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Smooth singular flows in dimension 2 with the minimal self-joining property

By

K. Frączek and **M. Lemańczyk**

Nicolaus Copernicus University, Toruń, Poland

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Abstract. It is proved that some velocity changes in flows on the torus determined by quasi-periodic Hamiltonians on $\mathbb{R}^2$:

$$H(x+m, y+n) = H(x, y) + m\alpha_1 + n\alpha_2,$$

where α_1/α_2 is an irrational number with bounded partial quotients, lead to singular flows on $\mathbb{T}^2$ with an ergodic component having a minimal set of self-joinings.

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Introduction

One of the classical problems of ergodic theory is, given a dynamical system $\mathscr{S} = (S_t)_{t\in\mathbb{R}}$ acting on a standard probability Borel space $(X, \mathscr{B}, \mu)$, to understand possible interactions between $\mathscr{S}$ and all other systems $\mathscr{T} = (T_t)_{t\in\mathbb{R}}$. Being more precise, we are interested in a description of all possible situations in which $\mathscr{S}$ and $\mathscr{T}$ are seen (as factors) in their common extension $\mathscr{U} = (U_t)_{t\in\mathbb{R}}$. Clearly, we can restrict ourselves to the class of "smallest" common extensions, that is we will assume that the sub-σ-algebras corresponding to $\mathscr{S}$ and $\mathscr{T}$ generate the σ-algebra of measurable sets for $\mathscr{U}$ – in this case $\mathscr{U}$ is called a joining of $\mathscr{S}$ and $\mathscr{T}$ (see Section 1 for a formal definition). If for $\mathscr{U}$ we take the product system $\mathscr{S}\times\mathscr{T} = (S_t \times T_t)_{t\in\mathbb{R}}$ (acting on the product space) then, obviously, $\mathscr{U}$ is a joining of $\mathscr{S}$ and $\mathscr{T}$. If this is the only way to join $\mathscr{S}$ and $\mathscr{T}$ then, following Furstenberg [8], we say that $\mathscr{S}$ and $\mathscr{T}$ are disjoint. Another easy observation is that given $\mathscr{S}$ there are always systems which are not disjoint from $\mathscr{S}$; indeed a system is never disjoint from itself and more generally two systems with a non-trivial common factor cannot be disjoint (there are however non-disjoint systems without common factors, see e.g. [30]). For a general $\mathscr{S}$, especially in the positive entropy case, a

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description of all possible joinings with an arbitrary $\mathscr{T}$ seems to be an impossible task – this requires a full description of all *infinite* self-joinings of $\mathscr{S}$, see [25]. However, there is at least one class of zero entropy flows for which such a description exists. This is the case of so called simple flows introduced by Veech ([34], only $\mathbb{Z}$-actions are considered there) and del Junco-Rudolph [16] (see Section 1 below). If $\mathscr{S}$ is simple and $\mathscr{T}$ is ergodic, then a non-product ergodic joining between $\mathscr{T}$ and $\mathscr{S}$ is possible only if $\mathscr{T}$ has a factor which is given by a symmetric factor of a finite product of a factor of $\mathscr{S}$ with itself (and such joinings are fully described, see [33]). This result is even more impressive when we restrict ourselves to a subclass of simple flows, namely to flows with the minimal self-joining property (MSJ) – these are ergodic flows for which ergodic self-joinings are products of graphs of their time-t automorphisms, see Section 1 below. Such a flow $\mathscr{S}$ has no non-trivial factors, and factors of a direct product $\underbrace{\mathscr{S}\times\cdots\times\mathscr{S}}_{n}$ are determined only by symmetries given by subgroups of the group of permutations on an n-element set. Hence either an ergodic flow $\mathscr{T}$ is disjoint from $\mathscr{S}$ or $\mathscr{T}$ is extremely "close" to $\mathscr{S}$ in the sense, that $\mathscr{T}$ is an ergodic extension of a symmetric factor $\mathscr{A}$ of $\underbrace{\mathscr{S}\times\cdots\times\mathscr{S}}_{n}$ and an ergodic joining is given by the restriction of the relative product (over $\mathscr{A}$) to the first copy of $\mathscr{S}$ in $\underbrace{\mathscr{S}\times\cdots\times\mathscr{S}}_{n}$ and $\mathscr{T}$. We should also notice that ergodic systems with pure point spectrum are simple, and that the considerations above are interesting only in the weak mixing case (we recall that the MSJ property implies weak mixing).

All the considerations above, although of abstract nature, seem to be also interesting from the smooth point of view. Indeed, assume that M_i ($i=1,2$) is a compact smooth manifold and let $A_i : M \to TM$ be a smooth vector-field. Denote by $\Phi^{(i)} = (\phi_t^{(i)})_{t\in\mathbb{R}}$ the flow given by the solution of the differential equation

$$\frac{d\phi_t^{(i)}x}{dt} = A_i(\phi_t^{(i)}x).$$

By compactness of M_i, stationary states (i.e. probability invariant measures) for $\Phi^{(i)}$ exist. If now, on $M_1\times M_2$ we consider the product vector field $A_1\times A_2$ then any stationary state for the corresponding (product) flow on $M_1\times M_2$ is a joining of some stationary states of $\Phi^{(1)}$ and $\Phi^{(2)}$. This approach will be fruitful if systems under considerations are uniquely ergodic or if we have finitely many invariant measures (recall that if M is an orientable manifold then every area – preserving smooth flow on M has at most genus(M) nontrivial ergodic invariant measures; see Theorem 14.7.6 in [17]). By what was said above, once $\Phi^{(1)}$ is uniquely ergodic and has the MSJ property we are able to describe stationary states of the system given by the vector-field $A_1\times A_2$.

For horocycle flows the problem of self-joinings was solved by Ratner in a series of remarkable papers ([27]–[29]) in the 1980s. Some horocycle flows turn out to be simple, or even to have the MSJ property, e.g. if $\Gamma\subset \mathrm{SL}(2,\mathbb{R})$ is maximal and not arithmetic lattice then the horocycle flow on $\mathrm{SL}(2,\mathbb{R})/\Gamma$ has MSJ (see [29]). Thouvenot in [33] has shown that horocycle flows are always factors of simple systems (in the cocompact case this was already shown by Glasner and

Weiss in [11]). Hence in dimension 3 the MSJ property appears quite naturally. It is an open question whether it can also be seen in dimension 2, that is on surfaces.

The present paper brings, in a sense, a positive answer to this question, however the flows that appear in the paper are singular flows – they will have finitely many points at which a smooth vector-field defining our system is not defined. Let us pass now to a more precise description of the main result of the paper.

Let $H:\mathbb{R}^2\to\mathbb{R}$ be a C^∞-quasi-periodic function, i.e.

$$H(x+m,y+n)=H(x,y)+m\alpha_1+n\alpha_2$$

for all $(x,y)\in\mathbb{R}^2$ and $m,n\in\mathbb{Z}$, and $\alpha=\alpha_1/\alpha_2$ is irrational. Clearly, $H(x,y)=\widetilde{H}(x,y)+\alpha_1x+\alpha_2y$, where $\widetilde{H}(x,y):\mathbb{R}^2\to\mathbb{R}$ is a periodic function of period 1 in each coordinate. Then H determines a (quasi-periodic) Hamiltonian flow $(h_t)_{t\in\mathbb{R}}$ on the torus associated with the following differential equation

$$\frac{d\bar{x}}{dt}=X_H(\bar{x}),\ \text{where } X_H=\left(\frac{\partial H}{\partial y},-\frac{\partial H}{\partial x}\right).$$

If H has no critical point then (h_t) is isomorphic to a special flow built over the rotation by α on the circle and under a positive C^∞-function (see [4], Ch. 16). Moreover, if α is Diophantine (there exist $\nu\geqslant 1$ and $C>0$ such that $|q\alpha-p|\geqslant Cq^{-\nu}$ for all integer numbers p,q with $q\geqslant 1$) then (h_t) is isomorphic to a linear flow on the torus.

Now suppose that H has critical points. Let us recall some terminology and results proved by Arnold in [3]. Suppose that H is in the general position, i.e. H has no degenerate critical points and has all critical values distinct. In particular, each critical point is either a non-degenerate saddle point or a non-degenerate center. Moreover critical points repeat periodically (with period 1 in each coordinate) but their critical values are distinct. Then any superlevel $\{(x,y)\in\mathbb{R}^2:H(x,y)>c\}$ has exactly one unbounded connected component which contains a half-plane. Any connected component of a level set of H passing through a critical point is either bounded (a point or a lemniscate-like curve) or it has the shape of a folium of Descartes. In the unbounded case, the critical value level set of H separates the plane into two unbounded components and a disk; the closure of the disk is called a *trap*. A trap is homeomorphic to a closed disk and has a critical point on the boundary, called the vertex of the trap (the same terminology applies when we pass to $\mathbb{T}^2$). Traps with distinct vertices are disjoint. The phase space of $(h_t)_{t\in\mathbb{R}}$ decomposes into traps filled with fixed points, separatrices and periodic orbits, and an ergodic component EC of positive Lebesgue measure.

Now we will change velocity in the flow $(h_t)_{t\in\mathbb{R}}$. Let $\{\bar{x}_1,\ldots,\bar{x}_r\}$ be vertices of all traps. Suppose $p:\mathbb{T}^2\to\mathbb{R}$ is a non-negative C^∞-function which is positive on the torus except of the points $\{\bar{x}_1,\ldots,\bar{x}_r\}$. Let us consider the flow $(\varphi_t)_{t\in\mathbb{R}}$ on $\mathbb{T}^2\setminus\{\bar{x}_1,\ldots,\bar{x}_r\}$ associated with the following differential equation

$$\frac{d\bar{x}}{dt}=X(\bar{x}),\quad\text{where}\quad X(\bar{x})=\frac{X_H(\bar{x})}{p(\bar{x})}.$$

Since the orbits of (φ_t) and (h_t) are the same (modulo fixed points of (h_t)), the phase space of $(\varphi_t)_{t\in\mathbb{R}}$ decomposes into traps filled with critical points, separatrices and periodic orbits, and the ergodic component EC with positive Lebesgue measure.

Let us denote by $\omega = \omega_X$ the 1-form of class C^∞ on $\mathbb{T}^2 \setminus \{\bar{x}_1, \dots, \bar{x}_r\}$ given by $\omega(Y) = \langle X, Y\rangle / \langle X, X\rangle$.

Theorem 1. *If α has bounded partial quotients and $\int_{EC} d\omega \neq 0$, then $(\varphi_t)_{t\in\mathbb{R}}$ is simple, and it is a finite extension of an MSJ-factor.*

Our approach to prove Theorem 1 will be a detailed analysis of the special representation of the Hamiltonian flow (h_t) obtained by Arnold, and applied to (φ_t). In fact, the first step will be to prove the following result whose proof is presented in the Appendix.

Proposition 2. *The action of (φ_t) in EC is isomorphic to a special flow built over the rotation by α and under a roof function f which is piecewise absolutely continuous with $f' \in L^2(\mathbb{T})$. Moreover, the sum of jumps $S(f)$ of f is equal to $\int_{EC} d\omega$.*

Hence, we have to study special flows over irrational rotations, with particular roof functions. In fact, such flows were already considered by von Neumann in 1932 [26], where he proved weak mixing property whenever $S(f) \neq 0$. The same flows were considered by the authors of the present paper in [6], where under von Neumann's assumption $S(f) \neq 0$ and boundness of partial quotients of α a certain combinatorial property, similar to the famous Ratner's property from [27], on the orbits of T^f has been proved. This property implies some strong rigidity property on joinings between T^f and an arbitrary ergodic system. The approach in the present paper is completely different. We have to show some minimality property for the set of ergodic self-joinings, that is we study invariant measures for the product system $T^f \times T^f$ (with "right" marginals), and the key argument consists in showing that such measures are in one-to-one correspondence with some locally finite measures of some $\mathbb{Z}^2$-cylindrical actions. Then the mathematical construction of the main steps in the paper goes back rather to a use of ideas from non-singular ergodic theory: close to the concept of Mackey actions (see [24] or [23]), a use of the concept of Maharam extension (see [2]) and also we will substantially use some recent results by Sarig [32].

1. Joinings

Assume that $\mathscr{T} = (T_t)_{t\in\mathbb{R}}$ and $\mathscr{S} = (S_t)_{t\in\mathbb{R}}$ are Borel ergodic flows on standard probability spaces $(X, \mathscr{B}, \mu)$ and $(Y, \mathscr{C}, \nu)$ respectively. By a *joining* between $\mathscr{T}$ and $\mathscr{S}$ we mean any probability $(T_t \times S_t)_{t\in\mathbb{R}}$-invariant measure on $(X \times Y, \mathscr{B} \otimes \mathscr{C})$ whose projections on X and Y are equal to μ and ν respectively. The set of joinings between $\mathscr{T}$ and $\mathscr{S}$ is denoted by $J(\mathscr{T}, \mathscr{S})$. The subset of ergodic joinings is denoted by $J^e(\mathscr{T}, \mathscr{S})$. Ergodic joinings are exactly extremal points in the simplex $J(\mathscr{T}, \mathscr{S})$. Of course, the product measure $\mu \otimes \nu \in J(\mathscr{T}, \mathscr{S})$, moreover, if $\mathscr{T}$ or $\mathscr{S}$ is weakly mixing then $\mu \otimes \nu \in J^e(\mathscr{T}, \mathscr{S})$.

We denote by $C(\mathscr{T})$ the *centralizer* of the flow $\mathscr{T}$, this is the group of Borel automorphisms $R : (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ such that $T_t \circ R = R \circ T_t$ for every $t \in \mathbb{R}$.

For every $R\in C(\mathscr{T})$ by $\mu_R\in J(\mathscr{T},\mathscr{T})$ we will denote the *graph joining* determined by $\mu_R(A\times B)=\mu(A\cap R^{-1}B)$ for $A,B\in\mathscr{B}$. Then μ_R is concentrated on the graph of R and $\mu_R\in J^e(\mathscr{T},\mathscr{T})$.

Remark 1. Suppose that flows $\mathscr{T}$ and $\mathscr{S}$ are uniquely ergodic. Then any finite $(T_t\times S_t)_{t\in\mathbb{R}}$-invariant measure on $(X\times Y,\mathscr{B}\otimes\mathscr{C})$ is a multiple of a joining from $J(\mathscr{T},\mathscr{S})$.

If $\mathscr{T}_i=(T_t^{(i)})_{t\in\mathbb{R}}$ is a Borel flow on $(X_i,\mathscr{B}_i,\mu_i)$ for $i=1,\dots,k$ then by a *k-joining* of $\mathscr{T}_1,\dots,\mathscr{T}_k$ we mean any probability $(T_t^{(1)}\times\ldots\times T_t^{(k)})_{t\in\mathbb{R}}$-invariant measure on $(\prod_{i=1}^k X_i,\bigotimes_{i=1}^k\mathscr{B}_i)$ whose projection on X_i is equal to μ_i for $i=1,\dots,k$.

Suppose that $\mathscr{T}$ is an ergodic flow on $(X,\mathscr{B},\mu)$ and $\mathscr{T}_i=\mathscr{T}$ for $i=1,\dots,k$. If $R_1,\dots,R_k\in C(\mathscr{T})$ then the image of μ via the map

$$X\ni x\mapsto(R_1x,\dots,R_kx)\in X^k$$

is called an *off-diagonal* joining. Of course, any off-diagonal joining is an ergodic k-self-joining. Suppose that the set of indices $\{1,\dots,k\}$ is now partitioned into some subsets and let on each of these subsets an off-diagonal joining be given. Then clearly the product of these off-diagonal joinings is a k-self-joining of $\mathscr{T}$.

Definition 1 (see [30]). We say that $\mathscr{T}$ is *k-fold simple* if every ergodic k-self-joining is a product of off-diagonal joinings. $\mathscr{T}$ is *simple* if it is k-fold simple for every $k\in\mathbb{N}$. If additionally $C(\mathscr{T})=\{T_t:t\in\mathbb{R}\}$ then we say that $\mathscr{T}$ has *minimal self-joining* (*MSJ*).

Proposition 3 (see [31]). *If $\mathscr{T}$ is a weakly mixing flow then 2-fold simplicity implies simplicity.*

Recall that this result is unknown for automorphisms.

2. Borel group actions and invariant measures

Let (X,d) be a Polish metric space and let $\mathscr{B}=\mathscr{B}_X$ denote the σ-algebra of Borel subsets of X. Denote by $\mathrm{Aut}(X,\mathscr{B})$ the group of all Borel automorphisms of X. Let G be a Polish Abelian locally compact group. Suppose that T is a *Borel G-action* on $(X,\mathscr{B})$, i.e.

$$G\ni g\mapsto T_g\in\mathrm{Aut}(X,\mathscr{B})\text{ is a group homomorphism and}$$
$$G\times X\ni(g,x)\mapsto gx=T_gx\in X\text{ is a Borel map}$$

($G\times X$ is endowed with the product Borel structure). We will say that the G-action T is free if for every $x\in X$ the map $G\ni g\mapsto gx\in X$ is one-to-one. We say that a measure m on $(X,\mathscr{B})$ is T-quasi-invariant, or G-quasi-invariant if no confusion arises, if

$$m(T_gA)=0\Longleftrightarrow m(A)=0\quad\text{for every }g\in G\quad\text{and }A\in\mathscr{B},$$

that is $m\circ g\sim m$ for every $g\in G$. A quasi-invariant G-action on $(X,\mathscr{B},m)$ (or the measure m) is called *ergodic* if for every G-invariant set $A\in\mathscr{B}$ (i.e. $T_gA=A$ mod

m for every $g\in G$) we have $m(A)=0$ or $m(A^c)=0$. A measure m on $(X,\mathscr{B})$ is said to be T-invariant, or G-invariant if no confusion arises, if

$$m(T_gA)=m(A)\quad\text{for every } g\in G\quad\text{and}\quad A\in\mathscr{B},$$

that is $m\circ g=m$ for every $g\in G$. Recall that a measure m on $(X,\mathscr{B})$ is called *locally finite* if every point in X has a neighborhood of finite measure (notice that if (X,d) is locally compact then m is locally finite iff $m(K)<+\infty$ for each compact $K\subset X$). We will denote by $\mathscr{M}_\sigma(X,T)$, $\mathscr{L}\mathscr{F}(X,T)$ and $\mathscr{F}(X,T)$ the sets of T-invariant measures on $(X,\mathscr{B})$ that are σ-finite, locally finite and finite respectively. By $\mathscr{M}^e_\sigma(X,T)$, $\mathscr{L}\mathscr{F}^e(X,T)$ and $\mathscr{F}^e(X,T)$ we will denote subsets of respective set consisting of ergodic measures.

Let $(X,\mathscr{B})$ and $(Y,\mathscr{C})$ be standard Borel spaces. Let G be a Polish Abelian locally compact group which acts on $(X,\mathscr{B})$ and $(Y,\mathscr{C})$ in a Borel way. Suppose that $\pi:(X,\mathscr{B})\to(Y,\mathscr{C})$ is a Borel factor (G-equivariant) map, i.e.

$$\pi(gx)=g\pi(x)$$

for every $x\in X$ and $g\in G$. Assume that $m\in\mathscr{M}_\sigma(X,T)$. Let μ be a probability measure on $(X,\mathscr{B})$ which is equivalent to m ($\mu\sim m$) and such that $f:=\frac{d\mu}{dm}\in L^1(X,\mathscr{B},m)$ is a Borel function with $f(x)>0$ for all $x\in X$. By the G-invariance of m we have

$$\frac{d\mu\circ g}{d\mu}(x)=\frac{f(gx)}{f(x)}$$

for μ–a.e. $x\in X$ and for every $g\in G$.

Let $\rho:=\pi_*(\mu)$, i.e. $\rho(A)=\mu(\pi^{-1}A)$ for every $A\in\mathscr{C}$. Then there exist $Y_0\in\mathscr{C}$ with $\rho(Y_0)=1$ and a measurable map $Y_0\ni y\mapsto\mu_y\in\mathscr{P}(X,\mathscr{B})$ ($\mathscr{P}(X,\mathscr{B})$ is the space of probability measures on $(X,\mathscr{B})$) such that $\mu_y(\pi^{-1}\{y\})=1$ for all $y\in Y_0$ and

$$\int_X h(x)\,d\mu(x)=\int_Y\left(\int_X h(x)\,d\mu_y(x)\right)d\rho(y)$$

for every $h\in L^1(X,\mathscr{B},\mu)$ (see e.g. [9]). For every $y\in Y_0$ let m_y denote the measure on $(X,\mathscr{B})$ given by

$$m_y(A)=\int_A\frac{1}{f(x)}\,d\mu_y(x)\quad\text{for } A\in\mathscr{B}.$$

Then

$$m(A)=\int_Y m_y(A)\,d\rho(y)\quad\text{for every } A\in\mathscr{B}.$$

Notice that m_y is σ-finite for ρ–a.e. $y\in Y$. Moreover if m is additionally locally finite then m_y is locally finite as well for ρ–a.e. $y\in Y$ (it is a consequence of the fact that the topology on X has a countable basis).

We will now show that $\rho\circ g\sim\rho$ and $\mu_{gy}\circ g\sim\mu_y$ for ρ–a.e. $y\in Y$ and for every $g\in G$, moreover

$$\frac{d\rho\circ g}{d\rho}(y)=\int_X\frac{f(gx)}{f(x)}\,d\mu_y(x)$$

and

$$\frac{d\mu_{gy} \circ g}{d\mu_y} = \frac{f \circ g}{f} \Big/ \frac{d\rho \circ g}{d\rho}(y)$$

for ρ–a.e. $y \in Y$ and for every $g \in G$. Indeed, suppose that $h : (X, \mathscr{B}) \to \mathbb{R}$ and $k : (Y, \mathscr{C}) \to \mathbb{R}$ are bounded Borel functions. Then

$$\begin{aligned}\int_X k(g^{-1}\pi(x))h(g^{-1}x)\, d\mu(x) &= \int_Y k(g^{-1}y)\left(\int_X h(g^{-1}x)\, d\mu_y(x)\right) d\rho(y)\\ &= \int_Y k(y)\left(\int_X h(x)\, d(\mu_{gy} \circ g)(x)\right) d(\rho \circ g)(y).\end{aligned}$$

On the other side

$$\begin{aligned}\int_X k(g^{-1}\pi(x))h(g^{-1}x)\, d\mu(x) &= \int_X k(\pi(x))h(x)\, d(\mu \circ g)(x)\\ &= \int_X k(\pi(x))h(x)\frac{f(gx)}{f(x)}\, d\mu(x)\\ &= \int_Y k(y)\left(\int_X h(x)\frac{f(gx)}{f(x)}\, d\mu_y(x)\right) d\rho(y).\end{aligned}$$

Letting $h = 1$ we obtain

$$\int_Y k(y)\, d(\rho \circ g)(y) = \int_Y k(y)\left(\int_X \frac{f(gx)}{f(x)}\, d\mu_y(x)\right) d\rho(y)$$

for every bounded Borel function $k : (Y, \mathscr{C}) \to \mathbb{R}$. It follows that $\rho \circ g \sim \rho$ and

$$\frac{d\rho \circ g}{d\rho}(y) = \int_X \frac{f(gx)}{f(x)}\, d\mu_y(x)$$

for ρ–a.e. $y \in Y$ and for all $g \in G$. Therefore ρ is a G-quasi-invariant measure on $(Y, \mathscr{C})$. Moreover,

$$\begin{aligned}&\int_Y k(y)\left(\int_X h(x)\frac{f(gx)}{f(x)}\, d\mu_y(x)\right) d\rho(y)\\ &\quad= \int_Y k(y)\left(\int_X h(x)\, d(\mu_{gy} \circ g)(x)\right) d(\rho \circ g)(y)\\ &\quad= \int_Y k(y)\left(\int_X h(x)\frac{d\rho \circ g}{d\rho}(y)\, d(\mu_{gy} \circ g)(x)\right) d\rho(y).\end{aligned}$$

It follows that

$$\frac{d(\mu_{gy} \circ g)}{d\mu_y} = \frac{f \circ g}{f} \Big/ \frac{d(\rho \circ g)}{d\rho}(y) \tag{1}$$

for all $g \in G$ and for ρ–a.e. $y \in Y$. However by replacing the Radon-Nikodym cocycle $(g, y) \mapsto \frac{d(\rho \circ g)}{d\rho}(y)$ by a strict cocycle and proceeding as in Appendix B

[35] we obtain that (1) holds for a.e. $y \in Y$ and for all $g \in G$. Hence

$$\frac{d\rho \circ g}{d\rho}(y) \cdot (m_{gy} \circ g) = m_y \tag{2}$$

for ρ–a.e. $y \in Y$ and for all $g \in G$.

Now let us consider a particular case where $G = G_1 \oplus G_2$ is the direct sum of Polish Abelian locally compact group G_1 and G_2. Since G_1 and G_2 can be treated as subgroups of G they yield Borel subactions of G_1 and G_2 (on $(X, \mathscr{B})$ and $(Y, \mathscr{C})$) which are commuting.

Suppose that the group G_2 acts on $(Y, \mathscr{C})$ as the identity, i.e. $g_2 y = y$ for all $g_2 \in G_2$ and $y \in Y$. Since π is a G_2-equivariant map, $g_2(X_y) = X_y$ for every $g_2 \in G_2$ and $y \in Y$, where $X_y = \pi^{-1}(\{y\})$. Then from (2) we have

$$m_y \circ g_2 = m_y \tag{3}$$

for ρ–a.e. $y \in Y$ and for every $g_2 \in G_2$. Therefore for ρ–a.e. $y \in Y$ we can consider a measure-preserving Borel action of the group G_2 on $(X_y, \mathscr{B}(X_y), m_y)$ and a quasi-invariant Borel action of the group G_1 on $(Y, \mathscr{C}, \rho)$.

Lemma 4. *If the G-action on $(X, \mathscr{B}, m)$ is ergodic then the quasi-invariant G_1-action on $(Y, \mathscr{C}, \rho)$ is ergodic as well.*

Proof. Let us consider the G-action on $(Y, \mathscr{C}, \rho)$. Since this action is a factor (in the non-singular framework) of the G-action on $(X, \mathscr{B}, m)$, it is ergodic. Moreover, $(g_1, g_2)y = g_1 y$ for all $g_1 \in G_1$, $g_2 \in G_2$. Suppose that $A \in \mathscr{C}$ is a G_1-invariant subset. Of course, A must be also G-invariant and consequently $\rho(A) = 0$ or $\rho(A^c) = 0$. □

Let (X, d) be a Polish metric space and let $(X, \mathscr{B})$ be its standard Borel space. Let T_1 and T_2 be Borel actions on $(X, \mathscr{B})$ of Polish Abelian locally compact groups G_1 and G_2 respectively. Suppose that the actions T_1 and T_2 commute and the G_2-action T_2 is free and of type I, i.e. there exists a Borel subset $Y \in \mathscr{B}$ such that for every $x \in X$ there exists a unique $g_2 \in G_2$ such that $g_2 x \in Y$. The set Y is said to be a *fundamental domain* for the action T_2. Then $\{g_2 Y : g_2 \in G_2\}$ is a Borel partition of X. Let $G = G_1 \oplus G_2$. The actions T_1 and T_2 determine the action $T = T_1 \oplus T_2$ of the group G on $(X, \mathscr{B})$ by $T_{(g_1, g_2)} = (T_1)_{g_1} \circ (T_2)_{g_2}$ for $(g_1, g_2) \in G$. We will always consider Y with the topology induced by the metric space (X, d). Thus $(Y, \mathscr{B}_Y)$ is a standard Borel space. Then $\Phi : (Y \times G_2, \mathscr{B}_Y \otimes \mathscr{B}_{G_2}) \to (X, \mathscr{B})$ given by $\Phi(y, g_2) = g_2 y$ establishes a Borel isomorphism.

Denote by $p_1 : Y \times G_2 \to Y$ and $p_2 : Y \times G_2 \to G_2$ the projection maps. Let $\pi : (X, \mathscr{B}) \to (Y, \mathscr{B}_Y)$ and $\zeta : (X, \mathscr{B}) \to (G_2, \mathscr{B}_{G_2})$ be given by $\pi = p_1 \circ \Phi^{-1}$ and $\zeta = p_2 \circ \Phi^{-1}$. Then $\pi(x) = y$ iff there exists $g_2 \in G_2$ such that $g_2 x = y$. This map determines a new Borel G-action on $(Y, \mathscr{B}_Y)$ given by $gy = \pi(gx)$ if $y = \pi(x)$. It is easy to see that this action is well defined and $g_2 y = y$ for any $g \in G_2$. Of course, the map $\pi : (X, \mathscr{B}) \to (Y, \mathscr{B}_Y)$ is G-equivariant. The restriction of this action to the group G_1 we will denote by T_1/T_2. Then for every $y \in Y$ and $g_1 \in G_1$ there exists a unique element $g_2 \in G_2$ such that

$$(T_1/T_2)_{g_1} y = (T_2)_{g_2} (T_1)_{g_1} y. \tag{4}$$

Moreover the G-action T on $(X,\mathscr{B})$ is Borel isomorphic (via Φ) to the G-action on $(Y\times G_2,\mathscr{B}_Y\otimes\mathscr{B}_{G_2})$ given by

$$(g_1,g_2)(y,g_2') = ((T_1/T_2)_{g_1}y, g_2\cdot g_2'\cdot\zeta((T_1)_{g_1}y)). \tag{5}$$

Then $p_1 : Y\times G_2\to Y$ is G-equivariant map and the fiber over $y\in Y$ equals

$$p_1^{-1}\{y\} = \{y\}\times G_2\simeq G_2.$$

Of course, the G_2-subaction acts inside each fiber. Moreover, since $\zeta(y)=0$ for every $y\in Y$, the G_2-subaction on each fiber is topologically conjugate to the action by translations G_2.

Suppose that m is a σ-finite $T_1\oplus T_2$-invariant measure on $(X,\mathscr{B})$. Then $\bar{m} = m\circ\Phi$ is a G-invariant σ-finite measure on $(Y\times G_2,\mathscr{B}_Y\otimes\mathscr{B}_{G_2})$. Applying now the reasoning preceding Lemma 4 for the measure $\bar{m}$ and the G-equivariant map $p_1 : Y\times G_2\to Y$, and using the identification of each fiber $p_1^{-1}\{y\}$ with G_2 we obtain

$$\bar{m}(A_1\times A_2) = \int_{A_1}\bar{m}_y(A_2)\,d\rho(y)\quad\text{for all}\quad A_1\in\mathscr{B}_Y, A_2\in\mathscr{B}_{G_2},$$

where ρ is a probability measure on $(Y,\mathscr{B}_Y)$ and $\{\bar{m}_y : y\in Y_0\}$ ($Y_0\in\mathscr{B}_Y$ and $\rho(Y_0)=1$) is a family of σ-finite measures on $(G_2,\mathscr{B}_{G_2})$ which are invariant under all translations on the group G_2. It was proved in [13] (see Remark 7, p. 265) such measures are necessarily multiples of a fixed Haar measure λ_{G_2} on G_2. Then there exists a measurable function $c : (Y,\mathscr{B}_Y,\rho)\to\mathbb{R}^+$ such that

$$\bar{m}_y = c(y)\lambda_{G_2}\quad\text{for } \rho\text{−a.e. } y\in Y.$$

Then from (2) we have

$$\bar{m}_y = \frac{d\rho\circ g}{d\rho}(y)\,\bar{m}_{gy}\circ g = \frac{d\rho\circ g}{d\rho}(y)\frac{c(gy)}{c(y)}\,\bar{m}_y,$$

and hence

$$\frac{d\rho\circ g}{d\rho}(y)\frac{c(gy)}{c(y)} = 1\quad\text{for } \rho\text{−a.e. } y\in Y\quad\text{and for all } g\in G.$$

Let ν be a measure on $(Y,\mathscr{B}_Y)$ given by

$$\nu(A) = \int_A c(y)\,d\rho(y)\quad\text{for } A\in\mathscr{B}_Y.$$

Then ν is σ-finite and

$$\begin{aligned}\nu(g^{-1}A) &= \int_{g^{-1}A}c(y)\,d\rho(y) = \int_A c(gy)\,d\rho\circ g(y)\\ &= \int_A c(gy)\frac{d\rho\circ g}{d\rho}(y)\,d\rho(y) = \int_A c(y)\,d\rho(y) = \nu(A)\end{aligned}$$

for every $g\in G_1$ and $A\in\mathscr{B}_Y$. It follows that T_1/T_2 is a measure-preserving G_1-action on $(Y,\mathscr{B}_Y,\nu)$. Moreover

$$\bar{m}(A_1\times A_2) = \int_{A_1}\bar{m}_y(A_2)\,d\rho(y) = \lambda_{G_2}(A_2)\int_{A_1}c(y)\,d\rho(y) = \nu(A_1)\lambda_{G_2}(A_2)$$

for all $A_1\in\mathscr{B}_Y, A_2\in\mathscr{B}_{G_2}$, whence $\bar{m} = \nu\otimes\lambda_{G_2}$.

On the other hand suppose ν is a σ-finite T_1/T_2-invariant measure on $(Y,\mathscr{B}_Y)$. Then $m=(\nu\otimes\lambda_{G_2})\circ\Phi^{-1}$ is a $T_1\oplus T_2$-invariant σ-finite measure on $(X,\mathscr{B})$.

Let us denote by $\Lambda:\mathscr{M}_\sigma(Y,T_1/T_2)\to\mathscr{M}_\sigma(X,T_1\oplus T_2)$ the map

$$\Lambda(\nu)=(\nu\otimes\lambda_{G_2})\circ\Phi^{-1}. \tag{6}$$

Then Λ is an affine bijection. Moreover, for every $\nu\in\mathscr{M}_\sigma(Y,T_1/T_2)$ and for every $h\in L^1(X,\Lambda(\nu))$ we have

$$\int_X h(x)\,d\Lambda(\nu)(x)=\int_{G_2}\int_Y h((T_2)_{g_2}y)\,d\nu(y)\,d\lambda_{G_2}(g_2).$$

On the other hand for every $m\in\mathscr{M}_\sigma(X,T_1\oplus T_2)$, $h_1\in L^1(Y,\Lambda^{-1}(m))$ and $h_2\in L^1(G_2,\lambda_{G_2})$ we have

$$\int_X h_1(\pi(x))h_2(\zeta(x))\,dm(x)=\int_Y h_1(y)\,d(\Lambda^{-1}(m))(y)\int_{G_2} h_2(g_2)\,d\lambda_{G_2}(g_2). \tag{7}$$

Remark 2. In particular, if we assume that G_2 is a countable group and let $\lambda_{G_2}(C)=\#C$ $(C\subset G_2)$ then

$$\int_X h(x)\,d\Lambda(\nu)(x)=\sum_{g_2\in G_2}\int_Y h((T_2)_{g_2}y)\,d\nu(y) \tag{8}$$

and taking $h_1=\chi_A$ and $h_2=\chi_{\{0\}}$ in (7) we obtain

$$\Lambda^{-1}(m)(A)=m(A)\quad\text{for every } A\in\mathscr{B}_Y. \tag{9}$$

Lemma 5. $\Lambda(\mathscr{M}^e_\sigma(Y,T_1/T_2))=\mathscr{M}^e_\sigma(X,T_1\oplus T_2)$.

Proof. From Lemma 4 we have $\Lambda(\mathscr{M}^e_\sigma(Y,T_1/T_2))\supset\mathscr{M}^e_\sigma(X,T_1\oplus T_2)$. Assume that $\nu\in\mathscr{M}^e_\sigma(Y,T_1/T_2)$. It suffices to show that $\nu\otimes\lambda_{G_2}$ is an ergodic measure for the $G_1\oplus G_2$-action T on $Y\times G_2$ given by (5). Suppose that $A\in\mathscr{B}_Y\otimes\mathscr{B}_{G_2}$ is a $G_1\oplus G_2$-invariant subset. Let $A_y=\{g_2\in G_2:(y,g_2)\in A\}$ for any $y\in Y$. By the Fubini Theorem, $A_y\in\mathscr{B}_{G_2}$ for any $y\in Y$ and the function

$$Y\ni y\mapsto\lambda_{G_2}(A_y)\in\mathbb{R}^+\cup\{+\infty\}$$

is Borel. Moreover $g_2A_y=A_y$ mod λ_{G_2} for ν–a.e. $y\in Y$ and for all $g_2\in G_2$. Since the G_2-subaction on each fiber is transitive (in the algebraic sense), either $\lambda_{G_2}(A_y)=0$ or $\lambda_{G_2}(A^c_y)=0$ for ν–a.e. $y\in Y$. Let $B=\{y\in Y:\lambda_{G_2}(A_y)=0\}$. Since $(T_{g_1}A)_{(T_1/T_2)_{g_1}y}=\zeta((T_1/T_2)_{g_1}y)\cdot A_y$ for all $y\in Y$ and $g_1\in G_1$, the set $B\in\mathscr{B}_Y$ is T_1/T_2-invariant. By the ergodicity of the T_1/T_2-action on $(Y,\mathscr{B}_Y,\nu)$, either $\nu(B)=0$ or $\nu(B^c)=0$. It follows that either $\nu\otimes\lambda_{G_2}(A)=0$ or $\nu\otimes\lambda_{G_2}(A^c)=0$; consequently $\nu\otimes\lambda_{G_2}$ is an ergodic measure. □

Lemma 6. *If $\Phi:Y\times G_2\to X$ is a homeomorphism then*

$$\Lambda(\mathscr{LF}(Y,T_1/T_2))=\mathscr{LF}(X,T_1\oplus T_2).$$

Proof. Since λ_{G_2} is locally finite, the result follows immediately from the fact that ν is locally finite iff $\nu\otimes\lambda_{G_2}$ is locally finite. □

Lemma 7. *Assume that G_2 is a countable discrete group,*

$$0<\delta := \min\{d(g_2x, g_2'x) : x\in X, g_2, g_2'\in G_2, g_2 \neq g_2'\} \tag{10}$$

and the closure of Y in X is compact. Then $\Lambda(\mathscr{F}(Y,T_1/T_2)) = \mathscr{L}\mathscr{F}(X,T_1\oplus T_2)$.

Proof. Suppose that $m\in\mathscr{L}\mathscr{F}(X,T_1\oplus T_2)$. Then from (9) we have

$$\Lambda^{-1}(m)(Y) = m(Y) \leqslant m(\overline{Y}) < +\infty,$$

and hence $\Lambda^{-1}(m)\in\mathscr{F}(Y,T_1/T_2)$.

Now assume that $\nu\in\mathscr{F}(Y,T_1/T_2)$. Take $x\in X$ and let $U = \{x'\in X : d(x,x') < \delta/2\}$. For every $g_2\in G_2$ let $U_{g_2} = \{y\in Y : (T_2)_{g_2}y\in U\}$. By assumption, U_{g_2}, $g_2\in G_2$ are pairwise disjoint. Therefore from (8) we have

$$\begin{aligned}\Lambda(\nu)(U) &= \sum_{g_2\,\in\,G_2}\int_Y \chi_U((T_2)_{g_2}y)\,d\nu(y) = \sum_{g_2\,\in\,G_2}\nu(U_{g_2})\\ &= \nu(\bigcup_{g_2\,\in\,G_2} U_{g_2}) \leqslant \nu(Y) < \infty\end{aligned}$$

and hence $\Lambda(\nu)\in\mathscr{L}\mathscr{F}(X,T_1\oplus T_2)$. □

3. Special flow

Let (X,d) be a Polish metric space and let $\mathscr{B} = \mathscr{B}_X$ stand for the σ-algebra of Borel subsets of X. Let $T\in\mathrm{Aut}(X,\mathscr{B})$. Denote by λ Lebesgue measure on $\mathbb{R}$ and by $\mathscr{B}_\mathbb{R}$ the σ-algebra of Borel sets of $\mathbb{R}$. Assume that $f : X\to\mathbb{R}$ is an integrable positive Borel function which is bounded away from zero. Let $X^f = \{(x,t)\in X\times\mathbb{R} : 0\leqslant t<f(x)\}$. The set X^f will be always considered with the topology induced by the product topology on $X\times\mathbb{R}$. Denote by $\mathscr{B}^f$ the σ-algebra of Borel sets on X^f. The *special flow* $T^f = ((T^f)_t)_{t\in\mathbb{R}}$ built from T and f is defined on $(X^f,\mathscr{B}^f)$. Under the action of the special flow each point (x,r) in X^f moves up along $\{(x,s) : 0\leqslant s<f(x)\}$ at the unit speed, and we identify the point $(x,f(x))$ with $(Tx,0)$ (see e.g. [4], Chapter 11). If μ is a T-invariant measure on $(X,\mathscr{B})$ then the flow T^f preserves the restriction μ^f of the product measure $\mu\otimes\lambda$ of $X\times\mathbb{R}$ to X^f. Moreover, μ^f is ergodic iff μ is ergodic.

Given $m\in\mathbb{Z}$ we put

$$f^{(m)}(x) = \begin{cases} f(x)+f(Tx)+\ldots+f(T^{m-1}x) & \text{if} \quad m>0\\ 0 & \text{if} \quad m=0\\ -(f(T^mx)+\ldots+f(T^{-1}x) & \text{if} \quad m<0.\end{cases}$$

We will now represent the action T^f as a quotient action of the form (4), where T_1 is an $\mathbb{R}$-action σ (defined below) and T_2 is a $\mathbb{Z}$-action generated by the skew product $T_{-f} : (X\times\mathbb{R},\mathscr{B}\otimes\mathscr{B}_\mathbb{R})\to(X\times\mathbb{R},\mathscr{B}\otimes\mathscr{B}_\mathbb{R})$ given by

$$T_{-f}(x,r) = (Tx, r-f(x)).$$

The $\mathbb{Z}$-action generated by T_{-f} is given by

$$\mathbb{Z}\ni k\mapsto (T_{-f})^k\in\mathrm{Aut}(X\times\mathbb{R},\mathscr{B}\otimes\mathscr{B}_\mathbb{R}).$$

Notice that $(T_{-f})^k(x,r) = (T^kx, r - f^{(k)}(x))$ for each $k \in \mathbb{Z}$. Let $\sigma = (\sigma_t)_{t\in\mathbb{R}}$ stand for the $\mathbb{R}$-action on $(X \times \mathbb{R}, \mathscr{B} \otimes \mathscr{B}_{\mathbb{R}})$ given by

$$\sigma_t(x,r) = (x, r+t).$$

Notice that the $\mathbb{R}$-action σ commutes with the $\mathbb{Z}$-action T_{-f}. Now the $\mathbb{Z}$-action T_{-f} is free and of type I and X^f is a fundamental domain of this action. Let us consider the $\mathbb{R}$-action σ/T_{-f} on X^f. Then $(\sigma/T_{-f})_t = \pi \circ \sigma_t$, where $\pi : X \times \mathbb{R} \to X^f$ is given by

$$\pi(x,r) = (T_{-f})^n(x,r) \quad \text{if} \quad f^{(n)}(x) \leqslant r < f^{(n+1)}(x). \tag{11}$$

Therefore the $\mathbb{R}$-action σ/T_{-f} coincides with the action of the special flow T^f.

Remark 3. Now using results from Section 2 we can prove a well known result which says that if X is compact and f is bounded then T is uniquely ergodic iff T^f is uniquely ergodic. Indeed, notice that σ is a free action of type I and $Y = X \times \{0\}$ its fundamental domain. Moreover, the action T_{-f}/σ on Y is isomorphic via a homeomorphism to the action generated by the automorphism $T : X \to X$. Since f is bounded away from zero, by Lemmas 6 and 7, there exists an affine one-to-one correspondence between $\mathscr{F}(X^f, T^f)$ and $\mathscr{L}\mathscr{F}(X,T)$ which is equal to $\mathscr{F}(X,T)$ because X is compact. This gives our claim.

Remark 4. If $T : (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ is ergodic then a special flow T^f on (X^f, μ^f) is weakly mixing iff for every $r \in \mathbb{R}\setminus\{0\}$ and $\gamma \in \mathbb{C}$ with $|\gamma| = 1$ the equation

$$g(Tx) = \gamma e^{2\pi i r f(x)} g(x)$$

has no measurable solution $g : X \to \mathbb{T}$.

3.1. Continuous centralizer of topological special flows. Suppose that (X,d) is a compact connected topological manifold. Let $T : X \to X$ be a homeomorphism and let $f : X \to \mathbb{R}$ be a positive continuous function. Let us consider the metric $\bar{d}$ on X^f given by

$$\bar{d}((x,t),(y,s)) = \min\{d(x,y) + |t-s|, d(Tx,y) + f(x) - t + s, d(x,Ty) + f(y) - s + t\}.$$

Then $(X^f, \bar{d})$ is a compact manifold and T^f is a topological flow on $(X^f, \bar{d})$. Let us denote by $C_c(T^f)$ the continuous centralizer of T^f, i.e. the group of homeomorphisms of $(X^f, \bar{d})$ which commute with the action of the flow T^f. Let $\pi : X \times \mathbb{R} \to X^f$ be given by (11). Then π is a covering map ($X \times \mathbb{R}$ is considered with the product topology). Denote by $C_{lc}(T^f)$ the set of homeomorphisms from $C_c(T^f)$ which can be lifted to homeomorphisms of $X \times \mathbb{R}$. As it was proved in [18] each such homeomorphism is of the form

$$(x,r) \mapsto \pi(Sx, r - g(x)),$$

where S is a homeomorphism of X which commutes with T and $g : X \to \mathbb{R}$ is a continuous function satisfying

$$g(Tx) - g(x) = f(Sx) - f(x) \quad \text{or equivalently} \quad T_{-f} \circ S_{-g} = S_{-g} \circ T_{-f}.$$

Moreover, if T is a minimal rotation on a finite dimension torus then $C_c(T^f) = C_{lc}(T^f)$ (see Corollary 3.8 in [18]).

4. Joinings of special flows

Let (X,d_1) and (Y,d_2) be compact metric spaces. Denote by $\mathscr{B}$ and $\mathscr{C}$ the σ-algebras of all Borel subsets of X and Y respectively. Let $T\in\mathrm{Aut}(X,\mathscr{B})$ and $S\in\mathrm{Aut}(Y,\mathscr{C})$. Let $f:X\to\mathbb{R}$ and $g:Y\to\mathbb{R}$ be positive bounded away from zero and bounded Borel functions. Let T^f and S^g stand for special flows acting on X^f and Y^g respectively. Let us consider the product flow $(T^f_t\times S^g_t)_{t\in\mathbb{R}}$ on $(X^f\times Y^g, \mathscr{B}^f\otimes\mathscr{C}^f)$. Moreover, let us consider the Borel flow $\bar\sigma$ on $X\times\mathbb{R}\times Y\times\mathbb{R}$ (this space is considered with the product metric) given by

$$\bar\sigma_t(x,r_1,y,r_2)=(x,r_1+t,y,r_2+t)$$

and two skew product $\mathbb{Z}$-actions $\overline{T_{-f}}$ and $\overline{S_{-g}}$ on $X\times\mathbb{R}\times Y\times\mathbb{R}$ given by

$$\overline{T_{-f}}^k(x,r_1,y,r_2)=(T^kx,r_1-f^{(k)}(x),y,r_2),$$
$$\overline{S_{-g}}^k(x,r_1,y,r_2)=(x,r_1,S^ky,r_2-g^{(k)}(y)).$$

Of course, the actions $\bar\sigma$, $\overline{T_{-f}}$ and $\overline{S_{-g}}$ commute. Let us consider the $\mathbb{Z}^2$-action $\overline{T_{-f}}\oplus\overline{S_{-g}}$, i.e.

$$(\overline{T_{-f}}\oplus\overline{S_{-g}})_{(k_1,k_2)}=\overline{S_{-g}}^{k_1}\circ\overline{T_{-f}}^{k_2}.$$

This action is free and of type I; moreover, the set $X^f\times Y^g$ is its fundamental domain. Then the $\mathbb{R}$-action $\bar\sigma/\overline{T_{-f}}\oplus\overline{S_{-g}}$ on $X^f\times Y^g$ coincides with the product $\mathbb{R}$-action $(T^f_t\times S^g_t)_{t\in\mathbb{R}}$.

Let us consider the $\mathbb{R}\times\mathbb{Z}^2$-action $\overline{\sigma}\oplus\overline{T_{-f}}\oplus\overline{S_{-g}}$ on $X\times\mathbb{R}\times Y\times\mathbb{R}$, i.e.

$$(\overline{\sigma}\oplus\overline{T_{-f}}\oplus\overline{S_{-g}})_{(t,k_1,k_2)}=\overline{S_{-g}}^{k_1}\circ\overline{T_{-f}}^{k_2}\circ\sigma_t.$$

Let

$$\Lambda_1:\mathscr{M}_\sigma(X^f\times Y^g,\overline{\sigma}/\overline{T_{-f}}\oplus\overline{S_{-g}})\to\mathscr{M}_\sigma(X\times\mathbb{R}\times Y\times\mathbb{R},\overline{\sigma}\oplus\overline{T_{-f}}\oplus\overline{S_{-g}})$$

be the affine bijection determined by (6). Then if $\nu\in\mathscr{M}_\sigma(X^f\times Y^g,\bar\sigma/\overline{T_{-f}}\oplus\overline{S_{-g}})$ then, by (8), we have

$$\int_{X\times\mathbb{R}\times Y\times\mathbb{R}}h(x,r_1,y,r_2)\,d\Lambda_1(\nu)(x,r_1,y,r_2)$$
$$=\sum_{m,n\in\mathbb{Z}}\int_{X^f\times Y^g}h((T)^m_{-f}(x,r_1),(S_{-g})^n(y,r_2))\,d\nu(x,r_1,y,r_2)\qquad(12)$$

for every $h\in L^1(X\times\mathbb{R}\times Y\times\mathbb{R},\Lambda_1(\nu))$. Since f and g are bounded away from zero, the $\mathbb{Z}^2$-action $\overline{T_{-f}}\oplus\overline{S_{-g}}$ satisfies (10). Since f and g are bounded, the closure of $X^f\times Y^g$ in $X\times\mathbb{R}\times Y\times\mathbb{R}$ is compact. Therefore, by Lemma 7, we have

$$\Lambda_1(\mathscr{F}(X^f\times Y^g,(T^f_t\times S^g_t)_{t\in\mathbb{R}}))=\mathscr{L}\mathscr{F}(X\times\mathbb{R}\times Y\times\mathbb{R},\bar\sigma\oplus\overline{T_{-f}}\oplus\overline{S_{-g}}).$$

On the other side the $\mathbb{R}$-action $\bar\sigma$ on $X\times\mathbb{R}\times Y\times\mathbb{R}$ is also free and of type I and the set $W=\{(x,r,y,0):x\in X,y\in Y,r\in\mathbb{R}\}$ is its fundamental domain. Then the $\mathbb{Z}^2$-action $\overline{T_{-f}} \oplus \overline{S_{-g}}/\bar{\sigma}$ acts on W in the following way

$$\begin{aligned}(\overline{T_{-f}} \oplus \overline{S_{-g}}/\bar{\sigma}_{(k_1,k_2)})(x,r,y,0) &= \bar{\sigma}_{g^{(k_2)}(y)}(T^{k_1}x, r - f^{(k_1)}(x), S^{k_2}y, -g^{(k_2)}(y)) \\ &= (T^{k_1}x, r + g^{(k_2)}(y) - f^{(k_1)}(x), S^{k_2}y, 0).\end{aligned}$$

The set W is homeomorphic to $X \times Y \times \mathbb{R}$; therefore we will identify them. Moreover the $\mathbb{Z}^2$-action $\overline{T_{-f}} \oplus \overline{S_{-g}}/\bar{\sigma}$ we will identify with the $\mathbb{Z}^2$-action $T_{-f} \star S_{-g}$ on $X \times Y \times \mathbb{R}$ given by

$$(T_{-f} \star S_{-g})_{(k_1,k_2)}(x,y,r) = (T^{k_1}x, S^{k_2}y, r + g^{(k_2)}(y) - f^{(k_1)}(x)).$$

Let

$$\Lambda_2 : \mathscr{M}_\sigma(W, \overline{(T_{-f}} \oplus \overline{S_{-g})}/\bar{\sigma}) \to \mathscr{M}_\sigma(X \times \mathbb{R} \times Y \times \mathbb{R}, \bar{\sigma} \oplus \overline{T_{-f}} \oplus \overline{S_{-g}})$$

be the affine bijection determined by (6). Of course, we will constantly identify $\mathscr{M}_\sigma(W, \overline{(T_{-f}} \oplus \overline{S_{-g})}/\bar{\sigma})$ with $\mathscr{M}_\sigma(X \times Y \times \mathbb{R}, T_{-f} \star S_{-g})$. Then if $\nu \in \mathscr{M}_\sigma(X \times Y \times \mathbb{R}, T_{-f} \star S_{-g})$ then, by (7), we have

$$\begin{aligned}&\int_{X \times Y \times \mathbb{R}} h_1(x,y,r)\, d\nu(x,y,r) \int_{\mathbb{R}} h_2(s)\, ds \\ &\quad = \int_{X \times \mathbb{R} \times Y \times \mathbb{R}} h_1(x,y,r-s) h_2(s)\, d(\Lambda_2(\nu))(x,r,y,s)\end{aligned} \tag{13}$$

for every $h_1 \in L^1(X \times Y \times \mathbb{R}, \nu)$ and $h_2 \in L^1(\mathbb{R}, \lambda_{\mathbb{R}})$. Since $\Phi : W \times \mathbb{R} \to X \times \mathbb{R} \times Y \times \mathbb{R}$, $\Phi(x,r,y,0,t) = (x, r+t, y, t)$ is a homeomorphism, by Lemma 6,

$$\Lambda_2(\mathscr{LF}(X \times Y \times \mathbb{R}, T_{-f} \star S_{-g})) = \mathscr{LF}(X \times \mathbb{R} \times Y \times \mathbb{R}, \bar{\sigma} \oplus \overline{T_{-f}} \oplus \overline{S_{-g}}).$$

From this and from Lemma 5 we obtain the following conclusion.

Corollary 8.

$$\Lambda_2^{-1} \circ \Lambda_1 : \mathscr{M}_\sigma(X^f \times Y^g, (T_t^f \times S_t^g)_{t \in \mathbb{R}}) \to \mathscr{M}_\sigma(X \times Y \times \mathbb{R}, T_{-f} \star S_{-g})$$

is an affine bijection such that

$$\Lambda_2^{-1} \circ \Lambda_1(\mathscr{F}(X^f \times Y^g, (T_t^f \times S_t^g)_{t \in \mathbb{R}})) = \mathscr{LF}(X \times Y \times \mathbb{R}, T_{-f} \star S_{-g})$$

and

$$\Lambda_2^{-1} \circ \Lambda_1(\mathscr{M}_\sigma^e(X^f \times Y^g, (T_t^f \times S_t^g)_{t \in \mathbb{R}})) = \mathscr{M}_\sigma^e(X \times Y \times \mathbb{R}, T_{-f} \star S_{-g}).$$

Remark 5. Suppose that $T \in \mathrm{Aut}(X, \mathscr{B})$ and $S \in \mathrm{Aut}(Y, \mathscr{C})$ are uniquely ergodic with invariant probability measures μ and ν respectively. Then special flows T^f and S^g are uniquely ergodic with invariant measures μ^f and ν^g respectively (see Remark 3). Therefore the set $\mathscr{F}(X^f \times Y^g, (T_t^f \times S_t^g)_{t \in \mathbb{R}})$ coincides with the cone of positive multiples of joinings between special flows T^f on (X^f, μ^f) and S^g on (Y^g, ν^g).

Suppose that ν is a σ-finite measure on $(X, \mathscr{B})$ that is T-invariant. Assume that $S \in \mathrm{Aut}(X, \mathscr{B})$ commutes with T (then $S_*\nu$ is also T-invariant) and $u : X \to \mathbb{R}$ is a Borel function such that

$$f(Sx) - f(x) = u(Tx) - u(x) \quad \text{for } \nu\text{--}a.e.\ x \in X. \tag{14}$$

Then

$$(T_{-f})^n \circ S_{-u}(x,r) = S_{-u} \circ (T_{-f})^n(x,r) \quad \text{for } \nu\text{-a.e. } x \in X \quad \text{and all } r \in \mathbb{R}.$$

Now we can define a Borel map $\widetilde{S_{-u}} : X^f \to X^f$ as the composition of $S_{-u} : X^f \to X \times \mathbb{R}$ and the projection $\pi : X \times \mathbb{R} \to X^f$ given by (11). Since the skew product $S_{-u} : X \times \mathbb{R} \to X \times \mathbb{R}$ commutes with the flow σ, we have

$$\widetilde{S_{-u}} \circ T_t^f(x,r) = T_t^f \circ \widetilde{S_{-u}}(x,r) \quad \text{for } \nu^f\text{-a.e. } (x,r) \in X^f \quad \text{and for all } t \in \mathbb{R}. \tag{15}$$

Remark 6. Notice that if ν is ergodic then u in (14) is determined up to an additive constant. Moreover, if $u_c(x) = u(x) + c$ (for some $c \in \mathbb{R}$) then $\widetilde{S_{-u_c}} = \widetilde{S_{-u}} \circ T^f_{-c}$.

The map $\widetilde{S_{-u}} : X^f \to X^f$ determines a σ-finite measure $\nu^f_{\widetilde{S_{-u}}}$ on $(X^f \times X^f, \mathscr{B}^f \otimes \mathscr{B}^f)$ by the formula

$$\nu^f_{\widetilde{S_{-u}}}(A \times B) = \nu^f(A \cap \widetilde{S_{-u}}^{-1} B)$$

for every $A, B \in \mathscr{B}^f$. From (15) we have $\nu^f_{\widetilde{S_{-u}}} \in \mathscr{M}_\sigma(X^f \times X^f, (T_t^f \times T_t^f)_{t \in \mathbb{R}})$ and

$$\int_{X^f \times X^f} h(x_1, r_1, x_2, r_2)\, d\nu^f_{\widetilde{S_{-u}}}(x_1, r_1, x_2, r_2) = \int_{X^f} h(x, r, \widetilde{S_{-u}}(x,r))\, d\nu^f(x,r)$$

for every $h \in L^1(X^f \times X^f, \nu^f_{\widetilde{S_{-u}}})$.

Lemma 9. *For every $h \in L^1(X \times X \times \mathbb{R}, \Lambda_2^{-1} \circ \Lambda_1(\nu^f_{\widetilde{S_{-u}}}))$ we have*

$$\begin{aligned}
&\int_{X \times X \times \mathbb{R}} h(x,y,r)\, d\left(\Lambda_2^{-1} \circ \Lambda_1\left(\nu^f_{\widetilde{S_{-u}}}\right)\right)(x,y,r) \\
&\quad = \sum_{n \in \mathbb{Z}} \int_X h(T^n x, Sx, u(x) - f^{(n)}(x))\, d\nu(x).
\end{aligned}$$

Proof. For every $h \in L^1(X \times \mathbb{R} \times X \times \mathbb{R}, \Lambda_1(\nu^f_{\widetilde{S_{-u}}}))$ from (12) we have

$$\begin{aligned}
&\int_{X \times \mathbb{R} \times X \times \mathbb{R}} h(x_1, r_1, x_2, r_2)\, d\Lambda_1(\nu^f_{\widetilde{S_{-u}}})(x_1, r_1, x_2, r_2) \\
&\quad = \sum_{m,n \in \mathbb{Z}} \int_{X^f \times X^f} h((T_{-f})^n(x_1, r_1), (T_{-f})^m(x_2, r_2))\, d\nu^f_{\widetilde{S_{-u}}}(x_1, r_1, x_2, r_2) \\
&\quad = \sum_{m,n \in \mathbb{Z}} \int_{X^f} h((T_{-f})^n(x,r), (T_{-f})^m \circ \widetilde{S_{-u}}(x,r))\, d\nu^f(x,r) \\
&\quad = \sum_{m,n \in \mathbb{Z}} \int_{X^f} h((T_{-f})^{m+n}(x,r), S_{-u} \circ (T_{-f})^m(x,r))\, d\nu(x)\, dr \\
&\quad = \sum_{m,n \in \mathbb{Z}} \int_{(T_{-f})^m X^f} h((T_{-f})^n(x,r), S_{-u}(x,r))\, d\nu(x)\, dr \\
&\quad = \sum_{n \in \mathbb{Z}} \int_{X \times \mathbb{R}} h((T_{-f})^n(x,r), S_{-u}(x,r))\, d\nu(x)\, dr.
\end{aligned}$$

Moreover, for every $h_1 \in L^1(X\times X\times\mathbb{R},\Lambda_2^{-1}\circ\Lambda_1(\nu^f_{\widetilde{S_{-u}}}))$ and $h_2\in L^1(\mathbb{R},\lambda_{\mathbb{R}})$ from (13) we have

$$\begin{aligned}
&\int_{X\times X\times\mathbb{R}} h_1(x_1,x_2,r)\,d\Lambda_2^{-1}\circ\Lambda_1(\nu^f_{\widetilde{S_{-u}}})(x_1,x_2,r)\int_{\mathbb{R}} h_2(s)\,ds\\
&\quad=\int_{X\times\mathbb{R}\times X\times\mathbb{R}} h_1(x_1,x_2,r-s)h_2(s)\,d\Lambda_1(\nu^f_{\widetilde{S_{-u}}})(x_1,r,x_2,s)\\
&\quad=\sum_{n\in\mathbb{Z}}\int_{X\times\mathbb{R}} h_1(T^nx,Sx,u(x)-f^{(n)}(x))h_2(s-f^{(n)}(x))\,d\nu(x)\,ds\\
&\quad=\sum_{n\in\mathbb{Z}}\int_X h_1(T^nx,Sx,u(x)-f^{(n)}(x))\,d\nu(x)\int_{\mathbb{R}} h_2(s)\,ds.
\end{aligned}$$

Therefore for every $h\in L^1(X\times X\times\mathbb{R},\Lambda_2^{-1}\circ\Lambda_1(\nu^f_{\widetilde{S_{-u}}}))$ we have

$$\int_{X\times X\times\mathbb{R}} h(x_1,x_2,r)\,d\Lambda_2^{-1}\circ\Lambda_1(\nu^f_{\widetilde{S_{-u}}})(x_1,x_2,r)=\sum_{n\in\mathbb{Z}}\int_X h(T^nx,Sx,u(x)-f^{(n)}(x))\,d\nu(x).$$

□

Remark 7. Assume that $\nu=\mu$ is a probability T-invariant measure, $S=Id$ and $u\equiv -t$ ($t\in\mathbb{R}$). Then $\widetilde{S_{-u}}=T^f_t$ and it follows that

$$\Lambda_2^{-1}\circ\Lambda_1(\mu^f_{T^f_t})(A)=\sum_{n\in\mathbb{Z}}\int_X 1_A(T^nx,x,-t-f^{(n)}(x))\,d\mu(x)$$

for any bounded Borel subset $A\subset\mathbb{T}^2\times\mathbb{R}$.

Remark 8. Notice also that

$$\Lambda_2^{-1}\circ\Lambda_1(\mu^f\otimes\mu^f)=\mu\otimes\mu\otimes\lambda_{\mathbb{R}}.$$

Indeed, for every $h\in L^1(X\times\mathbb{R}\times X\times\mathbb{R},\Lambda_1(\mu^f\otimes\mu^f))$ from (12) we have

$$\begin{aligned}
&\int_{X\times\mathbb{R}\times X\times\mathbb{R}} h(x_1,r_1,x_2,r_2)\,d\Lambda_1(\mu^f\otimes\mu^f)(x_1,r_1,x_2,r_2)\\
&\quad=\sum_{m,n\in\mathbb{Z}}\int_{X^f\times X^f} h((T_{-f})^m(x_1,r_1),(T_{-f})^n(x_2,r_2))\,d\mu^f(x_1,r_1)\,d\mu^f(x_2,r_2)\\
&\quad=\sum_{m,n\in\mathbb{Z}}\int_{(T_{-f})^mX^f\times(T_{-f})^nX^f} h(x_1,r_1,x_2,r_2)\,d\mu(x_1)\,dr_1\,d\mu(x_2)\,dr_2\\
&\quad=\int_{X\times\mathbb{R}\times X\times\mathbb{R}} h(x_1,r_1,x_2,r_2)\,d\mu(x_1)\,dr_1\,d\mu(x_2)\,dr_2.
\end{aligned}$$

Therefore $\Lambda_1(\mu^f\otimes\mu^f)=\mu\otimes\lambda_{\mathbb{R}}\otimes\mu\otimes\lambda_{\mathbb{R}}$. Furthermore, for every $h_1\in L^1(X\times X\times\mathbb{R},\Lambda_2^{-1}(\mu\otimes\lambda_{\mathbb{R}}\otimes\mu\otimes\lambda_{\mathbb{R}}))$ and $h_2\in L^1(\mathbb{R},\lambda_{\mathbb{R}})$ from (13) we have

$$\begin{aligned}
&\int_{X\times X\times\mathbb{R}} h_1(x_1,x_2,r)\,d\Lambda_2^{-1}(\mu\otimes\lambda_{\mathbb{R}}\otimes\mu\otimes\lambda_{\mathbb{R}})(x_1,x_2,r)\int_{\mathbb{R}} h_2(s)\,ds\\
&\quad=\int_{X\times\mathbb{R}\times X\times\mathbb{R}} h_1(x_1,x_2,r-s)h_2(s)\,d\mu(x_1)\,dr\,d\mu(x_2)\,ds\\
&\quad=\int_{X\times X\times\mathbb{R}} h_1(x_1,x_2,r)\,d\mu(x_1)\,d\mu(x_2)\,dr\int_{\mathbb{R}} h_2(s)\,ds.
\end{aligned}$$

Therefore $\Lambda_2^{-1}(\mu\otimes\lambda_{\mathbb{R}}\otimes\mu\otimes\lambda_{\mathbb{R}})=\mu\otimes\mu\otimes\lambda_{\mathbb{R}}$. □

5. Cocycles and skew products

Let T be a Borel action of a countable Abelian discrete group G on a standard Borel space $(X,\mathscr{B})$. Let H be a locally compact Abelian group. An *H-valued cocycle* over the action T is a Borel function $G\times X\ni(g,x)\to\varphi_g(x)\in H$ such that

$$\varphi_{g_1+g_2}(x)=\varphi_{g_1}(x)+\varphi_{g_2}(g_1x)\quad\text{for all } g_1,g_2\in G, x\in X.$$

If $G=\mathbb{Z}$ then the $\mathbb{Z}$-action T we will identify with the automorphism T_1 and every cocycle φ is determined by the function φ_1, and we will identify them as well.

Every H-valued cocycle over T determines a skew product Borel G-action T_φ on $X\times H$ given by

$$(T_\varphi)_g(x,h)=(T_gx,\varphi_g(x)+h).$$

Suppose that $\mu\in\mathscr{M}_\sigma(X,T)$. Then the product measure $\mu\otimes\lambda_H$ (λ_H is a fixed Haar measure on H) is invariant under the action of the skew product T_φ. Two cocycles φ, ψ over the action T are said to be *cohomologous* mod μ if there exists a Borel function $u:X\to H$ such that

$$\psi_g(x):=\varphi_g(x)+u(x)-u(T_gx)$$

for μ–a.e. $x\in X$ and for all $g\in G$. The function u is called the *transfer function*. Then the map

$$X\times H\ni(x,h)\mapsto\vartheta_u(x,h)=(x,h-u(x))\in X\times H$$

establishes an isomorphism between the measurable G-actions T_φ and T_ψ on $(X\times H,\mu\otimes\lambda_H)$. Cocycles which are cohomologous mod μ to the zero cocycle are called *coboundaries* mod μ.

Let θ be an $\mathbb{R}$-valued Borel cocycle over the G-action T. A finite measure ν on $(X,\mathscr{B})$ is called *(e^θ,T)-conformal* if $\nu\circ T_g\sim\nu$ and $d\nu\circ T_g/d\nu=e^{\theta_g}$ ν–a.e. for every $g\in G$.

Let φ be an H-valued cocycle over T and $\alpha:H\to\mathbb{R}$ be a continuous group homomorphism. Suppose that ν is an $(e^{\alpha\circ\varphi},T)$-conformal measure. Let m_α stand for the measure on $(X\times H,\mathscr{B}\otimes\mathscr{B}_H)$ given by

$$dm_\alpha(x,h):=e^{-\alpha(h)}\,d\nu(x)\,d\lambda_H(h).$$

Then m_α is a locally finite measure and it is T_φ-invariant. Such measures are called *Maharam measures* (see e.g. [1]).

For every $h\in H$ let $Q_h:X\times H\to X\times H$ stand for the map $Q_h(x,h')=(x,h'+h)$. Then $T_\varphi\circ Q_h=Q_h\circ T_\varphi$ for every $h\in H$. If m is an ergodic T_φ-invariant σ-finite measure on $(X\times H,\mathscr{B}\otimes\mathscr{B}_H)$ then the measure $m\circ Q_h$ is also an ergodic T_φ-invariant measure. Therefore either $m\circ Q_h\perp m$ or $m\circ Q_h=cm$ for some $c>0$. Then, following [2], define

$$\mathscr{R}_m:=\{h\in H: m\circ Q_h\sim m\}.$$

Let $u:X\to H$ be a Borel function. Let us consider the Borel cocycle φ^u over the action T given by

$$\varphi^u_g(x):=\varphi_g(x)+u(x)-u(T_gx)$$

for every $g\in G$. Then the measure $m\circ\vartheta_u^{-1}$ is σ-finite ergodic T_{φ^u}-invariant with $\mathscr{R}_{m\circ\vartheta_u^{-1}}=\mathscr{R}_m$.

Proposition 10 (see [2]). *For every ergodic T_φ-invariant locally finite Borel measure m on $X\times H$ the set $\mathscr{R}_m$ is a closed subgroup of H. Moreover, if $\mathscr{R}_m = H$ then m is a Maharam measure.*

Proposition 11 (see Theorem 2 in [32]). *Let $H=\mathbb{R}$ and let m be an ergodic T_φ-invariant locally finite Borel measure on $X\times\mathbb{R}$. Then there exist a Borel function $u:X\to\mathbb{R}$ and a Borel subset $A\subset X\times\mathbb{R}$ with $m(A^c)=0$ such that for every $x\in X$ if there exists $r\in\mathbb{R}$ with $(x,r)\in A$ then*

$$\varphi_g(x)+u(x)-u(T_gx)\in\mathscr{R}_m$$

for every $g\in G$.

Proposition 12 (see Lemma 8 in [32]). *Let $\mathscr{R}\subset H$ be a closed subgroup and let m be an ergodic T_φ-invariant locally finite Borel measure on $X\times H$. Suppose that there exists a Borel function $u:X\to H$ and a Borel subset $A\subset X\times H$ with $m(A^c)=0$ such that for every $x\in X$ if there exists $h\in H$ with $(x,h)\in A$ then*

$$\varphi^u_g(x)=\varphi_g(x)+u(x)-u(T_gx)\in\mathscr{R}$$

for every $g\in G$. Then there exists $c\in H$ such that $m\circ\vartheta^{-1}_{u+c}$ is an ergodic T_{φ^u}-invariant σ-finite measure on $(X\times\mathscr{R},\mathscr{B}\otimes\mathscr{B}_{\mathscr{R}})$, and $\mathscr{R}_m=\mathscr{R}_{m\circ\vartheta^{-1}_{u+c}}\subset\mathscr{R}$. If u is bounded then $m\circ\vartheta^{-1}_{u+c}$ is locally finite.

5.1. Cocycles over irrational rotations. We denote by $\mathbb{T}$ the circle group $\mathbb{R}/\mathbb{Z}$ which we will constantly identify with the interval $[0,1)$ with addition mod 1. For a real number t denote by $\{t\}$ its fractional part and by $\|t\|$ its distance to the nearest integer number. For an irrational $\alpha\in\mathbb{T}$ denote by (q_n) its sequence of denominators (see e.g. [19]), that is we have

$$\frac{1}{2q_nq_{n+1}}<\left|\alpha-\frac{p_n}{q_n}\right|<\frac{1}{q_nq_{n+1}},$$

where

$$\begin{aligned} q_0&=1, \quad q_1=a_1, \quad q_{n+1}=a_{n+1}q_n+q_{n-1}\\ p_0&=0, \quad p_1=1, \quad p_{n+1}=a_{n+1}p_n+p_{n-1}\end{aligned}$$

and $[0;a_1,a_2,\ldots]$ stands for the continued fraction expansion of α. We say that α has *bounded partial quotients* if the sequence (a_n) is bounded, or equivalently, there exists $c>0$ such that $\|q\alpha\|>c/q$ for every $q\in\mathbb{N}$. By R_α we will denote the rotation by α on $\mathbb{T}$.

Remark 9. Let $f,g:\mathbb{T}\to\mathbb{R}$ be positive integrable functions which are cohomologous over R_α mod $\lambda_{\mathbb{T}}$. Then the special flows $(R_\alpha)^f$ on $(\mathbb{T}^f,(\lambda_{\mathbb{T}})^f)$ and $(R_\alpha)^g$ on $(\mathbb{T}^g,(\lambda_{\mathbb{T}})^g)$ are isomorphic.

Recall that if $f:\mathbb{T}\to\mathbb{R}$ is a piecewise absolutely continuous function for which $\beta_1,\ldots,\beta_k\in\mathbb{T}$ are all its discontinuities and $d(\beta)=\lim_{y\to\beta^-}f(y)-\lim_{y\to\beta^+}f(y)$ then

$$S(f)=\sum_{j=1}^{k}d(\beta_j)=\int_{\mathbb{T}}f'(u)\,du$$

is called the sum of jumps of f.

Remark 10. Suppose that α has bounded partial quotients. If $f:\mathbb{T}\to R$ is an absolutely continuous function with zero mean such that $f'\in L^2(\mathbb{T},\lambda_{\mathbb{T}})$ then by the classical small divisor argument f is a coboundary mod $\lambda_{\mathbb{T}}$. It follows that every piecewise absolutely continuous function $f:\mathbb{T}\to\mathbb{R}$ whose derivative is square integrable is cohomologous to a piecewise linear function whose derivative is equal to $S(f)$ a.e. Indeed, since $g(x)=\int_0^x f'(u)\,du-S(f)x$ is absolutely continuous on $\mathbb{T}$ and g' is square integrable, g is cohomologous to a constant function. On the other hand $f-g$ is piecewise linear function and its derivative is equal to $S(f)$ piecewisely. Moreover, discontinuities and jumps of $f-g$ and f are the same.

Let α be an irrational number with bounded partial quotients and let $f:\mathbb{T}\to\mathbb{R}$ be a piecewise linear function, where $B=\{\beta_1,\beta_2,\ldots,\beta_k\}$ is the set of all its discontinuities and $d(\beta)$ is the size of a jump at $\beta\in B$. Let $\sim\subset B\times B$ stand for the equivalence relation given by $x\sim y$ iff $y-x\in\alpha\mathbb{Z}$. For every equivalence class $C\in B/\sim$ put $S(f,C):=\sum_{\beta\in C}d(\beta)$.

Proposition 13. *Suppose that α is an irrational number with bounded partial quotients and $f:\mathbb{T}\to\mathbb{R}$ is a piecewise linear function with zero mean. Then f is a coboundary mod $\lambda_{\mathbb{T}}$ if and only if $S(f,C)=0$ for every $C\in B/\sim$.*

Proof. Suppose that $S(f,C)=0$ for every $C\in B/\sim$. In view of Remark 10 we can assume that f is piecewise constant. By $\varrho:\mathbb{T}\to\mathbb{R}$ denote the function $\varrho(x)=\{x\}$. For every $C\in B/\sim$ choose an element $\beta_C\in C$. Then for every $\beta\in C$ let $k(\beta)$ stand for the integer number such that $\beta-\beta_C=k(\beta)\alpha$. Set

$$g(x)=-\sum_{C\in B/\sim}\sum_{\beta\in C}d(\beta)\,\varrho^{(k(\beta))}(x-\beta).$$

Then

$$\begin{aligned}
g(x+\alpha)-g(x)&=\sum_{C\in B/\sim}\sum_{\beta\in C}d(\beta)(\varrho^{(k(\beta))}(x-\beta)-\varrho^{(k(\beta))}(x+\alpha-\beta))\\
&=\sum_{C\in B/\sim}\sum_{\beta\in C}d(\beta)(\varrho(x-\beta)-\varrho(x+k(\beta)\alpha-\beta))\\
&=\sum_{C\in B/\sim}\sum_{\beta\in C}d(\beta)(\varrho(x-\beta)-\varrho(x-\beta_C))\\
&=\sum_{C\in B/\sim}\sum_{\beta\in C}d(\beta)(\chi_{[0,\beta)}(x)-\chi_{[0,\beta_C)}(x)+\beta_C-\beta)\\
&=\sum_{\beta\in B}d(\beta)(\chi_{[0,\beta)}(x)-\beta)=f(x)
\end{aligned}$$

for all $x\in\mathbb{T}\setminus B$.

Assume that $S(f,C)\neq 0$ for some $C\in B/\sim$.

Case 1. Suppose that $S(f)\neq 0$. Let c be a positive numer such that $f+c$ is positive. As it was proved by J. von Neumann in [26], the special flow $(R_\alpha)^{f+c}$ is weakly mixing. In view of Remark 4 $\mathbb{T}\ni x\mapsto e^{2\pi i r f(x)}\in\mathbb{T}$ is not a multiplicative coboundary for every $r\in\mathbb{R}\setminus\{0\}$. It follows that $f:\mathbb{T}\to\mathbb{R}$ is not an additive coboundary.

Case 2. Suppose that $S(f)=0$. In view of Remark 10 we can assume again that f is piecewise constant. Recall that (see Corollary 1.6 in [10]) if $h:\mathbb{T}\to\mathbb{R}$ is a piecewise constant function such that $S(h,C)\notin\mathbb{Z}$ for some $C\in B/\sim$ then $\mathbb{T}\ni x\mapsto e^{2\pi i h(x)}\in\mathbb{T}$ is not a multiplicative coboundary. Since $S(f,C)\neq 0$ for some $C\in B/\sim$, we can find $r\in\mathbb{R}\setminus\{0\}$ such that $S(rf,C)\notin\mathbb{Z}$. It follows that $\mathbb{T}\ni x\mapsto e^{2\pi i r f(x)}\in\mathbb{T}$ is not a multiplicative coboundary for every $r\in\mathbb{R}\setminus\{0\}$, and consequently $f:\mathbb{T}\to\mathbb{R}$ is not an additive coboundary. □

Proposition 14 (Denjoy-Koksma inequliaty, see [14]). *If $f:\mathbb{T}\to\mathbb{R}$ is a function of bounded variation then*

$$\left|\sum_{k=0}^{q_n-1} f(R_\alpha^k x)-\int_{\mathbb{T}} f\,d\lambda_{\mathbb{T}}\right|\leqslant \operatorname{Var} f$$

for every $x\in\mathbb{T}$ and $n\in\mathbb{N}$. If f is absolutely continuous then the sequence

$$\left(\sum_{k=0}^{q_n-1} f(R_\alpha^k\,\cdot\,)-\int_{\mathbb{T}} f\,d\lambda_{\mathbb{T}}\right)_{n\in\mathbb{N}}$$

tends uniformly to zero.

Proposition 15. *Let α be an irrational number and let (q_n) be its sequence of denominators. Let $f:\mathbb{T}\to\mathbb{R}$ be a function of bounded variation with zero mean. Suppose that there exists a finite subset $E\subset\mathbb{R}$ such that*

$$\sup_{x\in\mathbb{T}}\min_{r\in E}|f^{(q_n)}(x)-r|\to 0.$$

Then for every locally finite $(R_\alpha)_f$-invariant Borel measure m on $\mathbb{T}\times\mathbb{R}$ we have $\mathscr{R}_m\cap E\neq\emptyset$.

The proof of this proposition can be obtained in much the same way as the proof of Theorem 1.6 in [2].

6. Self-joinings for special flows built over irrational rotations

Let α be an irrational number and let $f:\mathbb{T}\to\mathbb{R}$ be a positive bounded away from zero and bounded Borel function. Let us consider the $\mathbb{Z}^2$-action T on $\mathbb{T}^2$ given by

$$T_{(k_1,k_2)}(x,y)=(x+k_1\alpha,y+k_2\alpha).$$

Denote by φ the $\mathbb{R}$-valued cocycle over T defined by

$$\varphi_{(k_1,k_2)}(x,y)=f^{(k_2)}(y)-f^{(k_1)}(x).$$

By Corollary 8, there is a one-to-one correspondence between ergodic locally finite T_φ-invariant Borel measures on $\mathbb{T}^2\times\mathbb{R}$ (up to a positive multiple) and ergodic self-joinings of $(R_\alpha)^f$.

Proposition 16. *Suppose that m is a locally finite T_φ-invariant ergodic Borel measure on $\mathbb{T}\times\mathbb{T}\times\mathbb{R}$ such that $\mathscr{R}_m=\mathbb{R}$. Then $m=c\,\lambda_{\mathbb{T}^2}\times\lambda_{\mathbb{R}}$ for some $c>0$.*

Proof. By Proposition 10 there exist $a \in \mathbb{R}$ and a finite Borel measure μ on $\mathbb{T} \times \mathbb{T}$ such that

$$dm(x,y,r) = dm_a(x,y,r) = e^{-ar}\, d\mu(x,y)\, d\lambda_{\mathbb{R}}(r)$$

and for every $(k_1,k_2) \in \mathbb{Z}^2$ we have

$$\mu \circ T_{(k_1,k_2)} \sim \mu \text{ and } \frac{d\mu \circ T_{(k_1,k_2)}}{d\mu} = e^{a\varphi(k_1,k_2)}.$$

Therefore

$$\frac{d\mu \circ T_{(0,1)}}{d\mu}(x,y) = e^{af(y)}.$$

Since f is positive, if $a \neq 0$ then $d\mu \circ T_{(0,1)}/d\mu < 1$ or $d\mu \circ T_{(0,1)}/d\mu > 1$ depending on the sign of a, which contradicts the fact that μ is a finite measure. Thus $a = 0$. Since the $\mathbb{Z}^2$-action T is uniquely ergodic, $\mu = c\lambda_{\mathbb{T}^2}$ for some $c > 0$, and hence $m = \mu \otimes \lambda_{\mathbb{R}} = c\,\lambda_{\mathbb{T}^2} \times \lambda_{\mathbb{R}}$. □

Suppose that m is a locally finite T_φ-invariant ergodic Borel measure on $\mathbb{T} \times \mathbb{T} \times \mathbb{R}$. Let us consider two $\mathbb{Z}$-subactions of the $\mathbb{Z}^2$-action T_φ generated by automorphisms $U = (T_\varphi)_{(-1,0)}$ and $W = (T_\varphi)_{(1,1)}$. They jointly generate the action T_φ and

$$U(x,y,r) = (x-\alpha, y, r+f(x-\alpha)),\ W(x,y,r) = (x+\alpha, y+\alpha, r+f(y)-f(x)).$$

Let $\pi : \mathbb{T} \times \mathbb{T} \times \mathbb{R} \to \mathbb{T}$ be given by $\pi(x,y,r) = y - x$. Then

$$\pi \circ W = \pi \quad \text{and} \quad \pi \circ U = R_\alpha \circ \pi.$$

Since $\pi^{-1}(\{\theta\}) = \{(x, x+\theta, r) : x \in \mathbb{T}, r \in \mathbb{R}\}$ for every $\theta \in \mathbb{T}$, we will identify each fiber $\pi^{-1}(\{\theta\})$ with $\mathbb{T} \times \mathbb{R}$. W preserves the fibers of π and

$$W(x, x+\theta, r) = (x+\alpha, x+\alpha+\theta, r+f(x+\theta)-f(x)),$$

therefore the action of W on a fiber $\pi^{-1}(\{\theta\})$ can be identified with the action of a skew product $W_\theta : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ given by

$$W_\theta(x,r) = (x+\alpha, r+f(x+\theta)-f(x)).$$

In summary, we have $\mathbb{Z}^2$-action T_φ on $\mathbb{T} \times \mathbb{T} \times \mathbb{R}$ generated by U and V and $\mathbb{Z}^2$-action on $\mathbb{T}$ given by $(R_\alpha \oplus Id)_{(k_1,k_2)}(\theta) = \theta + k_1\alpha$. Then $\pi : \mathbb{T} \times \mathbb{T} \times \mathbb{R} \to \mathbb{T}$ is a $\mathbb{Z}^2$-equivariant map for which R_α is a factor of U and Id is a factor of V. Under these circumstances, arguments contained in Section 2 give the existence of a probability Borel measure ρ on $\mathbb{T}$, a Borel subset $\Theta \subset \mathbb{T}$ with $\rho(\Theta) = 1$ and a map $\Theta \ni \theta \mapsto m_\theta \in \mathscr{LF}(\mathbb{T} \times \mathbb{T} \times \mathbb{R})$ such that

$$\int_{\mathbb{T}^2 \times \mathbb{R}} h(x,y,r)\, dm(x,y,r) = \int_{\mathbb{T}} \left(\int_{\mathbb{T} \times \mathbb{T} \times \mathbb{R}} h(x,y,r)\, dm_\theta(x,y,r) \right) d\rho(\theta)$$

for every $h \in L^1(\mathbb{T}^2 \times \mathbb{R}, m)$. Since m_θ is concentrated on the fiber $\pi^{-1}(\{\theta\})$ and every fiber is homeomorphic to $\mathbb{T} \times \mathbb{R}$, the measure m_θ will be treated as the locally finite measure on $\mathbb{T} \times \mathbb{R}$. Then

$$\int_{\mathbb{T}^2 \times \mathbb{R}} h(x,y,r)\, dm(x,y,r) = \int_{\mathbb{T}} \left(\int_{\mathbb{T} \times \mathbb{R}} h(x, x+\theta, r)\, dm_\theta(x,r) \right) d\rho(\theta) \tag{16}$$

for every $h \in L^1(\mathbb{T}^2 \times \mathbb{R}, m)$. Moreover, $m_\theta \circ W_\theta = m_\theta$ for ρ–a.e. $\theta \in \mathbb{T}$ (see (3)), $\rho \circ R_\alpha \sim \rho$, ρ is an ergodic measure for the action of R_α (see Lemma 4) and

$$\frac{d\rho \circ g}{d\rho}(\theta) \cdot (m_{R_\alpha \theta} \circ \underline{U}) = m_\theta \quad \text{for } \rho\text{–a.e. } \theta \in \mathbb{T}, \tag{17}$$

where $\underline{U} : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ is given by $\underline{U}(x,r) = (x - \alpha, r + f(x - \alpha))$ (see (2)).

Lemma 17. *For ρ–a.e. $\theta \in \mathbb{T}$ there exists a locally finite W_θ-invariant and ergodic measure m'_θ on $\mathbb{T} \times \mathbb{R}$ such that $\mathscr{R}_{m'_\theta} \subset \mathscr{R}_m$.*

Proof. By Proposition 11, there exist a Borel function $u : \mathbb{T}^2 \to \mathbb{R}$ and a Borel subset $A \subset \mathbb{T}^2 \times \mathbb{R}$ with $m(A^c) = 0$ such that for every $(x,y) \in \mathbb{T}^2$ if there exists $r \in \mathbb{R}$ with $(x,y,r) \in A$ then

$$\varphi_{(1,1)}(x,y) + u(T_{(1,1)}(x,y)) - u(x,y) = f(y) - f(x) + u(x+\alpha, y+\alpha) - u(x,y) \in \mathscr{R}_m.$$

For every $\theta \in \Theta$ let $A_\theta = \{(x,r) \in \mathbb{T} \times \mathbb{R} : (x, x+\theta, r) \in A\}$. Then A_θ is a Borel subset for every $\theta \in \Theta$ and

$$0 = m(A^c) = \int_{\mathscr{T}} m_\theta(A^c_\theta)\, d\rho(\theta).$$

It follows that for ρ–a.e. $\theta \in \mathbb{T}$ we have $m_\theta(A^c_\theta) = 0$. Suppose that $m_\theta(A^c_\theta) = 0$. Applying the ergodic decomposition theorem (see e.g. [12]) for the automorphism $W_\theta \colon (\mathbb{T} \times \mathbb{R}, m_\theta) \to (\mathbb{T} \times \mathbb{R}, m_\theta)$ we conclude that there exists a locally finite Borel W_θ-invariant ergodic measure m'_θ on $\mathbb{T} \times \mathbb{R}$ such that $m'_\theta(A^c_\theta) = 0$. Let $u_\theta \colon \mathbb{T} \to \mathbb{R}$ stand for the Borel map $u_\theta(x) = u(x, x+\theta)$. Then for every $x \in \mathbb{T}$ if there exists $r \in \mathbb{R}$ with $(x,r) \in A_\theta$ then

$$f(x+\theta) - f(x) + u_\theta(x+\alpha) - u_\theta(x) \in \mathscr{R}_m.$$

Now an application of Proposition 12 for the cocycle generated by $x \mapsto f(x+\theta) - f(x)$ over the rotation R_α and the measure m'_θ gives $\mathscr{R}_{m'_\theta} \subset \mathscr{R}_m$. □

Let α be an irrational number with bounded partial quotients. Let $f : \mathbb{T} \to \mathbb{R}$ be a piecewise linear function. For every $\theta \in \mathbb{T}$ let

$$\kappa_{f,\theta}(x) = f(x+\theta) - f(x).$$

Theorem 18. *Let α be an irrational number with bounded partial quotients and let $f : \mathbb{T} \to \mathbb{R}$ be a piecewise linear function with non-zero sum of jumps. Suppose that $\theta \notin \mathbb{Q} + \alpha\mathbb{Q}$. If ν is a locally finite $(R_\alpha)_{\kappa_{f,\theta}}$-invariant ergodic Borel measure on $\mathbb{T} \times \mathbb{R}$ then $\mathscr{R}_\nu = \mathbb{R}$.*

Proof. By $\varrho : \mathbb{T} \to \mathbb{R}$ denote the function $\varrho(x) = \{x\}$. Then

$$\kappa_{\varrho,\theta}(x) = \varrho(x+\theta) - \varrho(x) = \theta 1_{[0,1-\theta)}(x) + (\theta - 1)1_{[1-\theta,1)}(x).$$

Since f and $x \mapsto \sum_{j=1}^k d_j\{x - \beta_i\}$ has the same discontinuities and the same values of jumps, there exists an absolutely continuous function $g : \mathbb{T} \to \mathbb{R}$ such that

$$f(x) = \sum_{j=1}^k d_j \varrho(x - \beta_i) + g(x).$$

Let us consider the function $\kappa_{\varrho,\theta}^{(q_n)}$. Since $\kappa_{\varrho,\theta}$ is piecewise constant and has two jumps: of size -1 at 0 and of size 1 at $-\theta$, $\kappa_{\varrho,\theta}^{(q_n)}$ is also piecewise constant and has the following jumps: of size -1 at $0, -\alpha, \ldots, -(q_n-1)\alpha$ and of size 1 at $-\theta, -\theta-\alpha, \ldots, -\theta-(q_n-1)\alpha$. Moreover, for some $s_n \in \mathbb{N}$ we have

$$\kappa_{\varrho,\theta}^{(q_n)}(0) = s_n\theta + (q_n - s_n)(\theta - 1) = q_n\theta + q_n - s_n.$$

Therefore $\kappa_{\varrho,\theta}^{(q_n)}(0) \in \{q_n\theta\} + \mathbb{Z}$ and hence

$$\kappa_{\varrho,\theta}^{(q_n)}(x) \in \{q_n\theta\} + \mathbb{Z}$$

for every $x \in \mathbb{T}$. In fact, we have $\kappa_{\varrho,\theta}^{(q_n)}(x) \in \{q_n\theta\} + \{-2,-1,0,1,2\}$ because $|\kappa_{\varrho,\theta}^{(q_n)}(x)| \leqslant \mathrm{Var}\kappa_{\varrho,\theta} = 2$ (see Proposition 14). It follows that

$$\begin{aligned}\sum_{j=1}^{k} d_j\varrho^{(q_n)}(x+\theta-\beta_i) - \sum_{j=1}^{k} d_j\varrho^{(q_n)}(x-\beta_i) &= \sum_{j=1}^{k} d_j\kappa_{\varrho,\theta}^{(q_n)}(x-\beta_i)\\ &\in (d_1+\cdots+d_k)\{q_n\theta\} + D\\ &= S(f)\{q_n\theta\} + D\end{aligned}$$

where $D = d_1\{-2,-1,0,1,2\} + \cdots + d_k\{-2,-1,0,1,2\}$.

Suppose that $\theta \notin \mathbb{Q} + \alpha\mathbb{Q}$. Then the set L of limit points of the sequence $(\{q_n\theta\})_{n\in\mathbb{N}}$ is infinite (see [22]). Let ν be a locally finite $(R_\alpha)_{\kappa_{f,\theta}}$-invariant ergodic Borel measure on $\mathbb{T}\times\mathbb{R}$. Suppose that $\mathscr{R}_\nu \subsetneq \mathbb{R}$. Then $\mathscr{R}_\nu = a\mathbb{Z}$ for some $a \in \mathbb{R}$. Since the set

$$\frac{1}{S(f)}(a\mathbb{Z} - D) \cap [0,1)$$

is finite, there exists $b \in L$ which does not belong to this set. Then $(S(f)b + D) \cap a\mathbb{Z} = \emptyset$. Let $(q_{k_n})_{n\in\mathbb{N}}$ be a subsequence of denominators such that $\{q_{k_n}\theta\} \to b$. Since

$$\begin{aligned}\kappa_{f,\theta}^{(q_{k_n})}(x) &= \kappa_{g,\theta}^{(q_{k_n})}(x) + \sum_{j=1}^{k} d_j\kappa_{\varrho,\theta}^{(q_n)}(x-\beta_i)\\ &\in \kappa_{g,\theta}^{(q_{k_n})}(x) + S(f)(\{q_{k_n}\theta\} - b) + S(f)b + D\end{aligned}$$

and $\kappa_{g,\theta}^{(q_{k_n})} \to 0$ uniformly (see Proposition 14), by Proposition 15, we have $\mathscr{R}_\nu \cap (S(f)b + D) \neq \emptyset$, contrary to $(S(f)b + D) \cap a\mathbb{Z} = \emptyset$. □

Lemma 19. *Suppose that m is a locally finite T_φ-invariant ergodic Borel measure on $\mathbb{T}^2\times\mathbb{R}$ such that $\mathscr{R}_m = a\mathbb{Z}$, $a \in \mathbb{R}$. Then the measure ρ is concentrated on the set $\beta_1 + \alpha\beta_2 + \alpha\mathbb{Z}$, where $\beta_1, \beta_2 \in \mathbb{Q}$ and for every $\theta \in \beta_1 + \alpha\beta_2 + \alpha\mathbb{Z}$ the skew product $W_\theta : (\mathbb{T}\times\mathbb{R}, m_\theta) \to (\mathbb{T}\times\mathbb{R}, m_\theta)$ is ergodic and $\mathscr{R}_{m_\theta} = a\mathbb{Z}$. Moreover, for every $h \in L^1(\mathbb{T}^2\times\mathbb{R}, m)$ we have*

$$\begin{aligned}&\int_{\mathbb{T}^2\times\mathbb{R}} h(x,y,r)\,dm(x,y,r)\\ &\quad = \rho(\{\theta\})\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}\times\mathbb{R}} h(R_\alpha^k x, x+\theta, r - f^{(k)}(x))\,dm_\theta(x,r).\end{aligned} \tag{18}$$

Proof. By Lemma 17 and Theorem 18, the measure ρ is concentrated on the set $\mathbb{Q}+\alpha\mathbb{Q}$, consequently, ρ is discrete. By the ergodicity of $R_\alpha:(\mathbb{T},\rho)\to(\mathbb{T},\rho)$, the measure ρ is concentrated on an orbit, i.e. on the set $\beta_1+\alpha\beta_2+\alpha\mathbb{Z}$ where $\beta_1,\beta_2\in\mathbb{Q}$. Moreover, using (16) and (17) for every $h\in L^1(\mathbb{T}^2\times\mathbb{R},m)$ we have

$$\begin{aligned}
\int_{\mathbb{T}^2\times\mathbb{R}} & h(x,y,r)\,dm(x,y,r)\\
&=\sum_{k\in\mathbb{Z}}\rho(\{\theta-k\alpha\})\int_{\mathbb{T}\times\mathbb{R}} h(x,x+\theta-k\alpha,r)\,dm_{\theta-k\alpha}(x,r)\\
&=\sum_{k\in\mathbb{Z}}\rho(\{\theta-k\alpha\})\int_{\mathbb{T}\times\mathbb{R}} h(R_\alpha^k x,x+\theta,r-f^{(k)}(x))\,d(m_{\theta-k\alpha}\circ\underline{U}^{-k})(x,r)\\
&=\rho(\{\theta\})\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}\times\mathbb{R}} h(R_\alpha^k x,x+\theta,r-f^{(k)}(x))\,dm_\theta(x,r).
\end{aligned}$$

We now show that for every $\theta\in\beta_1+\alpha\beta_2+\alpha\mathbb{Z}$ the skew product $W_\theta:(\mathbb{T}\times\mathbb{R},m_\theta)\to(\mathbb{T}\times\mathbb{R},m_\theta)$ is ergodic. Indeed, suppose that there exist $\theta\in\beta_1+\alpha\beta_2+\alpha\mathbb{Z}$ and a Borel W_θ-invariant subset $B\subset\mathbb{T}\times\mathbb{R}$ such that $m_\theta(B)>0$ and $m_\theta(B^c)>0$. Let

$$\overline{B}=\{(x,x+\theta,r)\in\mathbb{T}^2\times\mathbb{R}:(x,r)\in B\}$$

and

$$\overline{A}=\bigcup_{n\in\mathbb{Z}}(T_\varphi)_{(n,0)}\overline{B}.$$

By definition, the set $\overline{A}$ is $(T_\varphi)_{(-1,0)}$-invariant. Moreover, $\overline{A}$ is also $(T_\varphi)_{(1,1)}$-invariant. Indeed, every element of $\overline{A}$ is of the form $(T_\varphi)_{(n,0)}(x,x+\theta,r)$, where $(x,r)\in B$. Then

$$\begin{aligned}
(T_\varphi)_{(1,1)}(T_\varphi)_{(n,0)}(x,x+\theta,r)&=(T_\varphi)_{(n,0)}(T_\varphi)_{(1,1)}(x,x+\theta,r)\\
&=(T_\varphi)_{(n,0)}(x+\alpha,x+\theta+\alpha,r+f(x+\theta)-f(x))\in\overline{A},
\end{aligned}$$

because $(x+\alpha,r+f(x+\theta)-f(x))=W_\theta(x,r)\in B$. Moreover,

$$\begin{aligned}
m(\overline{A})\geqslant m(\overline{B})&=\sum_{k\in\mathbb{Z}}\left(\int_{\mathbb{T}\times\mathbb{R}} I_{\overline{B}}(x,x+\theta+k\alpha,r)\,dm_{\theta+k\alpha}(x)\right)\rho(\{\theta+k\alpha\})\\
&=\int_{\mathbb{T}\times\mathbb{R}} I_B(x,r)\,dm_\theta(x)\rho(\{\theta\})=m_\theta(B)\rho(\{\theta\})>0.
\end{aligned}$$

Similarly we can show that $m(\overline{A}^c)>0$, contrary to the ergodicity of m. □

Lemma 20. *Suppose that m is locally finite T_φ-invariant ergodic Borel measure on $\mathbb{T}^2\times\mathbb{R}$ such that $\mathscr{R}_m=a\mathbb{Z}$. Then $a=0$.*

Proof. By Lemma 19, there exist $\theta\in\mathbb{Q}+\mathbb{Q}\alpha$, a probability measure ρ on $\mathbb{T}$ concentrated on $\theta+\mathbb{Z}\alpha$ and a map $\theta+\mathbb{Z}\alpha\ni\theta+k\alpha\mapsto m_{\theta+k\alpha}\in\mathscr{L}\mathscr{F}^e(\mathbb{T}\times\mathbb{R},W_\theta)$ satisfying (18).

Suppose, contrary to our claim, that $\mathscr{R}_m = \mathscr{R}_{m_\theta} = a\mathbb{Z}$, where $a > 0$. Then there exists $c > 0$ such that $m \circ Q_{ka} = c^k m$ for every $k \in \mathbb{Z}$. Let $I \subset \mathbb{R}$ be an interval such that $m_\theta(\mathbb{T} \times I) > 0$. Let

$$A = \{(x, x+\theta, r) \in \mathbb{T}^2 \times \mathbb{R} : x \in \mathbb{T}, r \in I\}.$$

Then

$$m(A) = m_\theta(\mathbb{T} \times I)\rho(\{\theta\}) > 0.$$

For every $l \in \mathbb{Z}$ let $\varsigma(l) := [l \int f(x)dx/a]$. By the Denjoy-Koksma inequality

$$B_l := (T_\varphi)_{(0,l)} Q_{-\varsigma(l)a} A \subset \mathbb{T}^2 \times (I + [-\mathrm{Var} f, a + \mathrm{Var} f]),$$

whenever $l = \pm q_n$ and

$$m(B_l) = m(Q_{-\varsigma(l)a} A) = c^{-\varsigma(l)} m(A).$$

Since $B_l \subset \pi^{-1}(\{\theta + l\alpha\})$, the sets B_l, $l \in \mathbb{Z}$ are pairwise disjoint. It follows that

$$m\left(\biguplus_{n \in \mathbb{N}} (B_{q_n} \uplus B_{-q_n})\right) = \sum_{n \in \mathbb{N}} (c^{-\varsigma(q_n)} + c^{-\varsigma(-q_n)})\, m(A) = \infty.$$

On the other hand the set

$$\biguplus_{n \in \mathbb{N}} (B_{q_n} \uplus B_{-q_n}) \subset \mathbb{T}^2 \times (I + [-\mathrm{Var} f, a + \mathrm{Var} f])$$

has a compact closure in $\mathbb{T} \times \mathbb{T} \times \mathbb{R}$, and therefore, by the local finiteness of the measure m, has finite m-measure. Consequently, $a = 0$. □

Lemma 21. *Suppose that m is a locally finite T_φ-invariant ergodic Borel measure on $\mathbb{T}^2 \times \mathbb{R}$ such that $\mathscr{R}_m = \{0\}$. Then there exist $\theta \in \mathbb{Q} + \alpha\mathbb{Q}$ and a Borel function $u : \mathbb{T} \to \mathbb{R}$ such that*

$$f(x+\theta) - f(x) = u(x+\alpha) - u(x) \quad \text{for } \lambda_{\mathbb{T}} - a.e.\ x \in \mathbb{T}.$$

Moreover, m is a positive multiple of the measure $(\Lambda_2^{-1} \circ \Lambda_1)((\lambda_{\mathbb{T}}^f)_{\widetilde{S_{-u}}})$, where $Sx = x + \theta$.

Proof. By Lemma 19, there exist $\theta \in \mathbb{Q} + \mathbb{Q}\alpha$, a probability measure ρ on $\mathbb{T}$ concentrated on $\theta + \mathbb{Z}\alpha$ and $m_\theta \in \mathscr{LF}^e(\mathbb{T} \times \mathbb{R}, W_\theta)$ satisfying (18) and such that $\mathscr{R}_{m_\theta} = \mathscr{R}_m = \{0\}$. By Proposition 11, there exist a Borel function $v : \mathbb{T} \to \mathbb{R}$ and a Borel subset $A \subset \mathbb{T} \times \mathbb{R}$ with $m_\theta(A^c) = 0$ such that for every $x \in \mathbb{T}$ if there exists $r \in \mathbb{R}$ with $(x, r) \in A$ then

$$f(x+\theta) - f(x) = v(x+\alpha) - v(x).$$

Moreover, by Proposition 12, there exists $c \in \mathbb{R}$ such that the measure $m_\theta \circ \vartheta_{u+c}^{-1}$ is an ergodic measure on $\mathbb{T} \times \{0\}$ invariant under the action of the automorphism $(R_\alpha)_0(x, r) = (x + \alpha, r)$. Let $u := v + c$. Therefore $m_\theta \circ \vartheta_u^{-1} = \nu \otimes \delta_0$, where ν is an ergodic R_α-invariant measure on $\mathbb{T}$. Hence

$$f(x+\theta) - f(x) = u(x+\alpha) - u(x)\, \nu\text{−a.e.}$$

Since $\vartheta_u : (\mathbb{T}\times\mathbb{R}, m_\theta) \to (\mathbb{T}\times\mathbb{R}, m_\theta\circ\vartheta_u^{-1})$ is an isomorphism, the measure $m_\theta\circ\vartheta_u^{-1}$ and hence ν is σ-finite. Moreover, for any $h\in L^1(\mathbb{T}\times\mathbb{R}, m_\theta)$ we have

$$\begin{aligned}\int_{\mathbb{T}\times\mathbb{R}} h(x,r)\,dm_\theta(x,r) &= \int_{\mathbb{T}\times\mathbb{R}} h(x,r+u(x))\,dm_\theta\circ\vartheta_u^{-1}(x,r)\\ &= \int_{\mathbb{T}} h(x,u(x))\,d\nu(x).\end{aligned}$$

By (18), it follows that

$$\int_{\mathbb{T}^2\times\mathbb{R}} h(x,y,r)\,dm(x,y,r) = \rho(\{\theta\})\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}} h(x+k\alpha, x+\theta, u(x)-f^{(k)}(x))\,d\nu(x)$$

for every $h\in L^1(\mathbb{T}^2\times\mathbb{R}, m)$. By Lemma 9, m is a multiple of $(\Lambda_2^{-1}\circ\Lambda_1)(\nu^f_{\widetilde{S_{-u}}})$, where $Sx = x+\theta$. Notice that ν can not be an infinite measure, as otherwise, the measure ν^f on $\mathbb{T}^f$ would be infinite and therefore $\nu^f_{\widetilde{S_{-u}}}$ would be infinite and by Corollary 8, it would follow that $(\Lambda_2^{-1}\circ\Lambda_1)(\nu^f_{\widetilde{S_{-u}}})$ is not locally finite.

Since ν is finite and R_α-invariant, ν is a positive multiple of $\lambda_{\mathbb{T}}$. Consequently,

$$f(x+\theta)-f(x) = u(x+\alpha)-u(x) \quad \lambda_{\mathbb{T}}-\text{a.e.}$$

and m is a multiple of $(\Lambda_2^{-1}\circ\Lambda_1)((\lambda^f_{\mathbb{T}})_{\widetilde{S_{-u}}})$. □

Theorem 22. *Let α be an irrational number with bounded partial quotients and let $f:\mathbb{T}\to\mathbb{R}$ be a piecewise linear positive and bounded away from zero function with $S(f)\neq 0$. Then the special flow $(R_\alpha)^f$ is simple. Moreover, the centralizer of $(R_\alpha)^f$ consists of automorphisms of the form $\widetilde{S_{-u}}$, where $Sx = x+\theta$ and $u:\mathbb{T}\to\mathbb{R}$ satisfy*

$$f(x+\theta)-f(x) = u(x+\alpha)-u(x) \quad \lambda_{\mathbb{T}}-\text{a.e.} \tag{19}$$

Proof. Suppose that η is an ergodic self-joining of $(R_\alpha)^f$. Then, by Corollary 8, $(\Lambda_2^{-1}\circ\Lambda_1)(\eta)$ is a locally finite ergodic Borel measure on $\mathbb{T}^2\times\mathbb{R}$ invariant under the skew product $\mathbb{Z}^2$-action

$$(m,n)(x,y,r) = (x+m\alpha, y+n\alpha, r+f^{(n)}(y)-f^{(m)}(x)).$$

If $\mathscr{R}_{(\Lambda_2^{-1}\circ\Lambda_1)(\eta)} = \mathbb{R}$, then, by Proposition 16, $(\Lambda_2^{-1}\circ\Lambda_1)(\eta) = c\,\lambda_{\mathbb{T}^2}\otimes\lambda_{\mathbb{R}}$ for some $c>0$. An application of Remark 8 gives $\eta = c\,\lambda^f_{\mathbb{T}}\otimes\lambda^f_{\mathbb{T}}$. If $\mathscr{R}_{(\Lambda_2^{-1}\circ\Lambda_1)(\eta)} = a\mathbb{Z}$, $a\in\mathbb{R}$, then, by Lemma 20, $a=0$. Thus by Lemma 21, η is a multiple of $(\lambda^f_{\mathbb{T}})_{\widetilde{S_{-u}}}$, where $Sx = x+\theta$ and $u:\mathbb{T}\to\mathbb{R}$ satisfy

$$f(x+\theta)-f(x) = u(x+\alpha)-u(x) \quad \lambda_{\mathbb{T}}-\text{a.e.}$$

Then $\widetilde{S_{-u}}\in C((R_\alpha)^f)$. It follows that $(R_\alpha)^f$ is 2-fold simple. Since the flow $(R_\alpha)^f$ is weakly mixing (see e.g. [26]), an application of Proposition 3 completes the proof. □

Theorem 23. *Let α be an irrational number with bounded partial quotients and let $f:\mathbb{T}\to\mathbb{R}$ be a piecewise linear function with $S(f)\neq 0$ which is bounded away from zero. Then $C((R_\alpha)^f)$ is an Abelian group which is the direct sum of the subgroup $\{(R_\alpha)^f_t : t\in\mathbb{R}\}$ and a finite subgroup.*

Proof. Let $B = \{\beta_1, \beta_2, \ldots, \beta_k\}$ be the set of all discontinuities of f and $d(\beta_j)$ stand for the size of jump at β_j for $j = 1, \ldots, k$. We can assume that $\beta_j - \beta_i \notin \alpha\mathbb{Z}$ for $i \neq j$. Otherwise, by Proposition 13, f is cohomologous with a piecewise linear function satisfying the required property.

By Theorem 22, every element of the centralizer of $(R_\alpha)^f$ is of the form $(R_\theta)_{-u}$, where $\theta \in \mathbb{T}$ and $u : \mathbb{T} \to \mathbb{R}$ satisfy (19). Let us denote by Θ the set of all $\theta \in \mathbb{T}$ for which the equation

$$f(x+\theta) - f(x) = u(x+\alpha) - u(x) \;\; \lambda_{\mathbb{T}} - \text{a.e.} \tag{20}$$

has a Borel solution. Notice that u in (20) is unique up to an additive constant. Moreover $\Theta \subset \mathbb{T}$ is a subgroup for which $\alpha \in \Theta$.

Suppose that $\theta \in \Theta$. Then the set of discontinuities of $f(\cdot + \theta) - f(\cdot)$ is equal to $B = \{\beta_1, \beta_2, \ldots, \beta_k, \beta_1 - \theta, \beta_2 - \theta, \ldots, \beta_k - \theta\}$. By Proposition 13, there exists a permutation σ of the set $\{1, 2, \ldots, k\}$ such that

$$\beta_i - \beta_{\sigma(i)} + \theta \in \alpha\mathbb{Z} \text{ and } d(\beta_i) = d(\beta_{\sigma(i)}) \tag{21}$$

for every $i = 1, \ldots, k$. Summing up (21) from $i = 1$ to k we obtain that $k\theta \in \alpha\mathbb{Z}$, and hence $\Theta \subset \frac{1}{k}(\mathbb{Z} + \alpha\mathbb{Z})$. Therefore the group Θ has at most two generators. Suppose that $\theta = \frac{m}{k} + \frac{n}{k}\alpha \in \Theta$ (m, n are unique) and $u : \mathbb{T} \to \mathbb{R}$ is a solution of (20). Since $n\alpha = k\theta \mod 1$, we have

$$f^{(n)}(x+\alpha) - f^{(n)}(x) = f(x+n\alpha) - f(x) = f(x+k\theta) - f(x) = u^{(k)}(x+\alpha) - u^{(k)}(x)$$

for $\lambda_{\mathbb{T}}$–a.e. $x \in \mathbb{T}$, where $f^{(\cdot)}(\cdot)$ and $u^{(\cdot)}(\cdot)$ are considered as cocycles over the rotations by α and θ respectively. By the ergodicity of R_α, $f^{(n)}$ and $u^{(k)}$ differ by a constant. Therefore we can choose a unique solution $u_\theta : \mathbb{T} \to \mathbb{R}$ of (20) such that $f^{(n)} = u_\theta^{(k)}$, or equivalently $\int u_\theta \, d\lambda = \frac{n}{k} \int f \, d\lambda$. Next notice that

$$\Theta \ni \theta \mapsto A(\theta) = \widetilde{(R_\theta)_{-u_\theta}} \in C((R_\alpha)^f)$$

is a group homomorphism. Indeed, suppose that $\theta_1 = \frac{m_1}{k} + \frac{n_1}{k}\alpha$, $\theta_2 = \frac{m_2}{k} + \frac{n_2}{k}\alpha \in \Theta$ and let us consider

$$u := u_{\theta_1} + u_{\theta_2} \circ R_{\theta_1}$$

as a cocycle over $R_{\Theta_1 + \Theta_2}$. Then

$$\begin{aligned} u(x+\alpha) - u(x) &= u_{\theta_1}(x+\alpha) - u_{\theta_1}(x) + u_{\theta_2}(x+\theta_1+\alpha) - u_{\theta_2}(x+\theta_1) \\ &= f(x+\theta_1) - f(x) + f(x+\theta_1+\theta_2) - f(x+\theta_1) \\ &= f(x+\theta_1+\theta_2) - f(x). \end{aligned}$$

Moreover,

$$\int u \, d\lambda = \int u_{\theta_1} \, d\lambda + \int u_{\theta_2} \, d\lambda = \frac{n_1 + n_2}{k} \int f \, d\lambda,$$

hence $u = u_{\theta_1+\theta_2}$. If follows that $(R_{\theta_1+\theta_2})_{-u_{\theta_1+\theta_2}} = (R_{\theta_2})_{-u_{\theta_2}} \circ (R_{\theta_1})_{-u_{\theta_1}}$, which implies our claim.

Moreover

$$A(\theta)^k(x, r) = \pi(x + k\theta, r - u_\theta^{(k)}(x)) = \pi(x + n\alpha, r - f^{(n)}(x)) = (x, r)$$

for every $(x,r)\in(R_\alpha)^f$. Therefore $A(\Theta)$ is a finite Abelian group with at most two generators. Moreover, every element from $C((R_\alpha)^f)$ is of the form $\widetilde{(R_\theta)_{-u}}$, where $\theta\in\Theta$ and u satisfies (20). Clearly, $u=u_\theta-t$ and

$$\widetilde{(R_\theta)_{-u}}=A(\theta)\circ(R_\alpha)^f_t=(R_\alpha)^f_t\circ A(\theta).$$

Since $\{(R_\alpha)^f_t : t\in\mathbb{R}\}\cap A(\Theta)=\{Id\}$, it follows that $C((R_\alpha)^f)$ is an Abelian group which is the direct sum of the group $\{(R_\alpha)^f_t : t\in\mathbb{R}\}$ and the finite group $A(\Theta)$. □

Corollary 24. *If* $\#\{S(f,C) : C\in B/\sim\}>\#(B/\sim)/2$ *or* $\beta_1,\dots,\beta_k,\alpha,1$ *are independent over* $\mathbb{Q}$ *then* T^f *has MSJ. In particular, if f has only one discontinuity then* T^f *has MSJ.*

Proposition 25. *Assume that* $\mathscr{T}=(T_t)_{t\in\mathbb{R}}$ *is an ergodic simple flow on a standard probability space* $(X,\mathscr{B},\mu)$ *and* $C(\mathscr{T})$ *is the direct sum of the group of time-t automorphisms and a finite Abelian group* $H\subset C(\mathscr{T})$. *Then* $\mathscr{T}$ is a *finite extension of an MSJ-flow.*

Proof. Let

$$\mathscr{C}=\{A\in\mathscr{B} : h(A)=A \text{ for all } h\in H\}.$$

Then $\mathscr{C}$ is a $\mathscr{T}$-invariant σ-algebra and $\mathscr{T}$ is a finite group extension of the factor flow $\mathscr{T}/\mathscr{C}$ on $(X/\mathscr{C},\mathscr{C},\mu)$ (see e.g. Theorem 1.8.1 in [16]). Since $C(\mathscr{T})$ is Abelian, by Corollary 3.6 in [16], $\mathscr{T}/\mathscr{C}$ is simple. We now only need to show that

$$C(\mathscr{T}/\mathscr{C})=\{T_t : (X/\mathscr{C},\mathscr{C},\mu)\to(X/\mathscr{C},\mathscr{C},\mu); t\in\mathbb{R}\}.$$

Suppose that $S\in C(\mathscr{T}/\mathscr{C})$ and let $\mu_S\in J^e(\mathscr{T}/\mathscr{C},\mathscr{T}/\mathscr{C})$ be the corresponding graph joining. Let $\rho\in J^e(\mathscr{T},\mathscr{T})$ be an extension of μ_S, i.e. $\rho(A)=\mu_S(A)$ for all $A\in\mathscr{C}\otimes\mathscr{C}$. Since $\mathscr{T}$ is simple and ρ is not the product measure, there exists $R\in C(\mathscr{T})$ such that $\rho=\mu_R$. Then there exist a unique $t\in\mathbb{R}$ and $h\in H$ such that $R=h\circ T_t$. Therefore for every $A,B\in\mathscr{C}$ we have

$$\mu(A\cap S^{-1}B)=\mu_S(A\times B)=\mu_R(A\times B)=\mu(A\cap T_t^{-1}\circ h^{-1}B)=\mu(A\cap T_t^{-1}B),$$

hence $S=T_t$ as automorphisms of $X/\mathscr{C}$, and consequently $\mathscr{T}/\mathscr{C}$ has MSJ. □

Proof of Theorem 1. Now the claim follows immediately form Proposition 2, Remark 10, Theorems 22, 23, and Proposition 25. □

A. Special representation of $(\varphi_t)_{t\in\mathbb{R}}$

Proof of Proposition 2. As it was proved by Arnold in [3], on the torus there exists a closed C^∞-curve transversal to the orbits of $(h_t)_{t\in\mathbb{R}}$ on EC. Moreover, the first-return map (Poincaré map) is determined everywhere on the curve, except for a finite set F of points that are points of the last intersection of the incoming separatrices with the transversal curve. In the induced parameterization, this map is the circle rotation by α. Recall that if a smooth tangent vector field X on a surface M preserves a volume form μ, then a parameterization $\gamma:[a,b]\to M$ is induced if

$$\int_{\gamma(s_1)}^{\gamma(s_2)} i_X\mu=s_2-s_1 \quad \text{for all } s_1,s_2\in[a,b].$$

Moreover the return time is a C^∞-function of the parameter everywhere except of points form the set F. This function has logarithmic singularities at these points (see [21]). Thus, the ergodic component of (h_t) is isomorphic to a special flow built over the rotation by α and under a roof function with logarithmic singularities.

For the flow $(\varphi_t)_{t\in\mathbb{R}}$ on EC we will consider the same transversal. Hence the Poincaré map is naturally identified with the rotation by α on $\mathbb{T}$. Let $f(x)$ stand for the time of the first return of x (from the transversal) to the transversal. Then the action of (φ_t) in EC is isomorphic to the special flow built over the rotation by α on $\mathbb{T}$ and under the roof function $f:\mathbb{T}\to\mathbb{R}$. Let $\beta_1<\cdots<\beta_r<\beta_{r+1}=\beta_1$ be all discontinuities of f, i.e. they represent the points from the set F. Then f is of class C^∞ on (β_i,β_{i+1}) for $i=1,\ldots,r$. Fix $1\leqslant i\leqslant r$. By the Morse Lemma, there exist a neighborhood $(0,0)\in V=V_i\subset\mathbb{R}^2$ and C^∞-diffeomorphism $\Phi=\Phi_i:V\to\Phi_i(V)\subset\mathbb{T}^2$ such that $\Phi(0,0)=\bar{x}_i$ and if $\widehat{H}=H\circ\Phi$, then $\widehat{H}(x,y)=x\cdot y$ for all $(x,y)\in V_i$. Recall that

$$X_H=J\nabla H,\quad \text{where}\quad J=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

and

$$\det A\cdot(A^{-1}J)=JA^T\quad\text{for all }\ A\in GL(2,\mathbb{R}).$$

It follows that

$$J\nabla\widehat{H}=J(D\Phi)^T(\nabla H\circ\Phi)=\det(D\Phi)(D\Phi)^{-1}(J\nabla H\circ\Phi),$$

hence

$$(D\Phi)\frac{X_{\widehat{H}}}{\hat{p}}=X\circ\Phi,\quad\text{where}\quad \hat{p}(\bar{x})=\det(D\Phi(\bar{x}))\,p(\Phi(\bar{x})).\tag{22}$$

Let $(\hat{\varphi}_t)$ stand for the local flow on V given by $\hat{\varphi}_t=\Phi^{-1}\circ\varphi_t\circ\Phi$. In view of (22) $(\hat{\varphi}_t)$ is associated with the following differential equation

$$\begin{aligned}\frac{dx}{dt}&=\frac{x}{\hat{p}(x,y)}\\ \frac{dy}{dt}&=-\frac{y}{\hat{p}(x,y)}.\end{aligned}$$

Let $\delta=\delta_i$ be a positive number such that $[-\delta,\delta]\times[-\delta,\delta]\subset V$. Let us consider the C^∞-curve $l:[-\delta^2,\delta^2]\to\mathbb{T}^2$ given by $l(s)=\Phi(s/\delta,\delta)$. Notice that l establishes an induced parameterization with respect to the form $\mu(x,y)=p(x,y)\,dx\wedge dy$ and the vector field X. Indeed,

$$\begin{aligned}\int_{l(s_1)}^{l(s_2)}i_X\mu&=\int_{s_1}^{s_2}dx\wedge dy(X_H(l(u)),l'(u))\,du=\int_{s_1}^{s_2}dH(l(u))l'(u)\,du\\ &=\int_{s_1}^{s_2}\frac{d}{dl}(H\circ l)(u)\,du=H(\Phi(s_2/\delta,\delta))-H(\Phi(s_1/\delta,\delta))\\ &=\widehat{H}(s_2/\delta,\delta)-\widehat{H}(s_1/\delta,\delta)=s_2-s_1\end{aligned}$$

for all $s_1, s_2 \in [-\delta^2, \delta^2]$. Denote by $\tau : [-\delta^2, 0) \cup (0, \delta^2] \to \mathbb{R}$ the function for which $\tau(s)$ is the time of going out of the point $l(s)$ (for the flow (φ_t)) from the set $\Phi([-\delta, \delta] \times [-\delta, \delta])$. On the other hand $\tau(s)$ is the time of passage from the point $(s/\delta, \delta)$ to $(\mathrm{sgn}(s)\delta, \mathrm{sgn}(s)s/\delta)$ for the flow $(\hat{\varphi}_t)$. Therefore

$$\tau(s) = \int_{s/\delta}^{\mathrm{sgn}(s)\delta} \hat{p}\left(x, \frac{s}{x}\right) \frac{1}{x} dx.$$

Of course τ is of class C^∞ on $[-\delta^2, 0) \cup (0, \delta^2]$. We will prove that $\tau' \in L^2([-\delta^2, \delta^2])$. First let us consider τ only on $(0, \delta^2]$. Let us decompose $\tau = \tau_1 + \tau_2$, where

$$\tau_1(s) = \int_{\sqrt{s}}^{\delta} \hat{p}\left(x, \frac{s}{x}\right) \frac{1}{x} dx, \quad \tau_2(s) = \int_{s/\delta}^{\sqrt{s}} \hat{p}\left(x, \frac{s}{x}\right) \frac{1}{x} dx = \int_{\sqrt{s}}^{\delta} \hat{p}\left(\frac{s}{x}, x\right) \frac{1}{x} dx. \tag{23}$$

Then

$$\tau_1'(s) = -\hat{p}(\sqrt{s}, \sqrt{s}) \frac{1}{2s} + \int_{\sqrt{s}}^{\delta} \frac{\partial}{\partial y} \hat{p}\left(x, \frac{s}{x}\right) \frac{1}{x^2} dx.$$

Since $\hat{p}(0,0) = 0$, $D\hat{p}(0,0) = (0,0)$ and there exists $d > 0$ such that

$$\|D\hat{p}(\bar{x}) - D\hat{p}(\bar{y})\| \leqslant d\|\bar{x} - \bar{y}\| \quad \text{for all } \bar{x}, \bar{y} \in [-\delta, \delta] \times [-\delta, \delta],$$

we have

$$\|D\hat{p}(x,y)\| \leqslant d\|(x,y)\| \leqslant d(|x| + |y|) \quad \text{for all } x, y \in [-\delta, \delta], \tag{24}$$

hence

$$|\hat{p}(\bar{x})| = |\hat{p}(\bar{x}) - \hat{p}(0,0)| \leqslant \sup_{0 \leqslant \lambda \leqslant 1} \|D\hat{p}(\lambda\bar{x})\| \|\bar{x}\| \leqslant d\|\bar{x}\|^2. \tag{25}$$

It follows that

$$\begin{aligned} |\tau_1'(s)| &= \left| -\hat{p}(\sqrt{s}, \sqrt{s}) \frac{1}{2s} + \int_{\sqrt{s}}^{\delta} \frac{\partial}{\partial y} \hat{p}\left(x, \frac{s}{x}\right) \frac{1}{x^2} dx \right| \\ &\leqslant d\left(1 + \int_{\sqrt{s}}^{\delta} \left(x + \frac{s}{x}\right) \frac{1}{x^2} dx\right) = \frac{d}{2}\left(3 - \frac{s}{\delta^2} - \log \frac{s}{\delta^2}\right). \end{aligned}$$

Thus $\tau_1' \in L^2((0, \delta^2])$. In view of (23) the same conclusion can be drawn for τ_2, hence $\tau' \in L^2((0, \delta^2])$. An application the same arguments, with $(0, \delta^2]$ replaced by $[-\delta^2, 0)$, yields $\tau' \in L^2([-\delta^2, \delta^2])$. It follows that $\tau : [-\delta^2, 0) \cup (0, \delta^2] \to \mathbb{R}$ is absolutely continuous.

Now using some standard arguments we conclude that for some $\varepsilon > 0$ the function $f : [\beta_i - \varepsilon, \beta_i) \cup (\beta_i, \beta_i + \varepsilon] \to \mathbb{R}$ is absolutely continuous and its derivative is square integrable.

Finally we will show that $S(f) = \int_{EC} d\omega$. First we must prove that $\int_{EC} d\omega$ exists. It suffices to show $\int_{\Phi_i([-\delta_i,\delta_i] \times [-\delta_i,\delta_i]) \setminus \{\bar{x}_i\}} d\omega$ is finite for every $i = 1, \ldots, r$. Fix $1 \leqslant i \leqslant r$. Then for $\Phi = \Phi_i$ we have

$$\begin{aligned} \int_{\Phi([-\delta,\delta] \times [-\delta,\delta] \setminus \{(0,0)\})} d\omega &= \int_{[-\delta,\delta] \times [-\delta,\delta] \setminus \{(0,0)\}} \Phi^*(d\omega) \\ &= \int_{[-\delta,\delta] \times [-\delta,\delta] \setminus \{(0,0)\}} d(\Phi^*\omega). \end{aligned}$$

Moreover,

$$(\Phi^*\omega)_{\bar{x}}Y = \frac{\langle X(\Phi(\bar{x})), D\Phi(\bar{x})Y\rangle}{\langle X(\Phi(\bar{x})), X(\Phi(\bar{x}))\rangle} = \hat{p}(\bar{x})\frac{\langle D\Phi(\bar{x})X_{\widehat{H}}(\bar{x}), D\Phi(\bar{x})Y\rangle}{\langle D\Phi(\bar{x})X_{\widehat{H}}(\bar{x}), D\Phi(\bar{x})X_{\widehat{H}}(\bar{x})\rangle}.$$

Therefore

$$(\Phi^*\omega)_{(x,y)} = \frac{\hat{p}(x,y)}{c(x,y)}(a(x,y)dx + b(x,y)dy),$$

where $a,b,c : [-\delta,\delta]\times[-\delta,\delta] \to \mathbb{R}$ are C^∞-functions such that

$$a(x,y)dx + b(x,y)dy = \left\langle D\Phi(x,y)^T D\Phi(x,y)\begin{bmatrix} y \\ -x \end{bmatrix}, \cdot \right\rangle$$

and

$$c(x,y) = \left\| D\Phi(x,y)\begin{bmatrix} y \\ -x \end{bmatrix}\right\|^2.$$

It is easy to check that the following functions: $Da(\bar{x})$, $Db(\bar{x})$, $\frac{a(\bar{x})}{\|\bar{x}\|}$, $\frac{b(\bar{x})}{\|\bar{x}\|}$, $\frac{c(\bar{x})}{\|\bar{x}\|^2}$, $\frac{Dc(\bar{x})}{\|\bar{x}\|}$ are bounded and $\frac{|c(\bar{x})|}{\|\bar{x}\|^2}$ is bounded away from zero on $[-\delta,\delta]\times[-\delta,\delta]\setminus\{(0,0)\}$. From (24) and (25), the functions $\frac{\hat{p}(\bar{x})}{\|\bar{x}\|^2}$, $\frac{D\hat{p}(\bar{x})}{\|\bar{x}\|}$ are also bounded on $[-\delta,\delta]\times[-\delta,\delta]\setminus\{(0,0)\}$. Since

$$d(\Phi^*\omega) = \left(-\frac{a}{c}\frac{\partial\hat{p}}{\partial y} - \frac{\hat{p}}{c}\frac{\partial a}{\partial y} + \frac{a\hat{p}}{c^2}\frac{\partial c}{\partial y} + \frac{b}{c}\frac{\partial\hat{p}}{\partial x} + \frac{\hat{p}}{c}\frac{\partial b}{\partial x} + \frac{b\hat{p}}{c^2}\frac{\partial c}{\partial x}\right)dx\wedge dy,$$

it follows that the form $d(\Phi^*\omega)$ bounded on $[-\delta,\delta]\times[-\delta,\delta]\setminus\{(0,0)\}$. Thus $\int_{\Phi([-\delta,\delta]\times[-\delta,\delta]\setminus\{(0,0)\})} d\omega$ exists.

Denote by $\nu : \mathbb{T}\to\mathbb{T}^2$ the induced parameterization of the transversal curve. For every $n\in\mathbb{N}$ and $1\leqslant i\leqslant r$ let us denote by $\sigma_{i,n}$ a singular 1-chain on EC which is a formal sum of four curves: $\nu : [\beta_i+\alpha-1/n, \beta_i+\alpha+1/n]\to\mathbb{T}^2$ plus $\varphi_{(\cdot)}(\nu(\beta_i-1/n)) : [0, f(\beta_i-1/n)]\to\mathbb{T}^2$ minus $\nu : [\beta_i-1/n, \beta_i+1/n]\to\mathbb{T}^2$ minus $\varphi_{(\cdot)}(\nu(\beta_i+1/n)) : [0, f(\beta_i+1/n)]\to\mathbb{T}^2$. Clearly, $\sigma_{i,n}$ is closed but not exact. Let us denote by $A_{n,i}$ the part of EC which is inside the chain $\sigma_{i,n}$ ($A_{n,i}$ is homotopic with an annulus). By the Stokes Theorem, we have

$$\int_{EC} d\omega = \sum_{i=1}^{r}\left(\int_{A_{n,i}} d\omega + \int_{\sigma_{i,n}}\omega\right).$$

Since the measure of $A_{n,i}$ tends to zero as $n\to\infty$ for all $i=1,\dots,r$ and the form $d\omega$ is bounded on EC, we have

$$\sum_{i=1}^{r}\int_{A_{n,i}} d\omega \to 0.$$

On the other hand

$$\int_{\sigma_{i,n}}\omega = f(\beta_i-1/n) - f(\beta_i+1/n) + \int_{\nu(\beta_i+\alpha-1/n)}^{\nu(\beta_i+\alpha+1/n)}\omega - \int_{\nu(\beta_i-1/n)}^{\nu(\beta_i+1/n)}\omega.$$

As

$$\left|\int_{\nu(s_1)}^{\nu(s_2)}\omega\right| \leqslant \max\{\|\nu'(s)\|/\|X(\nu(s))\| : s\in\mathbb{T}\}\,|s_2-s_1|$$

for all $s_1, s_2 \in \mathbb{T}$, it follows that

$$\lim_{n\to\infty}\int_{\sigma_{i,n}}\omega = f_-(\beta_i) - f_+(\beta_i).$$

Consequently

$$\int_{EC} d\omega = \sum_{i=1}^{r}(f_-(\beta_i) - f_+(\beta_i)) = S(f).$$

□

B. Examples

In this section we will describe some examples of flows on the two-torus which have an ergodic component of positive Lebesgue measure satisfying the simplicity property. We will deal with quasi-periodic Hamiltonians $H : \mathbb{R}^2 \to \mathbb{R}$ having the form

$$H(x,y) = -\sum_{i=1}^{k} b_i \exp(-a_i(\sin^2\pi(x-x_i) + \sin^2\pi(y-y_i))) + \alpha x + y, \quad (26)$$

where $a_i > 0$, $b_i \neq 0$ for $i = 1, \ldots, k$, the points (x_i, y_i), $i = 1, \ldots, k$ are pairwise distinct and α has bounded partial quotients. Next take $p : \mathbb{T}^2 \to \mathbb{R}$ given by $p(x,y) = q(x,y)\|X_H(x,y)\|^2$, where $q : \mathbb{T}^2 \to \mathbb{R}$ is a positive C^∞-function. The function p is non-negative and is positive except of the set Crit of all critical points of H on $\mathbb{T}^2$. Let us consider the flow $(\varphi_t)_{t\in\mathbb{R}}$ on $\mathbb{T}^2\setminus$Crit associated with the vector field $X = X_H/p = X_H/(q\|X_H\|^2)$.

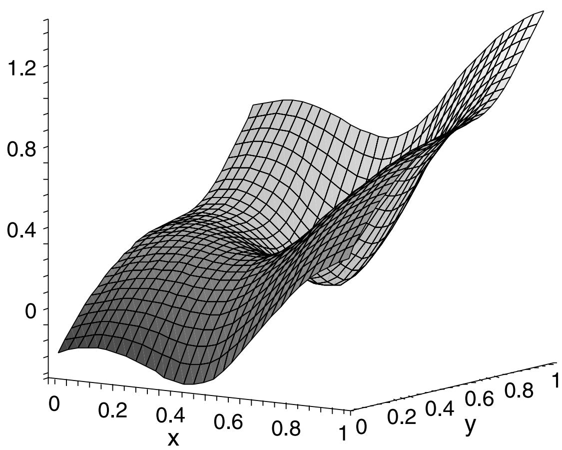

Figure 1. The graph of H

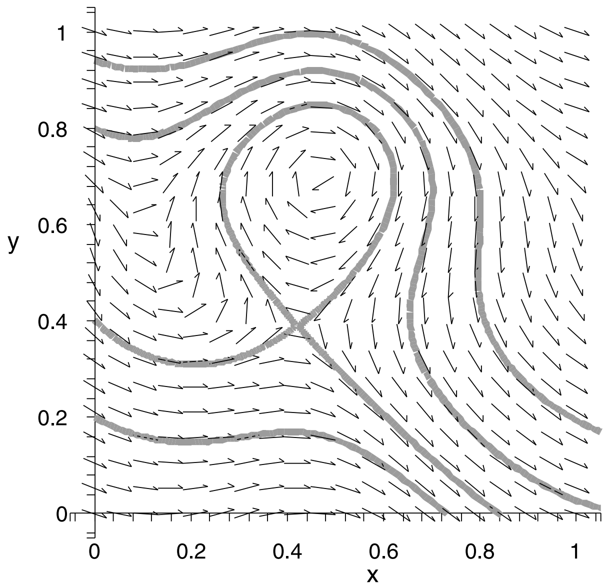

Figure 2. The phase space of (φ_t)

Now let us consider the special case

$$H(x,y) = -\exp(-(\sin^2\pi(x-0.5) + \sin^2\pi(y-0.75))) + \frac{\sqrt{5}-1}{2}\cdot x + y.$$

Here H has two critical points in the unit square: $\bar{x}_s = (0.4213\ldots, 0.3892\ldots)$ – a saddle and $\bar{x}_c = (0.4672\ldots, 0.6963\ldots)$ – a center (see Fig. 1). The phase space of (φ_t) decomposes into one trap

$$\text{Trap} = \{(x,y) \in [0,1)\times[0,1) : H(x,y) \leqslant H(x_s,y_s), y \geqslant y_s\}$$

and the ergodic component $EC = \mathbb{T}^2 \setminus \text{Trap}$ with positive Lebesgue measure (see Fig. 2). Denote by $\gamma : [0,l] \to \partial EC$ ($l = \text{length}(\partial EC)$) the unit speed parametrization of ∂EC (∂EC is oriented clockwise) such that $\gamma(0) = \gamma(l) = \bar{x}_s$. Then $\gamma'(t) = X_H(\gamma(t))/\|X_H(\gamma(t))\|$ for $0<t<l$. Since

$$\omega(Y) = \frac{\langle X, Y\rangle}{\langle X, X\rangle} = q\cdot\langle X_H, Y\rangle,$$

by the Stokes Theorem, we have

$$\begin{aligned}\int_{EC} d\omega &= \int_{\partial EC}\omega = \int_0^l q(\gamma(t))\langle X_H(\gamma(t)), \gamma'(t)\rangle\, dt\\ &= \int_0^l q(\gamma(t))\langle X_H(\gamma(t)), X_H(\gamma(t))/\|X_H(\gamma(t))\|\rangle\, dt\\ &= \int_0^l q(\gamma(t))\|X_H(\gamma(t))\|\, dt = \int_{\partial EC} q(s)\|X_H(s)\| ds > 0.\end{aligned}$$

Let us return to the general case where H has the form (26). For a_i, $i = 1,\dots,k$ large enough the flow (φ_t) has k traps: T_i, $i = 1,\dots,k$. Similar arguments to those above show that

$$I(q) := \int_{EC} d\omega = \sum_{i=1}^{k} \mathrm{sgn}(b_i) \int_{\partial T_i} q(s)\|X_H(s)\|ds.$$

Let $C_+^\infty(\mathbb{T}^2)$ stand for the set of positive C^∞ functions on $\mathbb{T}^2$ equipped with the topology induced from $C^\infty(\mathbb{T}^2)$.

If b_i, $i = 1,\dots,k$ have the same sign then $I(q) \neq 0$ for every $q \in C_+^\infty(\mathbb{T}^2)$, and hence the flow (φ_t) on EC is simple. In the general case the set Q of all parameters $q \in C_+^\infty(\mathbb{T}^2)$ for which $I(q) \neq 0$ is open and dense. Indeed, this is a consequence of the facts that the map $C_+^\infty(\mathbb{T}^2) \ni q \mapsto I(q) \in \mathbb{R}$ is continuous, the map

$$C_+^\infty(\mathbb{T}^2) \ni q \mapsto \int_{\partial T_i} q(s)\|X_H(s)\|ds \in \mathbb{R}$$

is strictly increasing for $i = 1,\dots,k$ and the traps T_i, $i = 1,\dots,k$ are pairwise disjoint. It follows that for a typical choice of the parameter $q \in C_+^\infty(\mathbb{T}^2)$ the flow (φ_t) on EC is also simple.

Some properties of the flow (φ_t) for which $\int_{EC} d\omega = 0$ are studied in [7].

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Authors' address: K. Frączek and M. Lemańczyk, Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland; e-mail: fraczek@mat.uni.torun.pl, mlem@mat.uni.torun.pl