(n)}(x) - p| < \varepsilon\} \geq L - dL/C_1 - 1.$$

In view of (28) and (29),

$$dL/C_1 + 1 \leq \varepsilon^3 C_1/2 \cdot L/C_1 + L\varepsilon^2 < \varepsilon L.$$

Consequently,

$$\#\{M \leq n < M + L : |f^{(n)}(y) - f^{(n)}(x) - p| < \varepsilon\} > (1 - \varepsilon)L,$$

which completes the proof. \square

We will consider now T an isometry of a (compact) metric space (X, d) which is ergodic with respect to a probability Borel measure μ . We will assume that $(\widehat{X}, \widehat{d})$ is another metric space. Moreover, we assume that $\pi : (\widehat{X}, \widehat{d}) \rightarrow (X, d)$ is a surjective function which, in addition, is uniformly locally isometric. More precisely, $\pi : B_{\widehat{d}}(\widehat{x}, 1/2) \rightarrow B_d(\pi(\widehat{x}), 1/2)$ is a bijective isometry for every $\widehat{x} \in \widehat{X}$. Let $\widehat{T} : \widehat{X} \rightarrow \widehat{X}$ be an isometry of $(\widehat{X}, \widehat{d})$ such that $\pi \circ \widehat{T} = T \circ \pi$.

Proposition 6.4. *Let $T : (X, \mu) \rightarrow (X, \mu)$ be an ergodic isometry of a metric space (X, d) . Suppose that $f : X \rightarrow \mathbb{R}$ is a bounded positive Borel function which is bounded away from zero. Let $\widehat{f} : \widehat{X} \rightarrow \mathbb{R}$ given by $\widehat{f} = f \circ \pi$. Assume that there exists a collection of real numbers $H = \{h_1, \dots, h_s\}$ and Borel functions $b : \widehat{X} \times \widehat{X} \rightarrow \mathbb{R}$, $N_j : \widehat{X} \times \widehat{X} \rightarrow \mathbb{Z}$, $j = 1, \dots, s + 1$, satisfying (23)-(27) for \widehat{f} and \widehat{T} . Then the special flow T^f satisfies weak Ratner's property.*

Proof. By Lemma 6.3 applied to \widehat{T} and \widehat{f} , for every $0 < \varepsilon < 1/2$ and $N \in \mathbb{N}$ there exist $\kappa = \kappa(\varepsilon) > 0$, $0 < \delta = \delta(\varepsilon, N) < \varepsilon$ such that if $\widehat{x}, \widehat{y} \in \widehat{X}$, $0 < \widehat{d}(\widehat{x}, \widehat{y}) < \delta$, then there are natural numbers $M = M(\widehat{x}, \widehat{y}) \geq N$, $L = L(\widehat{x}, \widehat{y}) \geq N$ such that $L/M \geq \kappa$ and there exists $p = p(\widehat{x}, \widehat{y})$ with $p_0 \leq |p| \leq p_1$ such that

$$\frac{1}{L} \#\left\{M \leq n < M + L : |\widehat{f}^{(n)}(\widehat{x}) - \widehat{f}^{(n)}(\widehat{y}) - p| < \varepsilon\right\} > 1 - \varepsilon.$$

Let $x, y \in X$ arbitrary distinct point such that $d(x, y) < \delta$. By assumption, there are distinct $\widehat{x}, \widehat{y} \in \widehat{X}$ such that $\pi(\widehat{x}) = x$, $\pi(\widehat{y}) = y$ and $\widehat{d}(\widehat{x}, \widehat{y}) = d(x, y) < \delta$. Since

$$\widehat{f}^{(n)}(\widehat{x}) - \widehat{f}^{(n)}(\widehat{y}) = f^{(n)}(x) - f^{(n)}(y),$$

it follows that T and f verify the assumptions of Proposition 6.1 with $P = [-p_1, -p_0] \cup [p_0, p_1]$. This gives $R(t_0, P)$ -property for all $t_0 \in \mathbb{R} \setminus \{0\}$ such that $T_{t_0}^f$ is ergodic and weak Ratner's property follows. \square

7. SPECIAL FLOWS OVER ROTATIONS ON THE TWO TORUS

In this section we will deal with special flows over ergodic rotations $T(x, y) = (x + \alpha, y + \beta)$ on \mathbb{T}^2 . We will constantly assume that both α and β have bounded partial quotients. We will consider roof functions of the form

$$f(x, y) = f_1(x) + f_2(y) + g(x, y) + \gamma h(x, y),$$

where $f_1, f_2 : \mathbb{T} \rightarrow \mathbb{R}$ are piecewise C^2 -functions which are not continuous, $g : \mathbb{T}^2 \rightarrow \mathbb{R}$ is C^2 , $h : \mathbb{T}^2 \rightarrow \mathbb{R}$ is given by

$$h(x, y) = \alpha\{y\} - (\{x\} + \alpha)[\{y\} + \beta]$$

and $\gamma \in \mathbb{R}$. The function h naturally appears when considering rotations on the nil-manifold which is the quotient of the Heisenberg group modulo its subgroup of matrices with integer coefficients.

In order to prove weak Ratner's property for the corresponding special flows, we will apply Proposition 6.4 in which $X = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, $\hat{X} = \mathbb{R}^2$, $\pi : \mathbb{R}^2 \rightarrow \mathbb{T}^2$ is defined naturally and \hat{T} is the translation on \mathbb{R}^2 by (α, β) .

Lemma 7.1. *Let $\alpha \in \mathbb{R}$ be an irrational number with bounded partial quotients. Let us consider the function $N : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{Z}$, $N(x, x') = [x'] - [x]$. Then there exist positive constants C_1, C_2 such that for any pair $x, x' \in \mathbb{R}$ of points with $0 < |x - x'| < 1/2$ the sequence $(N(x + n\alpha, x' + n\alpha))_{n \geq 0}$ is $(C_1/|x - x'|, C_2/|x - x'|)$ -sparse.*

Proof. Since α has bounded partial quotients, there are constants $C_1, C_2 > 0$ such that for each $m \in \mathbb{N}$ the lengths of intervals I in the partition of \mathbb{T} arisen from $0, \alpha, \dots, (m-1)\alpha$ satisfy

$$\frac{2C_1}{m} \leq |I| \leq \frac{C_2}{2m}.$$

Suppose that $x > x'$. Then $[x + n\alpha] - [x' + n\alpha] \in \{0, 1\}$ and

$$[x + n\alpha] - [x' + n\alpha] = 1 \text{ if and only if } n\alpha \in [-x, -x'] + \mathbb{Z}.$$

Suppose that $n_1 < n_2$ are natural numbers such that $n_1\alpha, n_2\alpha \in [-x, -x'] + \mathbb{Z}$ and $n\alpha \notin [-x, -x'] + \mathbb{Z}$ for $n_1 < n < n_2$. It follows that the interval $[-x, -x']$ (as an interval on \mathbb{T}) contains exactly one point of the sequence $n_1\alpha, \dots, (n_2-1)\alpha$, hence

$$|[-x, -x']| < 2 \frac{C_2}{2(n_2 - n_1)}.$$

Moreover, $[-x, -x']$ contains exactly two points of the sequence $n_1\alpha, \dots, n_2\alpha$, hence

$$|[-x, -x']| > \frac{2C_1}{n_2 - n_1 + 1} \geq \frac{C_1}{n_2 - n_1}.$$

Therefore,

$$\frac{C_1}{|x' - x|} < n_2 - n_1 < \frac{C_2}{|x' - x|},$$

which completes the proof. \square

Remark 7.2. Let us consider the function $u : \mathbb{R} \rightarrow \mathbb{R}$, $u(x) = \{x\}$. Then for the translation $x \mapsto x + \alpha$ on \mathbb{R}

$$u^{(n)}(x) = \sum_{k=0}^{n-1} \{x + k\alpha\} = \sum_{k=0}^{n-1} (x + k\alpha - [x + k\alpha])$$

and for distinct $x, x' \in \mathbb{R}$ we have

$$(40) \quad u^{(n)}(x') - u^{(n)}(x) = n(x' - x) + \sum_{k=0}^{n-1} ([x' + k\alpha] - [x + k\alpha]).$$

Remark 7.3. Let us consider the function $\widehat{h} : \mathbb{R}^2 \rightarrow \mathbb{R}$, $\widehat{h}(x, y) = \alpha\{y\} - (\{x\} + \alpha)[\{y\} + \beta]$. (Note that $\widehat{h} = h \circ \pi$.) Observe that

$$\widehat{h}(x, y) = \alpha y + x[y] - (x + \alpha)[y + \beta] + [x](\{y\} + \beta).$$

Then, for the translation $(x, y) \mapsto (x + \alpha, y + \beta)$ on \mathbb{R}^2 we have

$$\widehat{h}^{(n)}(x, y) = x[y] - (x + n\alpha)[y + n\beta] + \alpha \sum_{k=0}^{n-1} (y + k\beta) + \sum_{k=0}^{n-1} [x + k\alpha](\{y + k\beta\} + \beta).$$

It follows that

$$\begin{aligned} \widehat{h}^{(n)}(x', y') - \widehat{h}^{(n)}(x, y) &= x'[y'] - x[y] - (x' + n\alpha)[y' + n\beta] + (x + n\alpha)[y + n\beta] + \alpha n(y' - y) \\ &\quad + \sum_{k=0}^{n-1} ([x' + k\alpha](\{y' + k\beta\} + \beta) - [x + k\alpha](\{y + k\beta\} + \beta)), \end{aligned}$$

hence

$$\begin{aligned} \widehat{h}^{(n)}(x', y') - \widehat{h}^{(n)}(x, y) &= \alpha n(y' - y) - ([y + n\beta] - [y])(x' - x) \\ &\quad + x'([y'] - [y]) - (x' + n\alpha)([y' + n\beta] - [y + n\beta]) \\ &\quad + \sum_{k=0}^{n-1} ([x' + k\alpha] - [x + k\alpha])(\{y + k\beta\} + \beta) \\ &\quad + \sum_{k=0}^{n-1} [x' + k\alpha](\{y' + k\beta\} + \beta) - [x + k\alpha](\{y + k\beta\} + \beta). \end{aligned}$$

Moreover,

$$\begin{aligned} &\sum_{k=0}^{n-1} [x' + k\alpha](\{y' + k\beta\} + \beta) - [x + k\alpha](\{y + k\beta\} + \beta) \\ &= \sum_{k=0}^{n-1} [x' + k\alpha]([y' + (k+1)\beta] - [y + (k+1)\beta]) - ([y' + k\beta] - [y + k\beta]) \\ &= \sum_{k=0}^{n-1} ([x' + k\alpha] - [x' + (k+1)\alpha])([y' + (k+1)\beta] - [y + (k+1)\beta]) \\ &\quad + [x' + n\alpha]([y' + n\beta] - [y + n\beta]) - [x']([y'] - [y]). \end{aligned}$$

Consequently,

$$\begin{aligned} \widehat{h}^{(n)}(x', y') - \widehat{h}^{(n)}(x, y) &= \alpha n(y' - y) - ([\{y\} + n\beta])(x' - x) \\ &\quad + \{x'\}(\{y'\} - [y]) - \{x' + n\alpha\}([y' + n\beta] - [y + n\beta]) \\ (41) \quad &\quad - \sum_{k=0}^{n-1} [\{x' + k\alpha\} + \alpha]([y' + (k+1)\beta] - [y + (k+1)\beta]) \\ &\quad + \sum_{k=0}^{n-1} ([x' + k\alpha] - [x + k\alpha])(\{y + k\beta\} + \beta). \end{aligned}$$

Theorem 7.4. *Let $T(x, y) = (x + \alpha, y + \beta)$, $\alpha, \beta \in [0, 1)$, be an ergodic rotation on the torus \mathbb{T}^2 such that both α and β have bounded partial quotients. Let $f : \mathbb{T}^2 \rightarrow \mathbb{R}_+$ be of the form*

$$f(x, y) = f_1(x) + f_2(y) + g(x, y) + \gamma h(x, y),$$

where $f_1, f_2 : \mathbb{T} \rightarrow \mathbb{R}$ are piecewise C^2 -functions which are not continuous and $g : \mathbb{T}^2 \rightarrow \mathbb{R}$ is C^2 . Suppose that f_i has s_i discontinuities with jumps of size $d_{i,1}, \dots, d_{i,s_i}$ for $i = 1, 2$. Assume that $d_{1,1}, \dots, d_{1,s_1}, d_{2,1}, \dots, d_{2,s_2}$ are independent over \mathbb{Q} . Then T^f satisfies weak Ratner's property in the following two cases:

- (i) $\gamma, d_{1,1}, \dots, d_{1,s_1}, d_{2,1}, \dots, d_{2,s_2}$ are independent over \mathbb{Q} and $\sum_{j=1}^{s_1} d_{1,j} - \beta\gamma$ or $\sum_{j=1}^{s_2} d_{2,j} + \alpha\gamma$ is non-zero;
- (ii) $\gamma = 0$.

Proof. Since

$$\int_{\mathbb{T}^2} f_x(x, y) dx dy = \sum_{j=1}^{s_1} d_{1,j} - \beta\gamma \quad \text{and} \quad \int_{\mathbb{T}^2} f_y(x, y) dx dy = \sum_{j=1}^{s_2} d_{2,j} + \alpha\gamma,$$

by Theorem 3.2, the special flow T^f is weakly mixing.

Note that every piecewise C^2 -function $F : \mathbb{T} \rightarrow \mathbb{R}$ with s discontinuities $\Delta_1, \dots, \Delta_s$ with jumps d_1, \dots, d_s respectively, can be represented as

$$F(x) = \tilde{F}(x) + \sum_{j=1}^s d_j \{x - \Delta_j\},$$

where \tilde{F} is a continuous function which is piecewise C^2 . Therefore, we can assume that

$$f_i(x) = \sum_{j=1}^s d_{i,j} \{x - \Delta_{i,j}\} \quad \text{for } i = 1, 2$$

and g is a Lipschitz function.

We proceed to the proof of (i). On \mathbb{R}^2 and \mathbb{T}^2 we will consider the metrics $\hat{d}((x, y), (x', y')) = \max(|x' - x|, |y' - y|)$ and $d((x, y), (x', y')) = \max(\|x' - x\|, \|y' - y\|)$, respectively. Then the map $\pi : \mathbb{R}^2 \rightarrow \mathbb{T}^2$, $\pi(x, y) = (x, y) + \mathbb{Z}^2$ is surjective and uniformly locally isometric. Moreover, π is equivariant for the translation $\hat{T} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\hat{T}(x, y) = (x + \alpha, y + \beta)$ and T . Let $\hat{f} = f \circ \pi$ and $\hat{g} = g \circ \pi$. In view of (40) and (41)

$$\begin{aligned} \hat{f}^{(n)}(x', y') - \hat{f}^{(n)}(x, y) &= \hat{g}^{(n)}(x', y') - \hat{g}^{(n)}(x, y) \\ &+ n \left(\sum_{j=1}^{s_1} d_{1,j} - \gamma \frac{\{y\} + n\beta}{n} \right) (x' - x) + n \left(\sum_{j=1}^{s_2} d_{2,j} + \gamma\alpha \right) (y' - y) \\ &+ \sum_{i=1,2} \left(\sum_{j=1}^{s_i} N_{i,j}^{(n)}((x, y), (x', y')) d_{i,j} + N_i^{(n)}((x, y), (x', y')) \gamma \right) \\ &- b(\hat{T}^n(x, y), \hat{T}^n(x', y')) N(\hat{T}^n(x, y), \hat{T}^n(x', y')) \\ &+ b((x, y), (x', y')) N((x, y), (x', y')), \end{aligned}$$

where $N_{(\cdot)} : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{Z}$ and $b : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are given by

$$N_{1,j}((x, y), (x', y')) = [x' - \Delta_{1,j}] - [x - \Delta_{1,j}],$$

$$\begin{aligned}
N_{2,j}((x, y), (x', y')) &= [y' - \Delta_{2,j}] - [y - \Delta_{2,j}], \\
N_1((x, y), (x', y')) &= ([x'] - [x])[y] + \beta, \\
N_2((x, y), (x', y')) &= -[\{x'\} + \alpha]([y' + \beta] - [y + \beta]) \\
b((x, y), (x', y')) &= \gamma\{x'\}, \quad N((x, y), (x', y')) = [y'] - [y].
\end{aligned}$$

By Lemma 7.1, there exist positive constants C_1, C_2 such that for any pair of distinct point $(x, y), (x', y')$ in \mathbb{R}^2 with $d := \widehat{d}((x, y), (x', y')) \leq 1/2$ we have:

- $|N_{(\cdot)}((x, y), (x', y'))| \leq 1$ and $|b((x, y), (x', y'))| \leq |\gamma|$;
- each sequence $\left(N_{(\cdot)}(\widehat{T}^n(x, y), \widehat{T}^n(x', y'))\right)_{n \geq 0}$ is C_1/d -sparse;
- $\left(N_{1,j}(\widehat{T}^n(x, y), \widehat{T}^n(x', y'))\right)_{n \geq 0}$ is $(C_1/d, C_2/d)$ -sparse for all $j = 1, \dots, s_1$ whenever $|x' - x| \geq |y' - y|$;
- $\left(N_{2,j}(\widehat{T}^n(x, y), \widehat{T}^n(x', y'))\right)_{n \geq 0}$ is $(C_1/d, C_2/d)$ -sparse for all $j = 1, \dots, s_2$ whenever $|x' - x| \leq |y' - y|$.

Moreover, if L stands for the Lipschitz constant of g then

$$\begin{aligned}
&|\widehat{g}^{(n)}(x', y') - \widehat{g}^{(n)}(x, y)| \\
&+ \left| n \left(\sum_{j=1}^{s_1} d_{1,j} - \frac{[\{y\} + n\beta]}{n} \right) (x' - x) + n \left(\sum_{j=1}^{s_2} d_{2,j} + \alpha \right) (y' - y) \right| \\
&\leq nC_0 \widehat{d}((x, y), (x', y')),
\end{aligned}$$

where

$$C_0 = L + \sum_{j=1}^{s_1} |d_{1,j}| + \sum_{j=1}^{s_2} |d_{2,j}| + |\alpha| + |\beta| + 2.$$

Since $d_{1,1}, \dots, d_{1,s_1}, d_{2,1}, \dots, d_{2,s_2}, \gamma$ are independent over \mathbb{Q} , the assumptions of Proposition 6.4 are verified with $R = 1$ and $B = |\gamma|$. This completes the proof of weak Ratner's property for the special flow T^f in case (i).

The proof in case (ii) runs as before. \square

8. MILD MIXING

Using a result from [8] we will now show mild mixing property for the class of flows from the previous section.

Lemma 8.1 (see [8]). *Let $(S_t)_{t \in \mathbb{R}}$ be an ergodic flow on (X, \mathcal{B}, μ) which has finite fibers factor property. Then the flow $(S_t)_{t \in \mathbb{R}}$ is mildly mixing provided it is not partially rigid.* \square

Theorem 8.2. *Under the assumptions of Theorem 7.4, the special flow T^f is mildly mixing.*

Proof. In view of Theorem 5.9, since the special flow T^f has weak Ratner's property, it is a finite extension of each of its non-trivial factors. As $\int f_x(x, y) dx dy \neq 0$ or $\int f_y(x, y) dx dy \neq 0$, by Theorem 4.1, T^f is not partially rigid. An application of Lemma 8.1 completes the proof. \square

Example. Let us consider the roof function $f : \mathbb{T}^2 \rightarrow \mathbb{R}_+$ the form

$$\begin{aligned} f(x, y) &= a\{x\} + b\{y\} + c \text{ with } a/b \in \mathbb{R} \setminus \mathbb{Q} \text{ or} \\ f(x, y) &= a\{x\} + b\{y\} + c(\alpha\{y\} - (\{x\} + \alpha)\{y\} + \beta) + d, \end{aligned}$$

where a, b, c are independent over \mathbb{Q} with $a \neq c\beta$ or $b \neq -c\alpha$. By Theorem 8.2, the special flow T^f is mildly mixing provided that $T(x, y) = (x + \alpha, y + \beta)$ is an ergodic rotation on the torus \mathbb{T}^2 such that both α and β have bounded partial quotients.

In the light of next section it is not however clear whether flows from Theorem 8.2 are not mixing. We will now show that at least some of them are certainly not mixing. The main idea is to find $\alpha, \beta \in \mathbb{T}$ so that $1, \alpha, \beta$ are rationally independent, α and β have bounded partial quotients and the intersection of the sets of denominators of α and β are infinite. Examples of such α and β have been pointed out to us by M. Keane. Below, we present his argument.

Let $(a_n)_{n \geq 1}$ be a palindromic sequence in $\{1, \dots, N\}$ (for some fixed $N \geq 2$), i.e. we assume that $(a_n)_{n \geq 1}$ has infinitely many prefixes which are palindromes and (a_n) is not eventually periodic; if in the standard Thue-Morse sequence 01101001... we replace 0 by 1 and 1 by 2 the resulting sequence is palindromic for $N = 2$, see e.g. [1]. Let

$$\alpha := [0; a_1, a_2, \dots] \text{ and } \beta := \{1/\alpha\} = [0; a_2, a_3, \dots].$$

Since α is not quadratic irrational, $\alpha, 1/\alpha, 1$ cannot be rationally dependent. Moreover, if $a_1 \dots a_{k_n+1}$ is a palindrome then in fact

$$\alpha = [0; a_1, a_2, \dots, a_{k_n}, \dots] \text{ and } \beta = [0; a_{k_n}, a_{k_n-1}, \dots, a_1, \dots].$$

It is classical that

$$[0; a_1, a_2, \dots, a_{k_n}] = \frac{p_n}{q_n} \text{ and } [0; a_{k_n}, a_{k_n-1}, \dots, a_1] = \frac{r_n}{q_n},$$

so the k_n -th denominators of α and β are the same. In this way we have obtained an infinite sequence $(q_n)_{n \geq 1}$ for α and β (each q_n being the k_n -th denominator of α and β). Setting $f(x, y) = a\{x\} + b\{y\} + c$, by the Denjoy-Koksma inequality, $|f^{(q_n)}(x, y) - q_n \int f d\mu| \leq 2(|a| + |b|)$. Since (q_n) is a rigidity sequence for the ergodic rotation $T(x, y) = (x + \alpha, y + \beta)$, by standard arguments (see [18]), the special flow T^f is not mixing (in fact, it is not partially mixing, see Section 10).

9. MIXING

In this section we will show that von Neumann's special flows over ergodic two-dimensional rotations can be mixing. We will make use of the following criterion for mixing in which a partial partition of \mathbb{T} means a partition of a subset of \mathbb{T} .

Proposition 9.1 (see Proposition 3.3 in [5]). *Let T^f be the special flow built over an ergodic rotation $T : \mathbb{T}^2 \rightarrow \mathbb{T}^2$, $T(x, y) = (x + \alpha, y + \beta)$ and under a piecewise C^2 roof function $f : \mathbb{T}^2 \rightarrow \mathbb{R}_+$. Let (τ_n) , (ε_n) and (k_n) be sequences of real positive numbers such that $\tau_n \rightarrow \infty$, $\varepsilon_n \rightarrow 0$, $k_n \rightarrow \infty$ and let (η_n) be a sequence of partial partitions of \mathbb{T} , where $\eta_n = \{C_i^{(n)}\}$ and $C_i^{(n)}$ are intervals such that*

$$\sup_{C_i^{(n)} \in \eta_n} |C_i^{(n)}| \rightarrow 0 \text{ and } \sum_{C_i^{(n)} \in \eta_n} |C_i^{(n)}| \rightarrow 1.$$

Suppose that there exists n_0 such that if $n \geq n_0$ then

- for any $m \in [\tau_{2n}/2, 2\tau_{2n+1}]$, $y \in \mathbb{T}$ and $C_i^{(2n)} \in \eta_{2n}$ the map $C_i^{(2n)} \ni x \mapsto f^{(m)}(x, y) \in \mathbb{R}$ is of class C^2 and

$$k_{2n} \leq \inf_{x \in C_i^{(2n)}} |f_x^{(m)}(x, y)| |C_i^{(2n)}|,$$

$$\sup_{x \in C_i^{(2n)}} |f_{xx}^{(m)}(x, y)| |C_i^{(2n)}| \leq \varepsilon_{2n} \inf_{x \in C_i^{(2n)}} |f_x^{(m)}(x, y)|.$$

- for any $m \in [\tau_{2n+1}/2, 2\tau_{2n+2}]$, $x \in \mathbb{T}$ and $C_i^{(2n+1)} \in \eta_{2n+1}$ the map $C_i^{(2n+1)} \ni y \mapsto f^{(m)}(x, y) \in \mathbb{R}$ is of class C^2 and

$$k_{2n+1} \leq \inf_{y \in C_i^{(2n+1)}} |f_y^{(m)}(x, y)| |C_i^{(2n+1)}|,$$

$$\sup_{y \in C_i^{(2n+1)}} |f_{yy}^{(m)}(x, y)| |C_i^{(2n+1)}| \leq \varepsilon_{2n+1} \inf_{y \in C_i^{(2n+1)}} |f_y^{(m)}(x, y)|.$$

Then T^f is mixing. □

Remark 9.2. The above criterion for mixing has been formulated by Fayad [5] only for C^2 roof functions. Nevertheless, following word by word Fayad's proof we obtain that the assertion holds whenever f is piecewise C^2 .

Let $(\gamma(n))_{n \in \mathbb{N}}$ be an increasing sequence of positive real numbers such that $\gamma(1) \geq 1$ and $\gamma(n) \rightarrow \infty$. Choose a pair of irrational numbers $\alpha, \beta \in [0, 1)$ such that denoting by (q_n) and (r_n) the sequences of denominators for α and β respectively we have

$$(42) \quad 4\gamma(n-1)\gamma(n)q_n \leq r_n \text{ and } 4\gamma(n)^2 r_n \leq q_{n+1} \text{ for all } n \geq 1.$$

As it was observed by Yoccoz in [31, Appendix A] the set of all pairs satisfying (42) is uncountable. Note that the rotation $T : \mathbb{T}^2 \rightarrow \mathbb{T}^2$, $T(x, y) = (x + \alpha, y + \beta)$ is ergodic. Indeed, if T is not ergodic then there exist integer numbers $k \neq 0$, $l \neq 0$ and m such that $k\alpha + l\beta = m$. Next choose $n \in \mathbb{N}$ such that

$$(43) \quad \gamma(n) > \max(|k|, |l|).$$

In view of (42),

$$|lq_n| \leq \gamma(n)q_n < 4\gamma(n-1)\gamma(n)q_n \leq r_n \text{ and } 2r_n \leq \frac{q_{n+1}}{2\gamma(n)^2} \leq \frac{q_{n+1}}{2|k|}.$$

It follows that

$$\|lq_n\beta\| \geq \|r_{n-1}\beta\| \geq \frac{1}{2r_n} \geq \frac{2|k|}{q_{n+1}}.$$

Moreover,

$$\|kq_n\alpha\| \leq |k|\|q_n\alpha\| \leq \frac{|k|}{q_{n+1}}.$$

Therefore

$$0 = \|q_n m\| = \|q_n(k\alpha + l\beta)\| \geq \|lq_n\beta\| - \|kq_n\alpha\| \geq \frac{|k|}{q_{n+1}} > 0,$$

a contradiction.

Theorem 9.3. *Let $f : \mathbb{T}^2 \rightarrow \mathbb{R}_+$ be a piecewise C^2 -function satisfying (3). For every rotation $T : \mathbb{T}^2 \rightarrow \mathbb{T}^2$, $T(x, y) = (x + \alpha, y + \beta)$ satisfying (42) the special flow T^f is mixing.*

Proof. Let $0 \leq a_1 < \dots < a_N < 1$ and $0 \leq b_1 < \dots < b_M < 1$ be points determining the lines of discontinuities for f . Since $f_x, f_y : \mathbb{T}^2 \rightarrow \mathbb{R}$ are Riemann integrable function, by the unique ergodicity of T and (3), there exist $\theta > 0$ and $m_0 \in \mathbb{N}$ such that

$$(44) \quad m\theta \leq |(f_x)^{(m)}(x, y)| \text{ and } m\theta \leq |(f_y)^{(m)}(x, y)|$$

for all $(x, y) \in \mathbb{T}^2$ and $m \geq m_0$. Let

$$\Theta = \sup_{(x, y) \in \mathbb{T}^2} \max(|f_{xx}(x, y)|, |f_{yy}(x, y)|).$$

Then

$$(45) \quad |(f_{xx})^{(m)}(x, y)| \leq m\Theta \text{ and } |(f_{yy})^{(m)}(x, y)| \leq m\Theta.$$

Choose $n_0 \in \mathbb{N}$ such that $q_{n_0}, r_{n_0} \geq m_0$. Fix $n \geq n_0$. Let κ stand for the partition (into intervals) of \mathbb{T} determined by points $a_l - j\alpha$, $1 \leq l \leq N$, $0 \leq j < q_n \left\lceil \frac{q_{n+1}}{\gamma(n)q_n} \right\rceil$ ($\lceil x \rceil = \min\{n \in \mathbb{Z} : x \leq n\}$). Set

$$\{C_i^{(2n)}\} = \eta_{2n} = \left\{ I \in \kappa : |I| > \frac{1}{\sqrt{\gamma(n)q_n}} \right\}.$$

Recall that for every $1 \leq l \leq N$ the diameter of the partition \mathbb{T} determined by points $a_l - j\alpha$ for $0 \leq j < q_n$ is bounded by $\frac{1}{q_n} + \frac{1}{q_{n+1}}$. Since η_{2n} is finer than each such partition,

$$\max_{C_i^{(2n)} \in \eta_{2n}} |C_i^{(2n)}| < \frac{1}{q_n} + \frac{1}{q_{n+1}} < \frac{2}{q_n} \rightarrow 0.$$

For every pair l, j , where $1 \leq l \leq N$ and $0 \leq j < q_n$ let us consider the family of points

$$A_{l,j} = \left\{ a_l - (j + iq_n)\alpha : 0 \leq i < \left\lceil \frac{q_{n+1}}{\gamma(n)q_n} \right\rceil \right\}.$$

Note that $\bigcup_{1 \leq l \leq N} \bigcup_{0 \leq j < q_n} A_{l,j}$ coincides with the set determining κ . Moreover, for all $0 \leq i, i' < \left\lceil \frac{q_{n+1}}{\gamma(n)q_n} \right\rceil$ we have

$$\begin{aligned} \|(a_l - (j + iq_n)\alpha) - (a_l - (j + i'q_n)\alpha)\| &= \|(i - i')q_n\alpha\| \leq \frac{q_{n+1}}{\gamma(n)q_n} \|q_n\alpha\| \\ &\leq \frac{1}{\gamma(n)q_n} < \frac{1}{\sqrt{\gamma(n)q_n}}. \end{aligned}$$

It follows that for every pair l, j there exist $0 \leq i(l, j, 0), i(l, j, 1) < \left\lceil \frac{q_{n+1}}{\gamma(n)q_n} \right\rceil$ such that

$$A_{l,j} \subset I_{l,j} := [a_l - (j + i(l, j, 0)q_n)\alpha, a_l - (j + i(l, j, 1)q_n)\alpha]$$

and $|I_{l,j}| < 1/(\sqrt{\gamma(n)q_n})$. Denote by κ_1 the family of intervals $I \in \kappa$ such that $I \subset I_{l,j}$ for some $1 \leq l \leq N$ and $0 \leq j < q_n$. Since $|I| < 1/(\sqrt{\gamma(n)q_n})$ for every $I \in \kappa_1$, we have $\kappa_1 \subset \kappa \setminus \eta_{2n}$ and

$$\lambda_{\mathbb{T}}\left(\bigcup_{I \in \kappa_1} I\right) = \sum_{1 \leq l \leq N} \sum_{0 \leq j < q_n} |I_{l,j}| < \frac{Nq_n}{\sqrt{\gamma(n)q_n}} = \frac{N}{\sqrt{\gamma(n)}}.$$

Furthermore, the ends of every interval $I \in \kappa \setminus \kappa_1$ are of the form $a_l - (j + i(l, j, s)q_n)\alpha$ for some $1 \leq l \leq N$, $0 \leq j < q_n$ and $s = 0, 1$. It follows that $\#(\kappa \setminus \kappa_1) \leq Nq_n$.

Let κ_2 stand for the collection of all $I \in \kappa \setminus \kappa_1$ such that $|I| \leq \frac{1}{\sqrt{\gamma(n)q_n}}$. Since $\#\kappa_2 \leq \#(\kappa \setminus \kappa_1) \leq Nq_n$, we obtain

$$\lambda_{\mathbb{T}} \left(\bigcup_{I \in \kappa_2} I \right) \leq Nq_n \frac{1}{\sqrt{\gamma(n)q_n}} = \frac{N}{\sqrt{\gamma(n)}}.$$

By the definition of κ_1 and κ_2 , $\eta_{2n} = \kappa \setminus (\kappa_1 \cup \kappa_2)$, and hence

$$\sum_i |C_i^{(2n)}| = 1 - \lambda_{\mathbb{T}} \left(\bigcup_{I \in \kappa_1} I \right) - \lambda_{\mathbb{T}} \left(\bigcup_{I \in \kappa_2} I \right) \geq 1 - \frac{2N}{\sqrt{\gamma(n)}} \rightarrow 1.$$

Next let us consider the partition κ' of \mathbb{T} determined by points $b_l - (j + ir_n)\beta$, $1 \leq l \leq N$, $0 \leq j < r_n$, $0 \leq i \leq r_n \left\lceil \frac{r_{n+1}}{\gamma(n)r_n} \right\rceil$ and set

$$\{C_i^{(2n+1)}\} = \eta_{2n+1} = \left\{ I \in \kappa' : |I| > \frac{1}{\sqrt{\gamma(n)r_n}} \right\}.$$

Then

$$\max_{C_i^{(2n+1)} \in \eta_{2n+1}} |C_i^{(2n+1)}| < \frac{2}{r_n} \rightarrow 0 \text{ and } \sum_i |C_i^{(2n+1)}| \geq 1 - \frac{2M}{\sqrt{\gamma(n)}} \rightarrow 1.$$

Finally for every $n \geq n_0$ set

$$\begin{aligned} \tau_{2n} &= 2\gamma(n)q_n, \quad \tau_{2n+1} = 2\gamma(n)r_n, \\ \varepsilon_{2n} &= \frac{2\Theta}{\theta q_n}, \quad \varepsilon_{2n+1} = \frac{2\Theta}{\theta r_n}, \quad k_{2n} = k_{2n+1} = \theta\sqrt{\gamma(n)}. \end{aligned}$$

Assume that $m \in [\tau_{2n}/2, 2\tau_{2n+1}]$ ($n \geq n_0$) and fix $y \in \mathbb{T}$. From (42) we have

$$(46) \quad m_0 < \gamma(n)q_n \leq m \leq 4\gamma(n)r_n \leq q_{n+1}/\gamma(n) \leq q_n \left\lceil \frac{q_{n+1}}{\gamma(n)q_n} \right\rceil.$$

Then every discontinuity of $x \mapsto f^{(m)}(x, y)$ is of the form $a_l - j\alpha$ with $1 \leq l \leq N$, $0 \leq j < q_n \left\lceil \frac{q_{n+1}}{\gamma(n)q_n} \right\rceil$, and hence $C_i^{(2n)} \ni x \mapsto f^{(m)}(x, y) \in \mathbb{R}$ is of class C^2 for every $C_i^{(2n)} \in \eta_{2n}$. Since $\frac{1}{\sqrt{\gamma(n)q_n}} < |C_i^{(2n)}| < \frac{2}{q_n}$, by (44), (45) and (46),

$$\inf_{x \in C_i^{(2n)}} |f_x^{(m)}(x, y)| |C_i^{(2n)}| \geq \theta m \frac{1}{\sqrt{\gamma(n)q_n}} \geq \theta \gamma(n)q_n \frac{1}{\sqrt{\gamma(n)q_n}} = \theta\sqrt{\gamma(n)} = k_{2n},$$

$$\varepsilon_{2n} \inf_{x \in C_i^{(2n)}} |f_x^{(m)}(x, y)| \geq \frac{2\Theta}{\theta q_n} \theta m = \frac{2\Theta m}{q_n}$$

and

$$\sup_{x \in C_i^{(2n)}} |f_{xx}^{(m)}(x, y)| |C_i^{(2n)}| \leq \Theta m \frac{2}{q_n} \leq \varepsilon_{2n} \inf_{x \in C_i^{(2n)}} |f_x^{(m)}(x, y)|.$$

Similarly, if $m \in [\tau_{2n+1}/2, 2\tau_{2n+2}]$ and $n \geq n_0$ then $\gamma(n)r_n \leq m \leq r_{n+1}/\gamma(n)$. Moreover, for every $x \in \mathbb{T}$ and $C_i^{(2n+1)} \in \eta_{2n+1}$ the function $C_i^{(2n+1)} \ni y \mapsto f^{(m)}(x, y) \in \mathbb{R}$ is of class C^2 and

$$k_{2n+1} \leq \inf_{y \in C_i^{(2n+1)}} |f_y^{(m)}(x, y)| |C_i^{(2n+1)}|,$$

$$\sup_{y \in C_i^{(2n+1)}} |f_{yy}^{(m)}(x, y)| |C_i^{(2n+1)}| \leq \varepsilon_{2n+1} \inf_{y \in C_i^{(2n+1)}} |f_y^{(m)}(x, y)|.$$

Now an application of Proposition 9.1 completes the proof. \square

10. REMARKS

As we have already noticed in Section 8 certainly not all von Neumann's special flows over two-dimensional rotations are mixing. As a matter of fact, if assume that $f(x, y) = f_1(x) + f_2(y)$ (we assume tacitly that $f > 0$ and $\int_{\mathbb{T}^2} f d\lambda_{\mathbb{T}^2} = 1$ and we set $f_0 = f - 1$) and α and β have a common subsequence of denominators then basically we will copy results from the one-dimensional case. Indeed, the strong von Neumann's condition (3) is reduced to (1) for f_1 and f_2 separately (and f_i is piecewise C^2 , $i = 1, 2$). Denote by (q_n) and (t_n) the sequences of denominators of α and β respectively. If we assume additionally that α and β have a common subsequence of denominators $l_k := q_{n_k} = t_{m_k}$ for infinitely many $k \geq 1$ then it follows from [7] that the sequence of centered distributions $(f_0^{(l_k)})_* \rightarrow P$ weakly in the space of probability measures on \mathbb{R} (the probability measure P is concentrated on the interval $[-(\text{Var } f_1 + \text{Var } f_2), \text{Var } f_1 + \text{Var } f_2]$). Thus, by [7]

$$(47) \quad U_{T_{l_k}^f} \rightarrow \int_{\mathbb{R}} U_{T_t^f} dP(t)$$

in the space of Markov operators on $L^2((\mathbb{T}^2)^f, \lambda_{\mathbb{T}^2}^f)$, whence (again by [7]) T^f is spectrally disjoint from all mixing flows, which in particular rules out the possibility of T^f being mixing; here by $U_{T_t^f}$ we denote the corresponding Koopman representation: $U_{T_t^f} F = F \circ T_t^f$ for $t \in \mathbb{R}$. In fact, (47) implies even the absence of partial mixing for T^f . Indeed, recall that partial mixing means that there exists a constant $\kappa > 0$ such that

$$\liminf_{t \rightarrow \infty} \lambda_{\mathbb{T}^2}^f(A \cap T_t^f(B)) \geq \kappa \lambda_{\mathbb{T}^2}^f(A) \lambda_{\mathbb{T}^2}^f(B)$$

for each $A, B \in \mathcal{B}^f$. In terms of Markov operators it follows that for any convergent subsequence $U_{T_{s_k}^f} \rightarrow J$ we have $J = \kappa \Pi_{(\mathbb{T}^2)^f} + (1 - \kappa)K$ where $\Pi_{(\mathbb{T}^2)^f}(F) = \int_{\mathbb{T}^2} F d\lambda_{\mathbb{T}^2}$ and K is another Markov operator. Now, if we take $s_k = l_k$ we will obtain

$$\int_{\mathbb{R}} U_{T_t^f} dP(t) = \kappa \Pi_{(\mathbb{T}^2)^f} + (1 - \kappa)K$$

which is possible only if $\kappa = 0$ (indeed, otherwise by taking an ergodic decomposition of the joining corresponding to K we would obtain two different ergodic decompositions of the joining corresponding to the same Markov operator, see [7]).

The following natural questions easily follow:

1) *Is it possible to obtain a mixing strong von Neumann's flow over the rotation by (α, β) if α and β have a common subsequence of denominators?*

2) *Given $(\alpha, \beta) \in \mathbb{T}^2$ is there a large class of piecewise C^2 functions satisfying (3) for which mixing is excluded?* It seems that such a question makes sense even in case of smooth functions on \mathbb{T}^2 . We recall that mixing of T^f is excluded whenever the sequence $((f_0^{(n)})_*)_{n \geq 1}$ does not converge to δ_∞ in the space of probability measures on $\mathbb{R} \cup \{\infty\}$, see [24], [29].

Note in passing that the weak convergence of measures $(f_0^{(n)})_* \rightarrow \delta_\infty$ takes place for all examples coming from Theorem 9.3.

3) *Is it possible to obtain mixing for strong von Neumann's flows over the rotation by (α, β) where α, β have bounded partial quotients? More specifically, is mixing*

possible in the class of flows considered in Theorem 7.4? If the answer to the second question is positive then Theorem 7.4 would give the first examples of mixing special flows over rotations having (weak) Ratner's property. For such flows mixing of all order follows; indeed, flows having weak Ratner's property are quasi-simple in the sense of [27] and mixing implies mixing of all orders for such flows [27]. Another possibility to obtain mixing of all orders would be to show that for example if we take $f(x, y) = a\{x\} + b\{y\} + c$ and (α, β) satisfying (42) then the spectrum of U_{Tf} is singular: mixing of all orders would follow from [12]. We recall that Fayad in [6] has constructed a smooth reparametrization of a linear flow on \mathbb{T}^3 which is mixing and has simple singular spectrum. Such a reparametrization flow has a representation as the special flow over a two-dimensional rotation and under a smooth roof function.

Little is known about the spectrum of weak von Neumann's special flows. It seems to be completely open whether such flows can have an absolutely continuous component in the spectrum. This is impossible over rotations on \mathbb{T} (in fact, in the one dimensional case we have even spectral disjointness with all mixing flows [7]). It is neither clear whether such flows can have simple spectrum – this remains an open problem even in the one dimensional case.

Finally, it would be nice to decide whether there exists a weak von Neumann's special flow over two-dimensional rotations which is self-similar – this is impossible for von Neumann's special flows over rotations on the circle [10].

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