Monatsh. Math. 131, 279–307 (2000)

Monatshefte für
Mathematik

On Cocycles with Values in the Group $SU(2)$

By

Krzysztof Frączek

Nicholas Copernicus University, Toruń, Poland

(Received 8 February 2000; in revised form 26 September 2000)

Abstract. In this paper we introduce the notion of degree for C^1-cocycles over irrational rotations on the circle with values in the group $SU(2)$. It is shown that if a C^1-cocycle $\varphi : \mathbb{T} \to SU(2)$ over an irrational rotation by α has nonzero degree, then the skew product

$$\mathbb{T} \times SU(2) \ni (x, g) \mapsto (x + \alpha, g\ \varphi(x)) \in \mathbb{T} \times SU(2)$$

is not ergodic and the group of essential values of φ is equal to the maximal Abelian subgroup of $SU(2)$. Moreover, if φ is of class C^2 (with some additional assumptions) the Lebesgue component in the spectrum of the skew product has countable multiplicity. Possible values of degree are discussed, too.

2000 Mathematics Subject Classification: 37A05.
Key words: Skew products, degree of cocycles, Lebesgue spectrum

1. Introduction

Assume that $T : (X, \mathscr{B}, \lambda) \to (X, \mathscr{B}, \lambda)$ is an ergodic measure-preserving automorphism of standard Borel space. Let G be a compact Lie group, μ its Haar measure. For a given measurable function $\varphi : X \to G$ we study spectral properties of the measure-preserving automorphism of $X \times G$ (called skew product) defined by

$$T_\varphi : (X \times G, \lambda \otimes \mu) \to (X \times G, \lambda \otimes \mu), T_\varphi(x, g) = (Tx, g\varphi(x)).$$

A measurable function $\varphi : X \to G$ determines a measurable cocycle over the automorphism T given by

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x)\varphi(Tx)\ldots\varphi(T^{n-1}x) & \text{for} \quad n > 0 \\ e & \text{for} \quad n = 0 \\ (\varphi(T^n x)\varphi(T^{n+1}x)\ldots\varphi(T^{-1}x))^{-1} & \text{for} \quad n < 0, \end{cases}$$

which we will identify with the function φ. Then $T_\varphi^n(x, g) = (Tx, g\varphi^{(n)}(x))$ for any integer n. Two cocycles $\varphi, \psi : X \to G$ are *cohomologous* if there exists a measurable map $p : X \to G$ such that

$$\varphi(x) = p(x)^{-1}\psi(x)p(Tx).$$

Research partly supported by KBN grant 2 P03A 002 14(1998), by FWF grant P12250-MAT and by Foundation for Polish Science.

In this case, p will be called a *transfer function*. If φ and ψ are cohomologous, then the map $(x,g) \mapsto (x, p(x)g)$ establishes a metrical isomorphism of T_φ and T_ψ.

By $\mathbb{T}$ we will mean the circle group $\{z \in \mathbb{C}; |z| = 1\}$ which most often will be treated as the group $\mathbb{R}/\mathbb{Z}$; λ will denote Lebesgue measure on $\mathbb{T}$. We will identify functions on $\mathbb{T}$ with periodic functions of period 1 on $\mathbb{R}$. Assume that $\alpha \in \mathbb{T}$ is irrational. We will deal with the case where T is the ergodic rotation on $\mathbb{T}$ given by $Tx = x + \alpha$.

In the case where G is the circle and φ is a smooth cocycle, spectral properties of T_φ depend on the topological degree $d(\varphi)$ of φ. For example, in [5], Iwanik, Lemańczyk, Rudolph have proved that if φ is a C^2-cocycle with $d(\varphi) \neq 0$, then T_φ is ergodic and it has countable Lebesgue spectrum on the orthocomplement of the space of functions depending only on the first variable. On the other hand, in [3], Gabriel, Lemańczyk and Liardet have proved that if φ is absolutely continuous with $d(\varphi) = 0$, then T_φ has singular spectrum.

The aim of this paper is to find a spectral equivalent of topological degree in case $G = SU(2)$.

2. Degree of Cocycle

In this section we introduce the notion of degree in case $G = SU(2)$. For a given matrix $A = [a_{ij}]_{i,j=1,\dots,d} \in M_d(\mathbb{C})$ define $\|A\| = \sqrt{\frac{1}{d}\sum_{i,j=1}^{d} |a_{ij}|^2}$. Observe that if A is an element of the Lie algebra $\mathfrak{su}(2)$, i.e.

$$A = \begin{bmatrix} ia & b+ic \\ -b+ic & -ia \end{bmatrix},$$

where $a,b,c \in \mathbb{R}$, then $\|A\| = \sqrt{\det A}$. Moreover, if B is an element of the group $SU(2)$, i.e.

$$B = \begin{bmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{bmatrix},$$

where $z_1, z_2 \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1$, then $\mathrm{Ad}_B A = BAB^{-1} \in \mathfrak{su}(2)$ and $\|\mathrm{Ad}_B A\| = \|A\|$.

Consider the scalar product of $\mathfrak{su}(2)$ given by

$$\langle A, B\rangle = -\frac{1}{8}\mathrm{tr}(\mathrm{ad}A \circ \mathrm{ad}B).$$

Then $\|A\| = \sqrt{\langle A, A\rangle}$. By $L^2(X, \mathfrak{su}(2))$ we mean the space of all functions $f : X \to \mathfrak{su}(2)$ such that

$$\|f\|_{L^2} = \sqrt{\int_X \|f(x)\|^2 dx} < \infty.$$

For two $f_1, f_2 \in L^2(X, \mathfrak{su}(2))$ set

$$\langle f_1, f_2\rangle_{L^2} = \int_X \langle f_1(x), f_2(x)\rangle dx.$$

The space $L^2(X, \mathfrak{su}(2))$ endowed with the above scalar product is a Hilbert space.

By $L^1(X,\mathfrak{su}(2))$ we mean the space of all functions $f : X \to \mathfrak{su}(2)$ such that

$$\|f\|_{L^1} = \int_X \|f(x)\|dx < \infty.$$

The space $L^1(X,\mathfrak{su}(2))$ endowed with the norm $\|\ \|_{L^1}$ is a Banach space.

For a given measurable cocycle $\varphi : \mathbb{T} \to SU(2)$ consider the unitary operator

$$U : L^2(\mathbb{T},\mathfrak{su}(2)) \to L^2(\mathbb{T},\mathfrak{su}(2)), \quad Uf(x) = \mathrm{Ad}_{\varphi(x)} f(Tx). \tag{1}$$

Then $U^n f(x) = \mathrm{Ad}_{\varphi^{(n)}(x)} f(T^n x)$ for any integer n.

Lemma 2.1. *There exists an operator* $P : L^2(\mathbb{T},\mathfrak{su}(2)) \to L^2(\mathbb{T},\mathfrak{su}(2))$ *such that*

$$\frac{1}{n}\sum_{j=0}^{n-1} U^j f \to Pf \quad \text{in} \quad L^2(\mathbb{T},\mathfrak{su}(2))$$

for any $f \in L^2(\mathbb{T},\mathfrak{su}(2))$ *and* $U \circ P = P$. *Moreover,* $\|Pf\|$ *is constant* λ*-a.e..*

Proof. The first claim of the lemma follows from the von Neuman ergodic theorem. Since $U \circ P = P$, we have $\mathrm{Ad}_{\varphi(x)} Pf(Tx) = Pf(x)$, for λ-a.e. $x \in \mathbb{T}$. It follows that $\|Pf(Tx)\| = \|Pf(x)\|$, for λ-a.e. $x \in \mathbb{T}$. Hence $\|Pf(x)\| = c$, for λ-a.e. $x \in \mathbb{T}$, by the ergodicity of T. □

Lemma 2.2. *For every* $f \in L^2(\mathbb{T},\mathfrak{su}(2))$ *the sequence* $\frac{1}{n}\sum_{j=0}^{n-1} U^j f$ *converges* λ*-almost everywhere.*

Proof. Let $\tilde{f} \in L^2(\mathbb{T} \times SU(2),\mathfrak{su}(2))$ be given by $\tilde{f}(x,g) = \mathrm{Ad}_g f(x)$. Then

$$\tilde{f}(T^n_\varphi(x,g)) = \mathrm{Ad}_g(U^n f(x))$$

for any integer n. By the Birkhoff ergodic theorem, the sequence

$$\frac{1}{n}\sum_{j=0}^{n-1} \tilde{f}(T^n_\varphi(x,g)) = \mathrm{Ad}_g\left(\frac{1}{n}\sum_{j=0}^{n-1} U^j f(x)\right)$$

converges for $\lambda \otimes \mu$-a.e. $(x,g) \in \mathbb{T} \times SU(2)$. Hence there exists $g \in SU(2)$ such that $\mathrm{Ad}_g(\frac{1}{n}\sum_{j=0}^{n-1} U^j f(x))$ converges for λ-a.e. $x \in \mathbb{T}$, and the proof is complete. □

Recall that, if a function $\varphi : \mathbb{T} \to SU(2)$ is of class C^1, then $D\varphi(x)\varphi(x)^{-1} \in \mathfrak{su}(2)$ for every $x \in \mathbb{T}$.

Lemma 2.3. *For every* C^1*-cocycle* $\varphi : \mathbb{T} \to SU(2)$*, there exists* $\psi \in L^2(\mathbb{T},\mathfrak{su}(2))$ *such that*

$$\frac{1}{n} D\varphi^{(n)}(\varphi^{(n)})^{-1} \to \psi \text{ in } L^2(\mathbb{T},\mathfrak{su}(2)) \text{ and } \lambda\text{-almost everywhere.}$$

Moreover, $\|\psi\|$ *is a constant function* λ*-a.e. and* $\varphi(x)\psi(Tx)\varphi(x)^{-1} = \psi(x)$ *for* λ*-a.e.* $x \in \mathbb{T}$.

Proof. Since

$$D\varphi^{(n)}(x) = \sum_{j=0}^{n-1} \varphi(x)\ldots\varphi(T^{j-1}x)D\varphi(T^j x)\varphi(T^{j+1}x)\ldots\varphi(T^{n-1}x),$$

we have

$$\begin{aligned} &D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1} \\ &\quad = \sum_{j=0}^{n-1} \varphi(x)\dots\varphi(T^{j-1}x)D\varphi(T^jx)\varphi(T^jx)^{-1}\varphi(T^{j-1}x)^{-1}\dots\varphi(x)^{-1} \\ &\quad = \sum_{j=0}^{n-1} \varphi^{(j)}(x)D\varphi(T^jx)\varphi(T^jx)^{-1}(\varphi^{(j)}(x))^{-1} \\ &\quad = \sum_{j=0}^{n-1} U^j(D\varphi\varphi^{-1})(x), \end{aligned}$$

where U is the unitary operator given by (1). Put $\psi = P(D\varphi\varphi^{-1})$. Applying Lemmas 2.1 and 2.2, we conclude that

$$\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1} \to \psi \text{ in } L^2(\mathbb{T}, \mathfrak{su}(2)) \text{ and } \lambda\text{-almost everywhere.}$$

Moreover,

$$U\psi = U \circ P(D\varphi\varphi^{-1}) = P(D\varphi\varphi^{-1}) = \psi$$

and $\|\psi\| = \|P(D\varphi\varphi^{-1})\|$ is a constant function λ-a.e., by Lemma 2.1, which completes the proof. □

Definition 1. The number $\|\psi\|$ will be called the *degree* of the cocycle φ and denoted by $d(\varphi)$.

Lemma 2.3 shows that

$$\frac{1}{n}\|D\varphi^{(n)}(\varphi^{(n)})^{-1}\|_{L^1} \to d(\varphi).$$

On the other hand, $\|D\varphi^{(n)}(\varphi^{(n)})^{-1}\|_{L^1}$ is the length of the curve $\varphi^{(n)}$. Geometrically speaking, the degree of φ is the limit of length $(\varphi^{(n)})/n$.

A measurable cocycle $\delta : \mathbb{T} \to SU(2)$ is said to be *diagonal* if there exists a measurable function $\gamma : \mathbb{T} \to \mathbb{T}$ such that

$$\delta(x) = \begin{bmatrix} \gamma(x) & 0 \\ 0 & \dfrac{0}{\gamma(x)} \end{bmatrix}.$$

Theorem 2.4. *Suppose that $\varphi : \mathbb{T} \to SU(2)$ is a C^1-cocycle with $d(\varphi) \neq 0$. Then φ is cohomologous to a diagonal cocycle.*

Proof. For every nonzero $A \in \mathfrak{su}(2)$ there exists $B_A \in SU(2)$ such that

$$B_A A (B_A)^{-1} = \begin{bmatrix} i\|A\| & 0 \\ 0 & -i\|A\| \end{bmatrix}.$$

Indeed, if $A = \begin{bmatrix} ia & b+ic \\ -b+ic & -ia \end{bmatrix}$, then we can take

$$B_A = \begin{cases} \begin{bmatrix} i\sqrt{\dfrac{\|A\|+a}{2\|A\|}}\,\dfrac{b-ic}{|b+ic|} & \sqrt{\dfrac{\|A\|-a}{2\|A\|}} \\ -\sqrt{\dfrac{\|A\|-a}{2\|A\|}} & -i\sqrt{\dfrac{\|A\|+a}{2\|A\|}}\,\dfrac{b+ic}{|b+ic|} \end{bmatrix} & \text{if } |a| \neq \|A\|, \\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \text{if } a = -\|A\|, \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } a = \|A\|. \end{cases}$$

Set $p(x) = B_{\psi(x)}$. Then $p : \mathbb{T} \to SU(2)$ is a measurable function and

$$\psi(x) = p(x)^{-1} \begin{bmatrix} i\,d(\varphi) & 0 \\ 0 & -i\,d(\varphi) \end{bmatrix} p(x).$$

Since $\varphi(x)\psi(Tx)\varphi(x)^{-1} = \psi(x)$, we have

$$\varphi(x)p(Tx)^{-1} \begin{bmatrix} i\,d(\varphi) & 0 \\ 0 & -i\,d(\varphi) \end{bmatrix} p(Tx)\varphi(x)^{-1} = p(x)^{-1} \begin{bmatrix} i\,d(\varphi) & 0 \\ 0 & -i\,d(\varphi) \end{bmatrix} p(x).$$

Hence

$$p(x)\varphi(x)p(Tx)^{-1} \begin{bmatrix} i\,d(\varphi) & 0 \\ 0 & -i\,d(\varphi) \end{bmatrix} = \begin{bmatrix} i\,d(\varphi) & 0 \\ 0 & -i\,d(\varphi) \end{bmatrix} p(x)\varphi(x)p(Tx)^{-1}.$$

Since $d(\varphi) \neq 0$, we see that the cocycle $\delta : \mathbb{T} \to SU(2)$ defined by $\delta(x) = p(x)\varphi(x)p(Tx)^{-1}$ is diagonal. □

For a given C^1-cocycle $\varphi : \mathbb{T} \to SU(2)$ with nonzero degree let $\gamma = \gamma(\varphi) : \mathbb{T} \to \mathbb{T}$ be a measurable cocycle such that the cocycles φ and $\begin{bmatrix} \gamma & 0 \\ 0 & \overline{\gamma} \end{bmatrix}$ are cohomologous. It is easy to check that the choice of γ is unique up to a measurable cohomology with values in the circle and up to the complex conjugacy.

Theorem 2.4 shows that if $d(\varphi) \neq 0$, then the skew product T_φ is metrically isomorphic to a skew product of an irrational rotation on the circle and a diagonal cocycle. It follows that T_φ is not ergodic. However, in the next sections we show that if $d(\varphi) \neq 0$, then φ is not cohomologous to a constant cocycle. Moreover, the skew product $T_\gamma : \mathbb{T} \times \mathbb{T} \to \mathbb{T} \times \mathbb{T}$ is ergodic and it is mixing on the ortho-complement of the space of functions depending only on the first variable. We will prove also that (with some additional assumptions on φ) the Lebesgue component in the spectrum of T_γ has countable multiplicity. It follows that if $d(\varphi) \neq 0$, then:
- all ergodic components of T_φ are metrically isomorphic to T_γ,
- the spectrum of T_φ consists of two parts: discrete and mixing,
- (with some additional assumptions on φ) the Lebesgue component in the spectrum of T_φ has countable multiplicity.

In case $G = \mathbb{T}$, the topological degree of each C^1-cocycle is an integer number. An important question is: what can one say on values of degree in case $G = SU(2)$?

If a cocycle φ is cohomologous to a diagonal cocycle via a smooth transfer function, then $d(\varphi) \in 2\pi\mathbb{N}_0 = 2\pi(\mathbb{N} \cup \{0\})$.

We call a function $f : \mathbb{T} \to SU(2)$ absolutely continuous if $f_{ij} : \mathbb{T} \to \mathbb{C}$ is absolutely continuous for $i, j = 1, 2$. Suppose that φ is cohomologous to a diagonal cocycle via an absolutely continuous transfer function. Then φ can be represented as $\varphi(x) = p(x)^{-1}\delta(x)p(Tx)$, where $\delta, p : \mathbb{T} \to SU(2)$ are absolutely continuous and δ is diagonal. Since $\varphi^{(n)}(x) = p(x)^{-1}\delta^{(n)}(x)p(T^n x)$, we have

$$\begin{aligned}\frac{1}{n}D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1} &= \frac{1}{n}(-p(x)^{-1}Dp(x) + \varphi^{(n)}(x)p(T^n x)^{-1}Dp(T^n x)(\varphi^{(n)}(x))^{-1}\\ &\quad + p(x)^{-1}D\delta^{(n)}(x)(\delta^{(n)}(x))^{-1}p(x)).\end{aligned}$$

On the other hand, $\delta(x) = \begin{bmatrix} \gamma(x) & 0 \\ 0 & \overline{\gamma(x)} \end{bmatrix}$, where $\gamma : \mathbb{T} \to \mathbb{T}$ is an absolutely continuous cocycle of the form $\gamma(x) = \exp 2\pi i(\tilde{\gamma}(x) + kx)$, where k is the topological degree of γ and $\tilde{\gamma} : \mathbb{T} \to \mathbb{R}$ is an absolutely continuous function. Then

$$\frac{1}{n}D\gamma^{(n)}(x)(\gamma^{(n)}(x))^{-1} = 2\pi i\left(\frac{1}{n}\sum_{j=0}^{n-1} D\tilde{\gamma}(T^j x) + k\right) \to 2\pi ik$$

in $L^1(\mathbb{T}, \mathbb{R})$, by the Birkhoff ergodic theorem. It follows that

$$\frac{1}{n}D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1} \to p(x)^{-1}\begin{bmatrix} 2\pi ik & 0 \\ 0 & -2\pi ik \end{bmatrix}p(x)$$

in $L^1(\mathbb{T}, \mathfrak{su}(2))$. Hence $d(\varphi) = 2\pi|d(\gamma)| \in 2\pi\mathbb{N}_0$.

In Section 7, it is shown that if α is the golden ratio, then the degree of every C^2-cocycle belongs to $2\pi\mathbb{N}_0$, too.

3. Notation and Facts From Spectral Theory

Let U be a unitary operator on a separable Hilbert space $\mathcal{H}$. By the *cyclic space* generated by $f \in \mathcal{H}$ we mean the space $\mathbb{Z}(f) = \operatorname{span}\{U^n f; n \in \mathbb{Z}\}$. By the *spectral measure* σ_f of f we mean a Borel measure on $\mathbb{T}$ determined by the equalities

$$\hat{\sigma}_f(n) = \int_{\mathbb{T}} e^{2\pi inx} d\sigma_f(x) = \langle U^n f, f\rangle$$

for $n \in \mathbb{Z}$. Recall that there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that

$$\mathcal{H} = \oplus_{n=1}^{\infty}\mathbb{Z}(f_n) \quad \textit{and} \quad \sigma_{f_1} \gg \sigma_{f_2} \gg \ldots. \tag{2}$$

Moreover, for any sequence $\{f'_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ satisfying (2) we have $\sigma_{f_1} \equiv \sigma_{f'_1}$, $\sigma_{f_2} \equiv \sigma_{f'_2}, \ldots$. The above decompositions of $\mathcal{H}$ are called *spectral decompositions* of U.

The spectral type of σ_{f_1} (the equivalence class of measures) will be called the *maximal spectral type* of U. We say that U has *Lebesgue (continuous singular, discrete) spectrum* if σ_{f_1} is equivalent to Lebesgue (continuous singular, discrete) measure on the circle. An operator U is called *mixing* if

$$\hat{\sigma}_f(n) = \langle U^n f, f\rangle \to 0$$

for any $f \in \mathscr{H}$. We say that the spectrum of U has uniform multiplicity if either $\sigma_{f_n} \equiv \sigma_{f_1}$ or $\sigma_{f_n} \equiv 0$ for all natural n. We say that the Lebesgue component in the spectrum of U has countable multiplicity if $\lambda \ll \sigma_{f_n}$ for every natural n or equivalently if there exists a sequence $\{g_n\}_{n\in\mathbb{N}}$ in $\mathscr{H}$ such that the cyclic spaces $\mathbb{Z}(g_n)$ are pairwise orthogonal and $\sigma_{g_n} \equiv \lambda$ for every natural n.

For a skew product T_φ consider its Koopman operator

$$U_{T_\varphi} : L^2(\mathbb{T}\times G, \lambda\otimes\mu) \to L^2(\mathbb{T}\times G, \lambda\otimes\mu), U_{T_\varphi}f(x,g) = f(Tx, g\varphi(x)).$$

Denote by $\hat{G}$ the set of all equivalence classes of unitary irreducible representations of the group G. For any unitary irreducible representation $\Pi : G \to \mathscr{U}(\mathscr{H}_\Pi)$ by $\{\Pi_{ij}\}_{i,j=1}^{d_\Pi}$ we mean the matrix elements of Π, where $d_\Pi = \dim\mathscr{H}_\Pi$. Let us decompose

$$L^2(\mathbb{T}\times G) = \bigoplus_{\Pi\in\hat{G}} \bigoplus_{i=1}^{d_\pi} \mathscr{H}_i^\Pi,$$

where

$$\mathscr{H}_i^\Pi = \left\{ \sum_{j=1}^{d_\pi} \Pi_{ij}(g) f_j(x); f_j \in L^2(\mathbb{T},\lambda), j = 1,\ldots,d_\Pi \right\}$$
$$\simeq \overbrace{L^2(\mathbb{T},\lambda)\oplus\ldots\oplus L^2(\mathbb{T},\lambda)}^{d_\Pi}.$$

Observe that $\mathscr{H}_i^\Pi$ is a closed U_{T_φ}-invariant subspace of $L^2(\mathbb{T}\times G)$ and

$$U_{T_\varphi}^n \left(\sum_{j=1}^{d_\pi} \Pi_{ij}(g) f_j(x) \right) = \sum_{j,k=1}^{d_\pi} \Pi_{ik}(g)\Pi_{kj}(\varphi^{(n)}(x)) f_j(T^n x).$$

Consider the unitary operator $M_i^\Pi : \mathscr{H}_i^\Pi \to \mathscr{H}_i^\Pi$ given by

$$M_i^\Pi \left(\sum_{j=1}^{d_\pi} \Pi_{ij}(g) f_j(x) \right) = \sum_{j=1}^{d_\pi} e^{2\pi i x}\Pi_{ij}(g) f_j(x).$$

Then

$$U_{T_\varphi}^n M_i^\Pi f = e^{2\pi i n\alpha} M_i^\Pi U_{T_\varphi}^n f \tag{3}$$

for any $f \in \mathscr{H}_i^\Pi$. It follows that

$$\int_{\mathbb{T}} e^{2\pi i n x} d\sigma_{M_i^\Pi f}(x) = \langle U_{T_\varphi}^n M_i^\Pi f, M_i^\Pi f\rangle = e^{2\pi i n\alpha}\langle U_{T_\varphi}^n f, f\rangle = \int_{\mathbb{T}} e^{2\pi i n x} d(T^*\sigma_f)(x)$$

for any $f \in \mathscr{H}_i^\Pi$. Hence $\sigma_{M_i^\Pi f} = T^*\sigma_f$.

Lemma 3.1. *For every $\Pi \in \hat{G}$ and $i = 1,\ldots,d_\pi$ if the operator $U_{T_\varphi} : \mathscr{H}_i^\Pi \to \mathscr{H}_i^\Pi$ has absolutely continuous spectrum, then it has Lebesgue spectrum of uniform multiplicity.*

Proof. Let $\mathscr{H}_i^\Pi = \bigoplus_{n=1}^\infty \mathbb{Z}(f_n)$ be a spectral decomposition. Then

$$\mathscr{H}_i^\Pi = (M_i^\Pi)^m \mathscr{H}_i^\Pi = \bigoplus_{n=1}^\infty \mathbb{Z}((M_i^\Pi)^m f_n)$$

is a spectral decomposition for any integer m. Therefore $\sigma_{f_n} \equiv \sigma_{(M_i^\Pi)^m f_n} \ll \lambda$ for every natural n and integer m. Suppose that there exists a Borel set $A \subset \mathbb{T}$ such that $\sigma_{f_n}(A) = 0$ and $\lambda(A) > 0$. Then

$$\sigma_{f_n}\left(\bigcup_{m\in\mathbb{Z}} T^m A\right) = 0 \quad \text{and} \quad \lambda\left(\bigcup_{m\in\mathbb{Z}} T^m A\right) = 1,$$

by the ergodicity of T. It follows that $\sigma_{f_n} \equiv \lambda$ or $\sigma_{f_n} = 0$ for every natural n. □

Lemma 3.2. *If*

$$\sum_{n\in\mathbb{Z}} \left| \int_{\mathbb{T}} \Pi_{jj}(\varphi^{(n)}(x))dx \right|^2 < \infty$$

for $j = 1, \dots, d_\Pi$, *then* U_{T_φ} *has Lebesgue spectrum of uniform multiplicity on* $\mathscr{H}_i^\Pi$ *for* $i = 1, \dots, d_\Pi$.

Proof. Fix $1 \leqslant i \leqslant d_\Pi$. Note that

$$\langle U_{T_\varphi}^n \Pi_{ij}, \Pi_{ij}\rangle = \sum_{k=1}^{d_\Pi} \int_{\mathbb{T}} \int_G \langle \Pi_{ik}(g)\Pi_{kj}(\varphi^{(n)}(x)), \Pi_{ij}(g)\rangle dg dx = \frac{1}{d_\Pi}\int_{\mathbb{T}} \Pi_{jj}(\varphi^{(n)}(x))dx.$$

Since

$$\sum_{n\in\mathbb{Z}} |\langle U_{T_\varphi}^n \Pi_{ij}, \Pi_{ij}\rangle|^2 < \infty,$$

we have $\sigma_{\Pi_{ij}} \ll \lambda$ for $j = 1, \dots, d_\Pi$. From (3) we get $\sigma_{(M_i^\Pi)^m \Pi_{ij}} \ll \lambda$ for any integer m. Since $\{f \in \mathscr{H}_i^\Pi; \sigma_f \ll \lambda\}$ is a closed linear subspace of $L^2(\mathbb{T} \times G)$ and the set $\{(M_i^\Pi)^m \Pi_{ij}; j = 1, \dots, d_\Pi, m \in \mathbb{Z}\}$ generates the space $\mathscr{H}_i^\Pi$, it follows that U_{T_φ} has absolutely continuous spectrum on $\mathscr{H}_i^\Pi$. By Lemma 3.1, U_{T_φ} has Lebesgue spectrum of uniform multiplicity on $\mathscr{H}_i^\Pi$. □

Corollary 3.3. *For any* $\Pi \in \hat{G}$, *if*

$$\sum_{n\in\mathbb{Z}} \| \int_{\mathbb{T}} \Pi(\varphi^{(n)}(x))dx\|^2 < \infty,$$

then U_{T_φ} *has Lebesgue spectrum of uniform multiplicity on* $\oplus_{i=1}^{d_\Pi} \mathscr{H}_i^\Pi$.

Similarly one can prove the following result.

Theorem 3.4. *For any* $\Pi \in \hat{G}$, *if*

$$\lim_{n\to\infty} \int_{\mathbb{T}} \Pi(\varphi^{(n)}(x))dx = 0,$$

then U_{T_φ} *is mixing on* $\oplus_{i=1}^{d_\Pi} \mathscr{H}_i^\Pi$.

4. Representations of $SU(2)$

In this section, some basic information about the theory of representations of the group $SU(2)$ are presented. By $\mathscr{P}_k$ we mean the linear space of all homogeneous polynomials of degree $k \in \mathbb{N}_0$ in two variables u and v. Denote by Π_k the representation of the group $SU(2)$ in $\mathscr{P}_k$ given by

$$\left[\Pi_k\left(\begin{bmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{bmatrix}\right) f\right](u, v) = f(z_1 u - \overline{z_2} v, z_2 u + \overline{z_1} v).$$

Of course, all Π_k are unitary (under an appropriate inner product on $\mathscr{P}_k$) and the family $\{\Pi_0, \Pi_1, \Pi_2, \ldots\}$ is a complete family of continuous unitary irreducible representations of $SU(2)$. In the Lie algebra $\mathfrak{su}(2)$, we choose the following basis:

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Let V_k be a $k+1$-dimension linear space. For every natural k there exists a basis $v_0, \ldots, v_k$ of V_k such that the corresponding representation Π_k^* of $\mathfrak{su}(2)$ in V_k has the following form:

$$\begin{aligned} \Pi_k^*(e) v_i &= i(k - i + 1) v_{i-1}, \\ \Pi_k^*(f) v_i &= v_{i+1}, \\ \Pi_k^*(h) v_i &= (k - 2i) v_i \end{aligned}$$

for $i = 0, \ldots, k$. Then

$$\|A\| \leqslant \|\Pi_k^*(A)\| \leqslant k^2 \|A\| \tag{4}$$

for any $A \in \mathfrak{su}(2)$.

For abbreviation, we will write $(2k-1)!!$ instead of $1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-3) \cdot (2k-1)$ for any natural k.

Lemma 4.1.

$$\det \Pi_{2k-1}^*(A) = ((2k-1)!!)^2 (\det A)^k$$

for any $A \in \mathfrak{su}(2)$ and $k \in \mathbb{N}$.

Proof. For every $A \in \mathfrak{su}(2)$ there exists $B \in SU(2)$ and $d \in \mathbb{R}$ such that $A = \mathrm{Ad}_B \begin{bmatrix} id & 0 \\ 0 & -id \end{bmatrix}$. Then

$$\Pi_{2k-1}^*(A) = \Pi_{2k-1}^*\left(\mathrm{Ad}_B \begin{bmatrix} id & 0 \\ 0 & -id \end{bmatrix}\right) = \mathrm{Ad}_{\Pi_{2k-1}(B)} \Pi_{2k-1}^*\left(\begin{bmatrix} id & 0 \\ 0 & -id \end{bmatrix}\right).$$

It follows that

$$\begin{aligned} \det \Pi_{2k-1}^*(A) &= \det \Pi_{2k-1}^*\left(\begin{bmatrix} id & 0 \\ 0 & -id \end{bmatrix}\right) \\ &= ((2k-1)!!)^2 d^{2k} = ((2k-1)!!)^2 (\det A)^k. \end{aligned}$$

□

Lemma 4.2. *For any nonzero $A \in \mathfrak{su}(2)$ the matrix $\Pi^*_{2k-1}(A)$ is invertible. Moreover, for every natural k there exists a real constant $K_k > 0$ such that*

$$\|\Pi^*_{2k-1}(A)^{-1}\| \leqslant K_k \|A\|^{-1}$$

for every nonzero $A \in \mathfrak{su}(2)$.

Proof. The first claim of the lemma follows from Lemma 4.1. Set $C = \Pi^*_{2k-1}(A)$. Then

$$|[C]_{ij}| \leqslant (2k)^{4k}(2k-1)!\|A\|^{2k-1}$$

for $i, j = 1, \dots, 2k$. It follows that

$$|(C^{-1})_{ij}| = \frac{|[C]_{ij}|}{\det \Pi^*_{2k-1}(A)} \leqslant \frac{(2k)^{4k}(2k-1)!\|A\|^{2k-1}}{((2k-1)!!)^2\|A\|^{2k}} = \frac{(2k)^{4k}(2k-1)!}{((2k-1)!!)^2}\|A\|^{-1}.$$

Hence

$$\|C^{-1}\| \leqslant \frac{(2k)^{4k+1}(2k-1)!}{((2k-1)!!)^2}\|A\|^{-1}. \qquad \square$$

5. Ergodicity and Mixing of T_γ

Lemma 5.1. *Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $L^2(\mathbb{T}, \mathbb{C})$ such that $\int_0^x f_n(y)dy \to 0$ for any $x \in \mathbb{T}$. Let $g : \mathbb{T} \to \mathbb{C}$ be a bounded measurable function. Then*

$$\lim_{n\to\infty} \int_{\mathbb{T}} f_n(y)g(T^n y)dy = 0 \quad \textit{and} \quad \lim_{n\to\infty} \int_0^x f_n(y)g(y)dy = 0$$

for any $x \in \mathbb{T}$.

Proof. By assumption, the sequence $\{f_n\}_{n\in\mathbb{N}}$ tends to zero in the weak topology in $L^2(\mathbb{T}, \mathbb{C})$, which implies immediately the second claim of the lemma. Since $\{f_n\}_{n\in\mathbb{N}}$ converges weakly to zero, for every integer m we have

$$\lim_{n\to\infty} \int_{\mathbb{T}} f_n(T^{-n}y) \exp 2\pi i m y \, dy = \lim_{n\to\infty} \int_{\mathbb{T}} f_n(y) \exp 2\pi i m(y + n\alpha)dy = 0.$$

It follows that the sequence $\{f_n \circ T^{-n}\}_{n\in\mathbb{N}}$ converges weakly to zero. Therefore

$$\lim_{h\to\infty} \int_{\mathbb{T}} f_n(y)g(T^n y)dy = \lim_{h\to\infty} \int_{\mathbb{T}} f_n(T^{-n}y)g(y)dy = 0. \qquad \square$$

This gives immediately the following conclusion.

Corollary 5.2. *Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $L^2(\mathbb{T}, M_k(\mathbb{C}))$ (k is a natural number) such that $\int_0^x f_n(y)dy \to 0$ for any $x \in \mathbb{T}$. Let $g : \mathbb{T} \to M_k(\mathbb{C})$ be a bounded measurable function. Then*

$$\lim_{n\to\infty} \int_{\mathbb{T}} f_n(y)g(T^n y)dy = 0 \quad \textit{and} \quad \lim_{n\to\infty} \int_0^x f_n(y)g(y)dy = 0$$

for any $x \in \mathbb{T}$.

Theorem 5.3. *Let $\varphi : \mathbb{T} \to SU(2)$ be a C^1-cocycle with nonzero degree. Then the skew product $T_{\gamma(\varphi)} : \mathbb{T} \times \mathbb{T} \to \mathbb{T} \times \mathbb{T}$ is ergodic and it is mixing on the orthocomplement of the space of functions depending only on the first variable.*

Proof. By Theorem 3.4, it suffices to show that

$$\lim_{n\to\infty} \int_{\mathbb{T}} (\gamma^{(n)}(x))^k dx = 0$$

for every nonzero integer k. Fix $k \in \mathbb{N}$. Denote by $\psi : \mathbb{T} \to \mathfrak{su}(2)$ the limit (in $L^2(\mathbb{T}, \mathfrak{su}(2))$) of the sequence $\{\frac{1}{n} D\varphi^{(n)}(\varphi^{(n)})^{-1}\}_{n\in\mathbb{N}}$. Let $p : \mathbb{T} \to SU(2)$ be a measurable function such that

$$\begin{bmatrix} \gamma(x) & 0 \\ 0 & \overline{\gamma(x)} \end{bmatrix} = p(x)\varphi(x)p(Tx)^{-1} \quad \text{and} \quad \mathrm{Ad}_{p(x)}(\psi(x)) = \begin{bmatrix} id & 0 \\ 0 & -id \end{bmatrix},$$

where d is the degree of φ (see the proof of Theorem 2.4). Then

$$\begin{aligned} &\begin{bmatrix} (\gamma^{(n)})^k & & & & \\ & (\gamma^{(n)})^{k-2} & & 0 & \\ & & \ddots & & \\ & 0 & & (\gamma^{(n)})^{-k+2} & \\ & & & & (\gamma^{(n)})^{-k} \end{bmatrix} \\ &= \Pi_k(p)\Pi_k(\varphi^{(n)})\Pi_k(p \circ T^n)^{-1} \end{aligned} \tag{5}$$

for any natural n and

$$\begin{aligned} \mathrm{Ad}_{\Pi_k(p(x))}\Pi_k^*(\psi(x)) &= \Pi_k^*(\mathrm{Ad}_{p(x)}\psi(x)) \\ &= \Pi_k^*\left(\begin{bmatrix} id & 0 \\ 0 & -id \end{bmatrix}\right) \\ &= \begin{bmatrix} kid & & & & \\ & (k-2)id & & 0 & \\ & & \ddots & & \\ & 0 & & (-k+2)id & \\ & & & & -kid \end{bmatrix}. \end{aligned} \tag{6}$$

Recall that for any differentiable function $\xi : \mathbb{T} \to SU(2)$ and for any representation Π of $SU(2)$ we have

$$D(\Pi\xi(x))(\Pi\xi(x))^{-1} = \Pi^*(D\xi(x)\xi(x)^{-1}).$$

Therefore

$$\begin{aligned} \int_0^x \frac{1}{n}\Pi_k^*(D\varphi^{(n)}(y)(\varphi^{(n)}(y))^{-1})\Pi_k(\varphi^{(n)}(y))dy &= \int_0^x \frac{1}{n} D(\Pi_k\varphi^{(n)}(y))dy \\ &= \frac{1}{n}(\Pi_k(\varphi^{(n)}(x)) - \Pi_k(\varphi^{(n)}(0))) \end{aligned}$$

tends to zero for any $x \in \mathbb{T}$. Since

$$\frac{1}{n}\Pi_k^*(D\varphi^{(n)}(\varphi^{(n)})^{-1}) \to \Pi_k^*\psi$$

in $L^2(\mathbb{T}, M_{k+1}(\mathbb{C}))$, it follows that

$$\int_0^x \Pi_k^*(\psi(y))\Pi_k(\varphi^{(n)}(y))dy \to 0$$

for any $x \in \mathbb{T}$. By Corollary 5.2,

$$\int_{\mathbb{T}} \Pi_k(p(y))\Pi_k^*(\psi(y))\Pi_k(\varphi^{(n)}(y))\Pi_k(p(T^n y))^{-1}dy \to 0.$$

On the other hand,

$$\Pi_k(p(y))\Pi_k^*(\psi(y))\Pi_k(\varphi^{(n)}(y))\Pi_k(p(T^n y))^{-1}$$
$$= \begin{bmatrix} ikd(\gamma^{(n)}(y))^k & & 0 \\ & \ddots & \\ 0 & & -ikd(\gamma^{(n)}(y))^{-k} \end{bmatrix},$$

by (5) and (6). Therefore

$$\lim_{n\to\infty} \int_{\mathbb{T}} (\gamma^{(n)}(y))^m dy = 0$$

for any nonzero $m \in \{-k, -k+2, \ldots, k-2, k\}$, which completes the proof. $\square$

6. Spectral Analysis of Cocycles with Nonzero Degree

In this section, it is shown that for every cocycle $\varphi : \mathbb{T} \to SU(2)$ if $d(\varphi) \neq 0$ and if it satisfies some additional assumptions, then the Lebesgue component in the spectrum of T_φ has countable multiplicity.

Now we introduce a notation that is necessary to prove the main theory. Let $f, g : \mathbb{T} \to M_k(\mathbb{C})$ be functions of bounded variation (i.e. $f_{ij}, g_{ij} : \mathbb{T} \to \mathbb{C}$ have bounded variation for $i, j = 1, \ldots, k$) and let one of them be continuous. We will use the symbol $\int_{\mathbb{T}} f\, dg$ to denote the $k \times k$-matrix given by

$$\left(\int_{\mathbb{T}} f\, dg\right)_{ij} = \sum_{l=1}^{k} \int_{\mathbb{T}} f_{il}\, dg_{lj}$$

for $i, j = 1, \ldots, d$. It is clear that if g is absolutely continuous, then

$$\int_{\mathbb{T}} f\, dg = \int_{\mathbb{T}} f(x)Dg(x)dx. \tag{7}$$

Moreover, applying integration by parts, we have

$$\int_{\mathbb{T}} f\, dg = -\left(\int_{\mathbb{T}} g^T\, df^T\right)^T. \tag{8}$$

Theorem 6.1. *Let $\varphi : \mathbb{T} \to SU(2)$ be a C^2-cocycle with $d(\varphi) \neq 0$. Suppose that the sequence $\{\frac{1}{n} D\varphi^{(n)}(\varphi^{(n)})^{-1}\}_{n\in\mathbb{N}}$ is uniformly convergent and $\{D(\frac{1}{n} D\varphi^{(n)}(\varphi^{(n)})^{-1})\}_{n\in\mathbb{N}}$ is bounded in $L^1(\mathbb{T}, \mathfrak{su}(2))$. Then the Lebesgue component in the spectrum of T_φ has countable multiplicity. Moreover, the Lebesgue component in the spectrum of $T_{\gamma(\varphi)}$ has countable multiplicity, too.*

Proof. First, observe that it suffices to show that for every natural k there exists a real constant $C_k > 0$ such that

$$\left\| \int_{\mathbb{T}} \Pi_{2k-1}(\varphi^{(n)}(x))dx \right\| \leqslant \frac{C_k}{n} \tag{9}$$

for large enough natural n. Indeed, let $p : \mathbb{T} \to SU(2)$ be a measurable function such that

$$p(x)\varphi(x)p(Tx)^{-1} = \delta(x) = \begin{bmatrix} \gamma(x) & 0 \\ 0 & \overline{\gamma(x)} \end{bmatrix}.$$

Consider the unitary operator $V : \mathscr{H}_1^{\Pi_{2k-1}} \to \mathscr{H}_1^{\Pi_{2k-1}}$ given by

$$V\left(\sum_{i=1}^{d_{\Pi_{2k-1}}} \Pi_{1i}(g) f_i(x) \right) = \sum_{i,j=1}^{d_{\Pi_{2k-1}}} \Pi_{1i}(g)\Pi_{ji}(p(x)^{-1}) f_i(x).$$

Then

$$\begin{aligned} &V^{-1}U_{T_\varphi}V\left(\sum_{i=1}^{d_{\Pi_{2k-1}}} \Pi_{1i}(g) f_i(x) \right) \\ &= \sum_{i,j,l,m=1}^{d_{\Pi_{2k-1}}} \Pi_{1m}(g)\Pi_{ml}(p(x))\Pi_{lj}(\varphi(x))\Pi_{ji}(p(Tx)^{-1}) f_i(Tx) \\ &= \sum_{i=1}^{d_{\Pi_{2k-1}}} \Pi_{1i}(g)\Pi_{ii}(\delta(x)) f_i(Tx). \end{aligned}$$

From (9), $U_{T_\varphi} : \mathscr{H}_1^{\Pi_{2k-1}} \to \mathscr{H}_1^{\Pi_{2k-1}}$ has Lebesgue spectrum of uniform multiplicity, by Corollary 3.3. Hence $V^{-1}U_{T_\varphi}V$ has Lebesgue spectrum of uniform multiplicity and it is the product of the operators $U_j : L^2(\mathbb{T}, \mathbb{C}) \to L^2(\mathbb{T}, \mathbb{C})$ given by $U_j f(x) = (\gamma(x))^{2k-2j+1} f(Tx)$ for $j = 1, \dots, 2k$. Therefore U_j has absolutely continuous spectrum for $j = 1, \dots, 2k$. By Lemma 3.1, U_j has Lebesgue spectrum for all $j = 1, \dots, 2k$ and $k \in \mathbb{N}$. It follows that the Lebesgue component in the spectrum of $T_{\gamma(\varphi)}$ has countable multiplicity.

By assumption,

$$\left\| \frac{1}{n} D\varphi^{(n)}(\varphi^{(n)})^{-1} \right\| \to d(\varphi)$$

uniformly. Therefore

$$\left\| \frac{1}{n} D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1} \right\| \geqslant d(\varphi)/2 \tag{10}$$

for large enough natural n. For all $A,B\in M_k(\mathbb{C})$ we have $\|AB\|\leqslant\sqrt{k}\|A\|\,\|B\|$. Applying these facts, (7) and (8) we get

$$\begin{aligned}
&\left\|\int_{\mathbb{T}}\Pi_{2k-1}(\varphi^{(n)}(x))dx\right\|\\
&\quad=\left\|\int_{\mathbb{T}}\Pi_{2k-1}(\varphi^{(n)}(x))(D\Pi_{2k-1}(\varphi^{(n)}(x)))^{-1}d\Pi_{2k-1}(\varphi^{(n)}(x))\right\|\\
&\quad=\left\|\int_{\mathbb{T}}(\Pi^*_{2k-1}(D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}))^{-1}d\Pi_{2k-1}(\varphi^{(n)}(x))\right\|\\
&\quad=\left\|\int_{\mathbb{T}}(\Pi_{2k-1}(\varphi^{(n)}(x)))^T d((\Pi^*_{2k-1}(D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}))^{-1})^T\right\|\\
&\quad=\left\|\int_{\mathbb{T}}\Big[(\Pi_{2k-1}(\varphi^{(n)}(x)))^T((\Pi^*_{2k-1}(D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}))^{-1})^T\right.\\
&\qquad\left.(\Pi^*_{2k-1}D(D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}))^T((\Pi^*_{2k-1}(D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}))^{-1})^T\Big]dx\right\|\\
&\quad\leqslant 2k\int_{\mathbb{T}}[\|(\Pi^*_{2k-1}(D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}))^{-1}\|^2\\
&\qquad\|\Pi^*_{2k-1}D(D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1})\|]dx.
\end{aligned}$$

By Lemma 4.2, we have

$$\|(\Pi^*_{2k-1}(D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}))^{-1}\|\leqslant K_k\|D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}\|^{-1}.$$

From this, (4) and (10) we obtain

$$\begin{aligned}
&\left\|\int_{\mathbb{T}}\Pi_{2k-1}(\varphi^{(n)}(x))dx\right\|\\
&\quad\leqslant\frac{K_k^2(2k)^3}{n}\int_{\mathbb{T}}\left[\left\|\frac{1}{n}D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}\right\|^{-2}\left\|D\left(\frac{1}{n}D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}\right)\right\|\right]dx\\
&\quad\leqslant\frac{1}{n}\left(\frac{8K_kk^2}{d(\varphi)}\right)^2\left\|D\left(\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1}\right)\right\|_{L^1}
\end{aligned}$$

for large enough natural n. By assumption, there exists a real constant $M>0$ such that $\|D(\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1})\|_{L^1}\leqslant M$. Then

$$\left\|\int_{\mathbb{T}}\Pi_{2k-1}(\varphi^{(n)}(x))dx\right\|\leqslant\frac{C_k}{n}$$

for large enough natural n, where $C_k=(\frac{8K_kk^2}{d(\varphi)})^2M$. □

In this section we also present a class of cocycles satisfying the assumptions of Theorem 6.1. For $r=1,2$ let $L^r_+(\mathbb{T},\mathbb{R})=\{f\in L^r(\mathbb{T},\mathbb{R});f\geqslant 0\}$. We will need the following lemma.

Lemma 6.2. *Let $\{f_n:\mathbb{T}\to\mathbb{C}^d;n\in\mathbb{N}\}$ be a sequence of absolutely continuous functions. Assume that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges in $L^1(\mathbb{T},\mathbb{C}^d)$ to a function*

f and it is bounded for the sup norm. Suppose that there is a sequence $\{h_n\}_{n\in\mathbb{N}}$ convergent in $L^2_+(\mathbb{T},\mathbb{R})$ and a sequence $\{k_n\}_{n\in\mathbb{N}}$ bounded in $L^2_+(\mathbb{T},\mathbb{R})$ such that

$$\|Df_n(x)\| \leqslant h_n(x)k_n(x) \text{ for } \lambda\text{-a.e. } x\in\mathbb{T}$$

and for any natural n. Then $\{f_n\}_{n\in\mathbb{N}}$ converges to f uniformly.

Proof. Denote by $h\in L^2_+(\mathbb{T},\mathbb{R})$ the limit of the sequence $\{h_n\}_{n\in\mathbb{N}}$. Let $M>0$ be a real number such that $\|k_n\|_{L^2}\leqslant M$ for all natural n. First, observe that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is equicontinuous. Fix $\varepsilon>0$. Take $n_0\in\mathbb{N}$ such that $\|h_n-h\|_{L^2}<\varepsilon/2M$ for any $n\geqslant n_0$. Then for all $x,y\in\mathbb{T}$ and $n\geqslant n_0$ we have

$$\begin{aligned}\|f_n(x)-f_n(y)\| &= \left\|\int_x^y Df_n(t)dt\right\| \leqslant \int_x^y \|Df_n(t)\|dt \\ &\leqslant \int_x^y h_n(t)k_n(t)dt \leqslant \|k_n\|_{L^2}\sqrt{\int_x^y h_n^2(t)dt} \\ &\leqslant M\left(\sqrt{\int_x^y h^2(t)dt}+\|h_n-h\|_{L^2}\right) \leqslant M\left(\sqrt{\int_x^y h^2(t)dt}+\frac{\varepsilon}{2M}\right).\end{aligned}$$

Choose $\delta_1>0$ such that $|x-y|<\delta_1$ implies $\int_x^y h^2(t)dt<(\varepsilon/2M)^2$. Hence if $|x-y|<\delta_1$, then $\|f_n(x)-f_n(y)\|<\varepsilon$ for any $n\geqslant n_0$. Next choose $0<\delta\leqslant\delta_1$ such that $|x-y|<\delta$ implies $\|f_n(x)-f_n(y)\|<\varepsilon$ for any $n\leqslant n_0$. It follows that if $|x-y|<\delta$, then $\|f_n(x)-f_n(y)\|<\varepsilon$ for every natural n.

By the Arzela-Ascoli theorem, for any subsequence of $\{f_n\}_{n\in\mathbb{N}}$ there exists a subsequence convergent to f uniformly. Consequently, the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to f uniformly. □

This gives the following corollary.

Corollary 6.3. *Let $\{f_n:\mathbb{T}\to\mathbb{C}^d; n\in\mathbb{N}\}$ be a sequence of absolutely continuous functions. Assume that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges in $L^1(\mathbb{T},\mathbb{C}^d)$ to a function f and it is bounded in the sup norm. Suppose that there is a sequence $\{h_n\}_{n\in\mathbb{N}}$ convergent in $L^2_+(\mathbb{T},\mathbb{R})$, a sequence $\{k_n\}_{n\in\mathbb{N}}$ bounded in $L^2_+(\mathbb{T},\mathbb{R})$ and a sequence $\{l_n\}_{n\in\mathbb{N}}$ convergent in $L^1_+(\mathbb{T},\mathbb{R})$ such that*

$$\|Df_n(x)\| \leqslant l_n(x)+h_n(x)k_n(x) \text{ for } \lambda\text{-a.e. } x\in\mathbb{T}$$

and for any natural n. Then $\{f_n\}_{n\in\mathbb{N}}$ converges to f uniformly.

We will denote by $BV^2(\mathbb{T},SU(2))$ the set of all functions $f:\mathbb{T}\to SU(2)$ of bounded variation such that $Df(f)^{-1}\in L^2(\mathbb{T},\mathfrak{su}(2))$.

Lemma 6.4. *Let $\varphi:\mathbb{T}\to SU(2)$ be a C^2-cocycle. Suppose that φ is cohomologous to a diagonal cocycle with a transfer function in $BV^2(\mathbb{T},SU(2))$. Then the sequence $\{\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1}\}_{n\in\mathbb{N}}$ is uniformly convergent and $\{D(\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1})\}_{n\in\mathbb{N}}$ is bounded in $L^1(\mathbb{T},\mathfrak{su}(2))$.*

Proof. By Corollary 6.3, it suffices to show that there exist a sequence $\{h_n\}_{n\in\mathbb{N}}$ convergent in $L^2_+(\mathbb{T},\mathbb{R})$, a sequence $\{k_n\}_{n\in\mathbb{N}}$ bounded in $L^2_+(\mathbb{T},\mathbb{R})$ and a sequence

$\{l_n\}_{n\in\mathbb{N}}$ convergent in $L^1_+(\mathbb{T},\mathbb{R})$ such that

$$\left\|D\left(\frac{1}{n}D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}\right)\right\| \leqslant l_n(x)+h_n(x)k_n(x) \text{ for } \lambda\text{-a.e. } x\in\mathbb{T}.$$

By assumption, there exist $\delta, p\in BV^2(\mathbb{T},SU(2))$ such that $\varphi(x)=p(x)^{-1}\delta(x)p(Tx)$, where δ is a diagonal cocycle. Then

$$\begin{aligned}D\varphi(x)\varphi(x)^{-1} = {}& -p(x)^{-1}Dp(x)+p(x)^{-1}D\delta(x)\delta(x)^{-1}p(x)\\ &+\varphi(x)p(Tx)^{-1}Dp(Tx)\varphi(x)^{-1}\end{aligned}$$

for λ-a.e. $x\in\mathbb{T}$. Set

$$\tilde{\varphi}(x)=D\varphi(x)\varphi(x)^{-1}, \tilde{p}(x)=p(x)^{-1}Dp(x) \text{ and } \tilde{\delta}(x)=p(x)^{-1}D\delta(x)\delta(x)^{-1}p(x).$$

Then $\tilde{\varphi}(x)=-\tilde{p}(x)+U\tilde{p}(x)+\tilde{\delta}(x)$, where $\tilde{p},\tilde{\delta}\in L^2(\mathbb{T},\mathfrak{su}(2))$. We adopt the convention that $\sum_{j=0}^{-1}=0$. Since

$$\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1}=\frac{1}{n}\sum_{k=0}^{n-1}\varphi^{(k)}\tilde{\varphi}\circ T^k(\varphi^{(k)})^{-1},$$

we have

$$\begin{aligned}&D\left(\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1}\right)\\ &\quad=\frac{1}{n}\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}(\mathrm{Ad}_{\varphi^{(j)}}(\tilde{\varphi}\circ T^j)\mathrm{Ad}_{\varphi^{(k)}}(\tilde{\varphi}\circ T^k)-\mathrm{Ad}_{\varphi^{(k)}}(\tilde{\varphi}\circ T^k)\mathrm{Ad}_{\varphi^{(j)}}(\tilde{\varphi}\circ T^j))\\ &\qquad+\frac{1}{n}\sum_{k=0}^{n-1}\mathrm{Ad}_{\varphi^{(k)}}(D\tilde{\varphi}\circ T^k)\\ &\quad=\frac{1}{n}\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}[U^j\tilde{\varphi},U^k\tilde{\varphi}]+\frac{1}{n}\sum_{k=0}^{n-1}U^k(D\tilde{\varphi}).\end{aligned}$$

However,

$$\begin{aligned}\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}[U^j\tilde{\varphi},U^k\tilde{\varphi}] &= \sum_{k=0}^{n-1}\sum_{j=0}^{k-1}[U^{j+1}\tilde{p}-U^j\tilde{p}+U^j\tilde{\delta},U^k\tilde{\varphi}]\\ &=\sum_{k=0}^{n-1}[U^k\tilde{p}-\tilde{p},U^k\tilde{\varphi}]+\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}[U^j\tilde{\delta},U^{k+1}\tilde{p}-U^k\tilde{p}+U^k\tilde{\delta}]\\ &=\sum_{k=0}^{n-1}[U^k\tilde{p}-\tilde{p},U^k\tilde{\varphi}]+\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}[U^j\tilde{\delta},U^k\tilde{\delta}]\\ &\quad+\sum_{j=0}^{n-2}[U^j\tilde{\delta},U^n\tilde{p}-U^{j+1}\tilde{p}].\end{aligned}$$

Since

$$\begin{aligned} U^j\tilde{\delta}(x) &= \mathrm{Ad}_{\varphi^{(n)}(x)p(T^nx)^{-1}}(D\delta(T^nx)\delta(T^nx)^{-1}) \\ &= \mathrm{Ad}_{p(x)^{-1}\delta^{(n)}(x)}(D\delta(T^nx)\delta(T^nx)^{-1}) \\ &= \mathrm{Ad}_{p(x)^{-1}}(D\delta(T^nx)\delta(T^nx)^{-1}), \end{aligned}$$

we have $[U^j\tilde{\delta}, U^k\tilde{\delta}] = 0$ for any integers j,k. Observe that $\|[A,B]\| \leqslant 2\|A\|\,\|B\|$ for any $A,B \in \mathfrak{su}(2)$. It follows that

$$\begin{aligned} &\left\| D\left(\frac{1}{n} D\varphi^{(n)}(\varphi^{(n)})^{-1}\right)\right\| \\ &\quad \leqslant \frac{2}{n}\sum_{k=0}^{n-1}(\|D\tilde{\varphi}\circ T^k\| + \|\tilde{p}\circ T^k\|\,\|\tilde{\varphi}\circ T^k\| + \|\tilde{p}\|\,\|\tilde{\varphi}\circ T^k\| \\ &\quad\quad + \|\tilde{\delta}\circ T^k\|\,\|\tilde{p}\circ T^{k+1}\|) + \|\tilde{p}\circ T^n\|\frac{2}{n}\sum_{k=0}^{n-1}\|\tilde{\delta}\circ T^k\|. \end{aligned}$$

Set

$$\begin{aligned} h_n &= \frac{2}{n}\sum_{k=0}^{n-1}\|\tilde{\delta}\circ T^k\| \\ k_n &= \|\tilde{p}\circ T^n\| \\ l_n &= \frac{2}{n}\sum_{k=0}^{n-1}(\|D\tilde{\varphi}\circ T^k\| + \|\tilde{p}\circ T^k\|\,\|\tilde{\varphi}\circ T^k\| + \|\tilde{p}\|\,\|\tilde{\varphi}\circ T^k\| \\ &\quad + \|\tilde{\delta}\circ T^k\|\,\|\tilde{p}\circ T^{k+1}\|). \end{aligned}$$

By the Birkhoff ergodic theorem, the sequence $\{h_n\}_{n\in\mathbb{N}}$ converges in $L^2_+(\mathbb{T},\mathbb{R})$ and the sequence $\{l_n\}_{n\in\mathbb{N}}$ converges in $L^1_+(\mathbb{T},\mathbb{R})$. This completes the proof. □

Theorem 6.1 and Lemma 6.4 lead to the following conclusion.

Corollary 6.5. *Let $\varphi : \mathbb{T} \to SU(2)$ be a C^2-cocycle with $d(\varphi) \neq 0$. Suppose that φ is cohomologous to a diagonal cocycle with a transfer function in $BV^2(\mathbb{T}, SU(2))$. Then the Lebesgue component in the spectrum of T_φ has countable multiplicity. Moreover, the Lebesgue component in the spectrum of $T_{\gamma(\varphi)}$ has countable multiplicity, too.*

The following result will be useful in the next section of the paper.

Proposition 6.6. *For every C^2-cocycle $\varphi : \mathbb{T} \to SU(2)$, the sequence*

$$\frac{1}{n^2}\,D(D\varphi^{(n)}(\varphi^{(n)})^{-1})$$

converges to zero in $L^1(\mathbb{T}, \mathfrak{su}(2))$.

The following lemmas are some simple generalizations of some classical results. Their proofs are left to the reader.

Lemma 6.7. *Let $\{a_n\}_{n\in\mathbb{N}}$ be an increasing sequence of natural numbers and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in the Banach space $L^2(\mathbb{T},M_2(\mathbb{C}))$. Then*

$$\frac{f_{n+1}-f_n}{a_{n+1}-a_n}\to g \text{ in } L^2(\mathbb{T},M_2(\mathbb{C}))\Longrightarrow \frac{f_n}{a_n}\to g \text{ in } L^2(\mathbb{T},M_2(\mathbb{C})).$$

Lemma 6.8. *Let $\{g_k^n; n\in\mathbb{N}, 0\leqslant k<n\}$ be a triangular matrix of elements from $L^2(\mathbb{T},M_2(\mathbb{C}))$ such that $\|g_k^n\|=O(1/n)$ and*

$$g_0^n+g_1^n+\cdots+g_{n-1}^n\to g \quad \text{in} \quad L^2(\mathbb{T},M_2(\mathbb{C})).$$

Then $f_n\to f$ in $L^2(\mathbb{T},M_2(\mathbb{C}))$ implies

$$\sum_{k=0}^{n-1} g_k^n f_k\to gf \quad \text{and} \quad \sum_{k=0}^{n-1} f_k g_k^n\to f\,g \quad \text{in} \quad L^1(\mathbb{T},M_2(\mathbb{C})).$$

Proof of Proposition 6.6. First, recall that

$$\frac{1}{n^2}\,D(D\varphi^{(n)}(\varphi^{(n)})^{-1})=\frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}[U^j\tilde{\varphi},U^k\tilde{\varphi}]+\frac{1}{n^2}\sum_{k=0}^{n-1}U^k(D\tilde{\varphi}),$$

where $\tilde{\varphi}=D\varphi(\varphi)^{-1}$ and

$$\frac{1}{n}\sum_{k=0}^{n-1}U^k\tilde{\varphi}\to\psi \quad \text{in} \quad L^2(\mathbb{T},\mathfrak{su}(2)).$$

Since $\frac{1}{n^2}\sum_{k=0}^{n-1}U^k(D\tilde{\varphi})$ uniformly converges to zero, it suffices to show that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}U^j\tilde{\varphi}U^k\tilde{\varphi}=\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}U^k\tilde{\varphi}U^j\tilde{\varphi}=\frac{1}{2}\psi\psi \quad \text{in} \quad L^2(\mathbb{T},M_2(\mathbb{C})).$$

Set $f_n=\sum_{k=0}^{n-1}(n-k)U^k\tilde{\varphi}$ and $a_n=n^2$. Then

$$\frac{f_{n+1}-f_n}{a_{n+1}-a_n}=\frac{\sum_{k=0}^{n}(n+1-k)U^k\tilde{\varphi}-\sum_{k=0}^{n-1}(n-k)U^k\tilde{\varphi}}{(n+1)^2-n^2}=\frac{\sum_{k=0}^{n}U^k\tilde{\varphi}}{2n+1}\to\frac{1}{2}\psi$$

in $L^2(\mathbb{T},M_2(\mathbb{C}))$. Applying Lemma 6.7, we get

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(n-k)U^k\tilde{\varphi}\to\frac{1}{2}\psi \quad \text{in} \quad L^2(\mathbb{T},M_2(\mathbb{C})).$$

Therefore

$$\frac{1}{n^2}\sum_{k=0}^{n-1}kU^k\tilde{\varphi}=\frac{1}{n}\sum_{k=0}^{n-1}U^k\tilde{\varphi}-\frac{1}{n^2}\sum_{k=0}^{n-1}(n-k)\,U^k\tilde{\varphi}\to\psi-\frac{1}{2}\psi=\frac{1}{2}\psi$$

in $L^2(\mathbb{T},M_2(\mathbb{C}))$. Applying Lemma 6.8 with $g_k^n=\frac{k}{n^2}U^k\tilde{\varphi}$ and $f_k=\frac{1}{k}\sum_{j=0}^{k-1}U^j\tilde{\varphi}$, we conclude that

$$\sum_{k=0}^{n-1}g_k^nf_k=\frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}U^k\tilde{\varphi}U^j\tilde{\varphi}\to\frac{1}{2}\psi\psi$$

and

$$\sum_{k=0}^{n-1} f_k g_k^n = \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} U^j \tilde{\varphi} U^k \tilde{\varphi} \to \frac{1}{2} \psi\psi$$

in $L^2(\mathbb{T}, M_2(\mathbb{C}))$, which completes the proof. □

7. Possible Values of Degree

One may ask what we know about the set of possible values of degree. For $G = \mathbb{T}$ the degree of each smooth cocycle is an integer number. Probably, in the case of cocycles with values in $SU(2)$ the set of possible values of degree is more complicated. However, in this section, we show that if α is the golden ratio, then the degree of each smooth cocycle belongs to $2\pi\mathbb{N}_0$. The idea of renormalization, which is used to prove this result is due to Rychlik [8].

Let α be the golden ratio (i.e. the positive root of the equation $\alpha^2 + \alpha = 1$). It will be advantageous for our notation to consider the interval $[-\alpha^2, \alpha)$ to be the model of the circle. Then the map $T : [-\alpha^2, \alpha) \to [-\alpha^2, \alpha)$ given by

$$T(x) = \begin{cases} x + \alpha & \text{for } x \in [-\alpha^2, 0) \\ x - \alpha^2 & \text{for } x \in [0, \alpha) \end{cases}$$

is the rotation by α. Let $X = [-\alpha^2, \alpha^3)$. Then the first return time to X, which we call τ, satisfies the following formula

$$\tau(x) = \begin{cases} 1 & \text{for } x \in [0, \alpha^3) \\ 2 & \text{for } x \in [-\alpha^2, 0) \end{cases}$$

and the first return map $T_X : X \to X$ is equal to T up to a linear scaling. Indeed, if $M : \mathbb{T} \to X$ is the map given by $M(x) = -\alpha x$, then $T_X \circ M = M \circ T$.

By W^1 we mean the space of all cocycles $\varphi : \mathbb{T} \to SU(2)$ such that the functions $\varphi : [-\alpha^2, 0) \to SU(2), \varphi : [0, \alpha) \to SU(2)$ are both of class C^1 and

$$\lim_{x \to 0^-} D\varphi(x)\varphi(x)^{-1} \quad \text{and} \quad \lim_{x \to \alpha^-} D\varphi(x)\varphi(x)^{-1}$$

exist. The topology of W^1 is induced from $C^1((-\alpha^2, 0) \cup (0, \alpha))$. Consider the renormalization operator $\Phi : W^1 \to W^1$ defined by

$$\Phi\varphi(x) = \varphi^{(\tau(Mx))}(Mx).$$

Then

$$\Phi^n\varphi(x) = \begin{cases} \varphi^{(q_{n+1})}(M^n x) & \text{for} \quad x \in [-\alpha^2, 0) \\ \varphi^{(q_{n+2})}(M^n x) & \text{for} \quad x \in [0, \alpha) \end{cases}$$

for any natural n, where $\{q_n\}_{n \in \mathbb{N}}$ is the Fibonacci sequence. By W_0^1 we mean the set of all cocycles $\varphi \in W^1$ such that $\varphi^{(2)}$ is continuous at 0. The set W_0^1 is a closed subset of W^1 and

$$\Phi(W_0^1) \subset W_0^1 \tag{11}$$

(see [8]). It is easy to check that $\varphi \mapsto \|D\varphi(\varphi)^{-1}\|_{L^1}$ is a Lyapunov function for the renormalization map Φ, i.e. $\|D(\Phi\varphi)(\Phi\varphi)^{-1}\|_{L^1} \leqslant \|D\varphi(\varphi)^{-1}\|_{L^1}$ for any $\varphi \in W^1$. The following result is due to Rychlik [8].

Proposition 7.1. *If $\|D(\Phi^k\varphi)(\Phi^k\varphi)^{-1}\|_{L^1} = \|D\varphi(\varphi)^{-1}\|_{L^1}$ for all natural k, then*

$$D\varphi(x)(\varphi(x))^{-1} = \alpha\,\mathrm{Ad}_{\varphi(x)}[D\varphi(Tx)(\varphi(Tx))^{-1}]$$

for every $x \in [-\alpha^2, 0)$.

Lemma 7.2. *Let $\varphi : \mathbb{T} \to SU(2)$ be a C^2-cocycle. Assume that*

$$\frac{1}{n} D\varphi^{(n)}(0)(\varphi^{(n)}(0))^{-1} \to H \in \mathfrak{su}(2)$$

and there is an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of even numbers such that

$$\lim_{k\to\infty} \alpha^{n_k} \int_0^{\alpha^{n_k}} |D(D\varphi^{(q_{n_k+i})}(x)(\varphi^{(q_{n_k+i})}(x))^{-1})|dx = 0$$

for $i = 1$, 2. Then $\|H\| \in 2\pi\mathbb{N}_0$.

Proof. First, note that

$$D\Phi^n\varphi(x)(\Phi^n\varphi(x))^{-1} = \begin{cases} \alpha^n D\varphi^{(q_{n+1})}(M^n x)(\varphi^{(q_{n+1})}(M^n x))^{-1} & \text{for } x \in [-\alpha^2, 0) \\ \alpha^n D\varphi^{(q_{n+2})}(M^n x)(\varphi^{(q_{n+2})}(M^n x))^{-1} & \text{for } x \in [0, \alpha) \end{cases}$$

for any even n. Since

$$\begin{aligned} &\left|\frac{1}{q_{n+i}} D\varphi^{(q_{n+i})}(M^n x)(\varphi^{(q_{n+i})}(M^n x))^{-1} - \frac{1}{q_{n+i}} D\varphi^{(q_{n+i})}(0)(\varphi^{(q_{n+i})}(0))^{-1}\right| \\ &\quad\leqslant \frac{1}{q_{n+i}} \int_0^{\alpha^n x} |D(D\varphi^{(q_{n+i})}(\varphi^{(q_{n+i})})^{-1})|d\lambda \\ &\quad\leqslant \frac{1}{q_{n+i}\alpha^n} \alpha^n \int_0^{\alpha^n} |D(D\varphi^{(q_{n+i})}(\varphi^{(q_{n+i})})^{-1})|d\lambda \end{aligned}$$

for all even $n, i = 1$, 2 and

$$\lim_{n\to\infty} \alpha^n q_{n+1} = 1/(1+\alpha^2), \quad \lim_{n\to\infty} \alpha^n q_{n+2} = 1/(\alpha+\alpha^3),$$

it follows that

$$\begin{aligned} \lim_{k\to\infty} D\Phi^{n_k}\varphi(x)(\Phi^{n_k}\varphi(x))^{-1} &= \lim_{k\to\infty} \alpha^{n_k} q_{n_k+1} \frac{1}{q_{n_k+1}} D\varphi^{(q_{n_k+1})}(0)(\varphi^{(q_{n_k+1})}(0))^{-1} \\ &= \frac{1}{1+\alpha^2} H \end{aligned}$$

uniformly on $[-\alpha^2, 0)$ and

$$\begin{aligned} \lim_{k\to\infty} D\Phi^{n_k}\varphi(x)(\Phi^{n_k}\varphi(x))^{-1} &= \lim_{k\to\infty} \alpha^{n_k} q_{n_k+2} \frac{1}{q_{n_k+2}} D\varphi^{(q_{n_k+2})}(0)(\varphi^{(q_{n_k+2})}(0))^{-1} \\ &= \frac{1}{\alpha+\alpha^3} H \end{aligned}$$

uniformly on $[0, \alpha)$. Therefore we can assume that there exists $v \in W^1$ such that

$$\Phi^{n_k}\varphi \to v \quad \text{and} \quad D\Phi^{n_k}\varphi(\Phi^{n_k}\varphi)^{-1} \to Dv\, v^{-1}$$

uniformly. Then

$$Dv(x)(v(x))^{-1} = \begin{cases} \alpha A & \text{for } x \in [-\alpha^2, 0) \\ A & \text{for } x \in [0, \alpha), \end{cases}$$

where $A = 1/(\alpha + \alpha^3)H \in \mathfrak{su}(2)$. Therefore

$$v(x) = \begin{cases} e^{\alpha x A}B & \text{for } x \in [-\alpha^2, 0) \\ e^{xA}C & \text{for } x \in [0, \alpha), \end{cases}$$

where $B = v_-(0)$ and $C = v_+(0)$. Since the set $W_0^1 \subset W^1$ is closed and Φ-invariant, $v \in W_0^1$. It follows that

$$Ce^{-\alpha^3 A}B = Be^{\alpha A}C. \tag{12}$$

Since v is a limit point of the sequence $\{\Phi^n\varphi\}_{n\in\mathbb{N}}$ and $\varphi \mapsto \|D\varphi(\varphi)^{-1}\|_{L^1}$ is a Lyapunov function for the renormalization map Φ, we have $\|D\Phi^k v(\Phi^k v)^{-1}\|_{L^1} = \|Dvv^{-1}\|_{L^1}$ for any natural k. By Proposition 7.1,

$$\lim_{x\to 0^-} Dv(x)(v(x))^{-1} = \alpha \mathrm{Ad}_{v_-(0)} \lim_{x\to\alpha^-} Dv(x)(v(x))^{-1}.$$

Hence

$$\alpha A = \alpha \mathrm{Ad}_B(A)$$

and finally $AB = BA$. Therefore

$$\Phi v(x) = \begin{cases} e^{-\alpha x A}C & \text{for } x \in [-\alpha^2, 0) \\ e^{-xA+\alpha A}BC & \text{for } x \in [0, \alpha). \end{cases}$$

By Proposition 7.1,

$$\lim_{x\to 0^-} D\Phi v(x)(\Phi v(x))^{-1} = \alpha \mathrm{Ad}_{\Phi v_-(0)} \lim_{x\to\alpha^-} D\Phi v(x)(\Phi v(x))^{-1}.$$

Hence

$$-\alpha A = \alpha \mathrm{Ad}_C(-A)$$

and finally $AC = CA$. It follows that B and C commute, by (12). From (12), we obtain $e^{(\alpha+\alpha^3)A} = \mathrm{Id}$. Therefore $\|H\| = \|(\alpha + \alpha^3)A\| \in 2\pi\mathbb{N}_0$. □

Theorem 7.3. *Suppose that α is the golden ratio. Then for every C^2-cocycle $\varphi : \mathbb{T} \to SU(2)$, we have $d(\varphi) \in 2\pi\mathbb{N}_0$.*

Proof. Fix $n \in \mathbb{N}$ such that $2\alpha^{2n}[1/2\alpha^{2n}] \geqslant 4/5$. Set $I_j = [2(j-1)\alpha^{2n}, 2j\alpha^{2n}]$ for $j \in E = \{1, \ldots, [1/2\alpha^{2n}]\}$ and $\varepsilon_n = \frac{1}{q_n^2}\int_{\mathbb{T}} |D(D\varphi^{(q_n)}(\varphi^{(q_n)})^{-1})|d\lambda$. By Proposition 6.6, ε_n tends to zero. For $i = 1, 2$ define

$$E_i = \left\{ j \in E; \frac{1}{2\alpha^{2n}q_{2n+i}^2}\int_{I_j} |D(D\varphi^{(q_{2n+i})}(\varphi^{(q_{2n+i})})^{-1})|d\lambda \leqslant 10\varepsilon_{2n+i} \right\}.$$

Then

$$\begin{aligned}\varepsilon_{2n+i} &= \frac{1}{q_{2n+i}^2}\int_{\mathbb{T}} |D(D\varphi^{(q_{2n+i})}(\varphi^{(q_{2n+i})})^{-1})|d\lambda\\ &\geqslant \frac{1}{q_{2n+i}^2}\sum_{j\in E\setminus E_i}\int_{I_j}|D(D\varphi^{(q_{2n+i})}(\varphi^{(q_{2n+i})})^{-1})|d\lambda\cdot\\ &\geqslant 20\alpha^{2n}\varepsilon_{2n+i}([1/2\alpha^{2n}]-\#E_i)\end{aligned}$$

Hence

$$\#E_i \geqslant [1/2\alpha^{2n}]\left(1-\frac{1}{10}\frac{1/2\alpha^{2n}}{[1/2\alpha^{2n}]}\right)\geqslant \frac{7}{8}[1/2\alpha^{2n}]$$

for $i = 1, 2$. Therefore

$$\#(E_1\cap E_2)\geqslant \#E_1+\#E_2-\#E\geqslant \frac{3}{4}[1/2\alpha^{2n}].$$

Define

$$G_n = \bigcup_{j\in E_1\cap E_2}[(2j-2)\alpha^{2n},(2j-1)\alpha^{2n}].$$

Observe that $y\in G_n$ implies

$$\frac{1}{2\alpha^{2n}q_{2n+i}^2}\int_y^{y+\alpha^{2n}}|D(D\varphi^{(q_{2n+i})}(\varphi^{(q_{2n+i})})^{-1})|d\lambda\leqslant 10\varepsilon_{2n+i}$$

for $i = 1, 2$ and

$$\lambda(G_n)\geqslant \alpha^{2n}\#(E_1\cap E_2)\geqslant \frac{3}{8}2\alpha^{2n}[1/2\alpha^{2n}]\geqslant\frac{3}{10}.$$

Set $G' = \bigcap_{n\in\mathbb{N}}\bigcup_{k>n}G_k$. Then $\lambda(G')\geqslant 3/10$. Since $\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1}\to\psi$ almost everywhere, we see that the set

$$G = \left\{x\in G';\frac{1}{n}D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}\to\psi(x)\right\}$$

has positive measure.

For every $y\in\mathbb{T}$ denote by $\varphi_y:\mathbb{T}\to SU(2)$ the C^2-cocycle $\varphi_y(x)=\varphi(x+y)$. Suppose that $y\in G$. Then $\frac{1}{n}D\varphi_y^{(n)}(0)(\varphi_y^{(n)}(0))^{-1}\to\psi(y)$ and there exists an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers such that $y\in G_{n_k}$ for any natural k. Hence

$$\alpha^{2n_k}\int_0^{\alpha^{2n_k}}|D(D\varphi_y^{(q_{2n_k+i})}(\varphi_y^{(q_{2n_k+i})})^{-1})|d\lambda\leqslant 20(\alpha^{2n_k}q_{2n_k+i})^2\varepsilon_{2n_k+i}$$

for $i = 1, 2$. Since the sequence $\{\alpha^n q_{n+i}\}_{n\in\mathbb{N}}$ converges for $i = 1, 2$ and ε_n tends to zero, letting $k\to\infty$ we have

$$\lim_{k\to\infty}\alpha^{2n_k}\int_0^{\alpha^{2n_k}}|D(D\varphi_y^{(q_{2n_k+i})}(\varphi_y^{(q_{2n_k+i})})^{-1})|d\lambda = 0$$

for $i=1,2$. By Lemma 7.2, $\|\psi(y)\| \in 2\pi\mathbb{N}_0$ for every $y \in G$. Since $d(\varphi) = \|\psi(y)\|$ for a.e. $y \in \mathbb{T}$, we conclude that $d(\varphi) \in 2\pi\mathbb{N}_0$. □

8. The 2-Dimensional Case

This section will deal with properties of smooth cocycles over ergodic rotations on the 2-dimensional torus with values in $SU(2)$. By $\mathbb{T}^2$ we will mean the group $\mathbb{R}^2/\mathbb{Z}^2$. We will identify functions on $\mathbb{T}^2$ with functions on $\mathbb{R}^2$ periodic in each coordinate with period 1. Suppose that $T(x_1,x_2) = (x_1+\alpha, x_2+\beta)$ is an ergodic rotation on $\mathbb{T}^2$. Let $\varphi : \mathbb{T}^2 \to SU(2)$ be a C^1-cocycle over the rotation T. Analysis similar to that in Section 2 shows that there exists $\psi_i \in L^2(\mathbb{T}^2, \mathfrak{su}(2))$, $i=1,2$ such that

$$\frac{1}{n}\frac{\partial}{\partial x_i}\varphi^{(n)}(\varphi^{(n)})^{-1} \to \psi_i \quad \text{in} \quad L^2(\mathbb{T}^2, \mathfrak{su}(2)).$$

Moreover, $\|\psi_i\|$ is a $\lambda\otimes\lambda$-a.e. constant function and $\varphi(\bar{x})\psi_i(T\bar{x})\varphi(\bar{x})^{-1} = \psi_i(\bar{x})$ for $\lambda\otimes\lambda$-a.e. $\bar{x} \in \mathbb{T}\times\mathbb{T}$ for $i=1,2$.

Definition 2. The pair

$$(\|\psi_1\|, \|\psi_2\|) = \lim_{n\to\infty}\frac{1}{n}\left(\left\|\frac{\partial}{\partial x_1}\varphi^{(n)}(\varphi^{(n)})^{-1}\right\|_{L^1}, \left\|\frac{\partial}{\partial x_2}\varphi^{(n)}(\varphi^{(n)})^{-1}\right\|_{L^1}\right)$$

will be called the *degree* of the cocycle $\varphi : \mathbb{T}^2 \to SU(2)$ and denoted by $d(\varphi)$.

Similarly, one can prove the following

Theorem 8.1. *If $d(\varphi) \neq 0$, then φ is cohomologous to a diagonal cocycle $\mathbb{T}^2 \ni \bar{x} \mapsto \begin{bmatrix} \gamma(\bar{x}) & 0 \\ 0 & \overline{\gamma(\bar{x})} \end{bmatrix} \in SU(2)$, where $\gamma : \mathbb{T}^2 \to \mathbb{T}$ is measurable. Moreover, the skew product $T_\gamma : \mathbb{T}^2\times\mathbb{T} \to \mathbb{T}^2\times\mathbb{T}$ is ergodic and it is mixing on the orthocomplement of the space of functions depending only on the first two variables.*

Analysis similar to that in the proof of Theorem 6.1 gives

Theorem 8.2. *Let $\varphi : \mathbb{T}^2 \to SU(2)$ be a C^2-cocycle with $d(\varphi) \neq 0$. Suppose that the sequence $\{\frac{1}{n}\frac{\partial}{\partial x_i}\varphi^{(n)}(\varphi^{(n)})^{-1}\}_{n\in\mathbb{N}}$ is uniformly convergent and $\{\frac{\partial}{\partial x_j}(\frac{1}{n}\frac{\partial}{\partial x_i}\varphi^{(n)}(\varphi^{(n)})^{-1})\}_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{T}^2, \mathfrak{su}(2))$ for $i=1,2$. Then the Lebesgue component in the spectrum of T_φ has countable multiplicity.*

By $BV^{\mathscr{R}}(\mathbb{T}^2, SU(2))$ we mean the set of all measurable functions $f : \mathbb{T} \to SU(2)$ such that

- the functions $f(x,\cdot), f(\cdot,x) : \mathbb{T} \to SU(2)$ are of bounded variation for any $x \in \mathbb{T}$;
- the functions $\frac{\partial}{\partial x_1}f(f)^{-1}, \frac{\partial}{\partial x_2}f(f)^{-1} : \mathbb{T}^2 \to \mathfrak{su}(2)$ are Riemann integrable for $i=1,2$.

Then we immediately get the following

Lemma 8.3. *Let $\varphi : \mathbb{T}^2 \to SU(2)$ be a C^2-cocycle. Suppose that φ is cohomologous to a diagonal cocycle with a transfer function in $BV^{\mathscr{R}}(\mathbb{T}^2, SU(2))$.*

Then the sequence $\{\frac{1}{n}\frac{\partial}{\partial x_i}\varphi^{(n)}(\varphi^{(n)})^{-1}\}_{n\in\mathbb{N}}$ *is uniformly convergent and* $\{\frac{\partial}{\partial x_i}(\frac{1}{n}\frac{\partial}{\partial x_i}\varphi^{(n)}(\varphi^{(n)})^{-1})\}_{n\in\mathbb{N}}$ *is uniformly bounded for* $i = 1, 2$.

It is easy to check that if φ is cohomologous to a diagonal cocycle via a C^1 transfer function, then $d(\varphi) \in 2\pi(\mathbb{N}_0 \times \mathbb{N}_0)$. However, in the next section we show that for every ergodic rotation $T(x_1, x_2) = (x_1 + \alpha, x_2 + \beta)$ there exists a smooth cocycle whose degree is equal to $2\pi(|\beta|, |\alpha|)$.

9. Cocycles Over Flows

Let ω be an irrational number. By $S : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{T}^2$ we mean the ergodic flow defined by

$$S_t(x_1, x_2) = (x_1 + t\omega, x_2 + t). \tag{13}$$

Let $\varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2)$ be a smooth cocycle over S, i.e.

$$\varphi_{t+s}(\bar{x}) = \varphi_t(\bar{x})\varphi_s(S_t\bar{x})$$

for all $t, s \in \mathbb{R}, \bar{x} \in \mathbb{T}^2$ or equivalently, φ is the fundamental matrix solution for a linear differential system

$$\frac{d}{dt}y(t) = y(t)A(S_t\bar{x}),$$

where $A : \mathbb{T}^2 \to \mathfrak{su}(2)$, i.e. φ satisfies

$$\begin{cases} \frac{d}{dt}\varphi_t(\bar{x}) = \varphi_t(\bar{x})A(S_t\bar{x}) \\ \varphi_0(\bar{x}) = \mathrm{Id}. \end{cases}$$

Then

$$\frac{\partial}{\partial x_i}\varphi_{t+s}(\bar{x})\varphi_{t+s}(\bar{x})^{-1} = \frac{\partial}{\partial x_i}\varphi_t(\bar{x})\varphi_t(\bar{x})^{-1} + \mathrm{Ad}_{\varphi_t(\bar{x})}\frac{\partial}{\partial x_i}\varphi_s(S_t\bar{x})\varphi_s(S_t\bar{x})^{-1}.$$

Hence

$$\left\|\frac{\partial}{\partial x_i}\varphi_{t+s}(\varphi_{t+s})^{-1}\right\|_{L^1} \leqslant \left\|\frac{\partial}{\partial x_i}\varphi_t(\varphi_t)^{-1}\right\|_{L^1} + \left\|\frac{\partial}{\partial x_i}\varphi_s(\varphi_s)^{-1}\right\|_{L^1}.$$

It follows that the limit

$$\lim_{t\to\infty}\frac{1}{|t|}\left\|\frac{\partial}{\partial x_i}\varphi_t(\varphi_t)^{-1}\right\|_{L^1}$$

exists for $i = 1, 2$.

Definition 3. The pair

$$\lim_{t\to\infty}\frac{1}{|t|}\left(\left\|\frac{\partial}{\partial x_1}\varphi_t(\varphi_t)^{-1}\right\|_{L^1}, \left\|\frac{\partial}{\partial x_2}\varphi_t(\varphi_t)^{-1}\right\|_{L^1}\right)$$

will be called the *degree* of the cocycle $\varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2)$ and denoted by $d(\varphi)$.

For a given cocycle $\varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2)$ over the flow S, by $\hat{\varphi} : \mathbb{T} \to SU(2)$ we will mean the cocycle over the rotation $Tx = x + \omega$ defined by $\hat{\varphi}(x) = \varphi_1(x, 0)$. Then $\hat{\varphi}^{(n)}(x) = \varphi_n(x, 0)$.

Lemma 9.1. $d(\varphi) = (1, |\omega|)d(\hat{\varphi})$.

Proof. First, observe that

$$\begin{aligned}\varphi_{x_2}(x_1 - x_2\omega, 0)\varphi_n(x_1, x_2) &= \varphi_{n+x_2}(x_1 - x_2\omega, 0)\\ &= \varphi_n(x_1 - x_2\omega, 0)\varphi_{x_2}(x_1 - x_2\omega + n\omega, 0).\end{aligned}$$

Hence

$$\varphi_n(x_1, x_2) = \varphi_{x_2}(x_1 - x_2\omega, 0)^{-1}\hat{\varphi}^{(n)}(x_1 - x_2\omega)\varphi_{x_2}(x_1 - x_2\omega + n\omega, 0)$$

for all $x_1, x_2 \in \mathbb{R}$ and $n \in \mathbb{N}$. Fix $(x_1, x_2) \in [0,1] \times [0,1]$. Then

$$\begin{aligned}&\frac{\partial}{\partial x_1}\varphi_n(x_1, x_2)\varphi_n(x_1, x_2)^{-1}\\ &\quad = -\varphi_{x_2}(x_1 - x_2\omega, 0)^{-1}\frac{\partial}{\partial x_1}\varphi_{x_2}(x_1 - x_2\omega, 0)\\ &\qquad + \mathrm{Ad}_{\varphi_{x_2}(x_1 - x_2\omega, 0)^{-1}}(D\hat{\varphi}^{(n)}(x_1 - x_2\omega)\hat{\varphi}^{(n)}(x_1 - x_2\omega)^{-1})\\ &\qquad + \mathrm{Ad}_{\varphi_{x_2}(x_1 - x_2\omega, 0)^{-1}\hat{\varphi}^{(n)}(x_1 - x_2\omega)}\\ &\qquad\qquad \left(\frac{\partial}{\partial x_1}\varphi_{x_2}(x_1 - x_2\omega + n\omega, 0)\varphi_{x_2}(x_1 - x_2\omega + n\omega, 0)^{-1}\right).\end{aligned}$$

It follows that

$$\begin{aligned}&\left|\left\|\frac{\partial}{\partial x_1}\varphi_n(\varphi_n)^{-1}\right\|_{L^1} - \left\|D\hat{\varphi}^{(n)}(\hat{\varphi}^{(n)})^{-1}\right\|_{L^1}\right|\\ &\quad = \left|\left\|\frac{\partial}{\partial x_1}\varphi_n(\varphi_n)^{-1}\right\|_{L^1} - \int_0^1\int_0^1 \left\|D\hat{\varphi}^{(n)}(x_1 - x_2\omega)\hat{\varphi}^{(n)}(x_1 - x_2\omega)^{-1}\right\| dx_1 dx_2\right|\\ &\quad \leqslant 2\int_0^1\int_0^1 \left\|\frac{\partial}{\partial x_1}\varphi_{x_2}(x_1 - x_2\omega, 0)\varphi_{x_2}(x_1 - x_2\omega, 0)^{-1}\right\| dx_1 dx_2.\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\frac{1}{n}\left\|\frac{\partial}{\partial x_1}\varphi_n(\varphi_n)^{-1}\right\|_{L^1} = \lim_{n\to\infty}\frac{1}{n}\|D\hat{\varphi}^{(n)}(\hat{\varphi}^{(n)})^{-1}\|_{L^1} = d(\hat{\varphi}).$$

Similarly,

$$\begin{aligned}&\frac{\partial}{\partial x_2}\varphi_n(x_1, x_2)\varphi_n(x_1, x_2)^{-1}\\ &\quad = -\varphi_{x_2}(x_1 - x_2\omega, 0)^{-1}\frac{\partial}{\partial t}\varphi_{x_2}(x_1 - x_2\omega, 0)\\ &\qquad + \omega\varphi_{x_2}(x_1 - x_2\omega, 0)^{-1}\frac{\partial}{\partial x_1}\varphi_{x_2}(x_1 - x_2\omega, 0)\\ &\qquad - \omega\mathrm{Ad}_{\varphi_{x_2}(x_1 - x_2\omega, 0)^{-1}}(D\hat{\varphi}^{(n)}(x_1 - x_2\omega)\hat{\varphi}^{(n)}(x_1 - x_2\omega)^{-1})\end{aligned}$$

$$
\begin{aligned}
&+ \mathrm{Ad}_{\varphi_{x_2}(x_1-x_2\omega,0)^{-1}\hat{\varphi}^{(n)}(x_1-x_2\omega)}\\
&\qquad\left(\frac{\partial}{\partial t}\varphi_{x_2}(x_1-x_2\omega+n\omega,0)\varphi_{x_2}(x_1-x_2\omega+n\omega,0)^{-1}\right)\\
&- \omega\mathrm{Ad}_{\varphi_{x_2}(x_1-x_2\omega,0)^{-1}\hat{\varphi}^{(n)}(x_1-x_2\omega)}\\
&\qquad\left(\frac{\partial}{\partial x_1}\varphi_{x_2}(x_1-x_2\omega+n\omega,0)\varphi_{x_2}(x_1-x_2\omega+n\omega,0)^{-1}\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\left|\left\|\frac{\partial}{\partial x_2}\varphi_n(\varphi_n)^{-1}\right\|_{L^1} - |\omega|\|D\hat{\varphi}^{(n)}(\hat{\varphi}^{(n)})^{-1}\|_{L^1}\right|\\
&\leqslant 2\int_0^1\int_0^1\left\|\frac{\partial}{\partial t}\varphi_{x_2}(x_1-x_2\omega,0)\varphi_{x_2}(x_1-x_2\omega,0)^{-1}\right\|dx_1dx_2\\
&\quad + 2|\omega|\int_0^1\int_0^1\left\|\frac{\partial}{\partial x_1}\varphi_{x_2}(x_1-x_2\omega,0)\varphi_{x_2}(x_1-x_2\omega,0)^{-1}\right\|dx_1dx_2.
\end{aligned}
$$

Therefore

$$
\lim_{n\to\infty}\frac{1}{n}\left\|\frac{\partial}{\partial x_2}\varphi_n(\varphi_n)^{-1}\right\|_{L^1} = |\omega|\lim_{n\to\infty}\frac{1}{n}\left\|D\hat{\varphi}^{(n)}(\hat{\varphi}^{(n)})^{-1}\right\|_{L^1} = |\omega|d(\hat{\varphi}),
$$

and the proof is complete. □

Lemma 9.2. *For any C^2-cocycle $\psi : \mathbb{T} \to SU(2)$ over the rotation T there exists a C^2-cocycle $\varphi : \mathbb{R}\times\mathbb{T}^2 \to SU(2)$ over the flow S such that $\hat{\varphi} = \psi$.*

Proof. Since the fundamental group of $SU(2)$ is trivial, we can choose a C^2-homotopy $\psi : [0,1]\times\mathbb{T} \to SU(2)$ such that

$$
\psi(t,x) = \begin{cases} \mathrm{Id} & \text{for } t\in[0,1/4]\\ \psi(x) & \text{for } t\in[3/4,1].\end{cases}
$$

By $\varphi : \mathbb{R}\times\mathbb{T} \to SU(2)$ we mean the C^2-function determined by

$$
\psi(n+t,x) = \psi^{(n)}(x)\psi(t,x+n\omega)
$$

for any $t\in[0,1]$ and $n\in\mathbb{Z}$. Then it is easy to check that

$$
\psi(n+t,x) = \psi^{(n)}(x)\psi(t,x+n\omega) \tag{14}
$$

for any $t\in\mathbb{R}$ and $n\in\mathbb{Z}$. Let $\varphi : \mathbb{R}\times\mathbb{R}^2 \to SU(2)$ be defined by

$$
\varphi_t(x_1,x_2) = \psi(x_2, x_1-x_2\omega)^{-1}\psi(t+x_2, x_1-x_2\omega).
$$

It is easy to see that $\varphi_t(x_1+1,x_2) = \varphi_t(x_1,x_2)$ and $\varphi_t(x_1,x_2+1) = \varphi_t(x_1,x_2)$, by (14). Then $\varphi : \mathbb{R}\times\mathbb{T}^2 \to SU(2)$ is a C^2-function and

$$
\begin{aligned}
\varphi_{t+s}(\bar{x}) &= \psi(x_2,x_1-x_2\omega)^{-1}\psi(t+s+x_2,x_1-x_2\omega)\\
&= \psi(x_2,x_1-x_2\omega)^{-1}\psi(t+x_2,x_1-x_2\omega)\psi(x_2+t,(x_1+t\omega)-(x_2+t)\omega)^{-1}\\
&\quad\times\psi(s+(x_2+t),(x_1+t\omega)-(x_2+t)\omega)\\
&= \varphi_t(\bar{x})\varphi_s(S_t\bar{x}).
\end{aligned}
$$

Moreover,

$$\hat{\varphi}(x) = \varphi_1(x, 0) = \psi(0, x)^{-1}\psi(1, x) = \psi(x),$$

which completes the proof. □

Suppose that $\alpha, \beta, 1$ are independent over $\mathbb{Q}$. Set $\omega = \alpha/\beta$.

Theorem 9.3. *For every ergodic rotation $T(x_1, x_2) = (x_1 + \alpha, x_2 + \beta)$ and for every natural k there exists a C^2-cocycle over T whose degree is equal to $2\pi k(|\beta|, |\alpha|)$.*

Proof. Let S denote the ergodic flow given by (13). Suppose that $\varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2)$ is a C^2-cocycle over S such that $d(\hat{\varphi}) = 2\pi k$. Consider the cocycle $\varphi_\beta : \mathbb{T}^2 \to SU(2)$ over the rotation $T = S_\beta$. Then $\varphi_\beta^{(n)} = \varphi_{\beta_n}$ and

$$\lim_{n\to\infty} \frac{1}{n} \left\| \frac{\partial}{\partial x_i} \varphi_\beta^{(n)} (\varphi_\beta^{(n)})^{-1} \right\|_{L_1} = |\beta| \lim_{n\to\infty} \frac{1}{|\beta| n} \left\| \frac{\partial}{\partial x_i} \varphi_{\beta n} (\varphi_{\beta n})^{-1} \right\|_{L_1}.$$

It follows that

$$d(\varphi_\beta) = |\beta| d(\varphi) = |\beta|(1, |\omega|) d(\hat{\varphi}) = (|\beta|, |\alpha|) d(\hat{\varphi}),$$

which proves the theorem. □

Suppose that $\beta \in (0, 1)$. Let $\varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2)$ be a C^2-cocycle over S such that $\hat{\varphi}$ is a diagonal C^2-cocycle with nonzero degree. Set $T = S_\beta$ and $\psi = \varphi_\beta$. Let $p : \mathbb{T}^2 \to SU(2)$ be a $BV^{\mathcal{R}}$-function such that

$$p(x_1, x_2) = \varphi_{x_2}(x_1 - x_2\omega, 0)^{-1}$$

for $(x_1, x_2) \in \mathbb{R} \times [0, 1)$. Then

$$p(T(x_1, x_2)) = \begin{cases} \varphi_{x_2+\beta}(x_1 - x_2\omega, 0)^{-1} & \text{for} \quad x_2 \in [0, 1 - \beta) \\ \varphi_{x_2+\beta-1}(x_1 - (x_2 - 1)\omega, 0)^{-1} & \text{for} \quad x_2 \in [1 - \beta, 1). \end{cases}$$

Moreover,

$$\varphi_{x_2+\beta}(x_1 - x_2\omega, 0) = \varphi_{x_2}(x_1 - x_2\omega, 0)\varphi_\beta(x_1, x_2)$$

and

$$\begin{aligned} \varphi_{x_2+\beta-1}(x_1 - (x_2 - 1)\omega, 0) &= \varphi_{-1}(x_1 - (x_2 - 1)\omega, 0)\varphi_{x_2+\beta}(x_1 - x_2\omega, 0) \\ &= \varphi_1(x_1 - x_2\omega, 0)^{-1}\varphi_{x_2+\beta}(x_1 - x_2\omega, 0). \end{aligned}$$

It follows that $\psi(\bar{x}) = p(\bar{x})\delta(\bar{x})p(T\bar{x})^{-1}$, where $\delta : \mathbb{T}^2 \to SU(2)$ is the diagonal $BV^{\mathcal{R}}$-cocycle given by

$$\delta(x_1, x_2) = \begin{cases} \mathrm{Id} & \text{for} \quad x_2 \in [0, 1 - \beta) \\ \hat{\varphi}(x_1 - x_2\omega) & \text{for} \quad x_2 \in [1 - \beta, 1). \end{cases}$$

Lemma 9.4. *Let $\phi : \mathbb{T}^2 \to \mathbb{T}$ be a cocycle over the rotation $T(x_1, x_2) = (x_1 + \alpha, x_2 + \beta)$. Suppose that $\phi|\mathbb{T} \times [0, \gamma), \phi|\mathbb{T} \times [\gamma, 1)$ are C^1-functions, where γ is irrational. If $d(\phi(\cdot, 0)) \neq d(\phi(\cdot, \gamma))$, then ϕ is not a coboundary.*

Proof. Set $I_1=[0,\gamma), I_2=[\gamma,1), a_1=d(\phi(\cdot,0))$ and $a_2=d(\phi(\cdot,\gamma))$. Then there exists a function $\tilde{\phi}:\mathbb{T}^2\to\mathbb{R}$ such that $\tilde{\phi}|\mathbb{T}\times I_j$ is of class C^1 for $j=1,2$ and $\phi(x_1,x_2)=\exp 2\pi i(\tilde{\phi}(x_1,x_2)+a_jx_1)$ for any $(x_1,x_2)\in\mathbb{T}\times I_j$.

Clearly, it suffices to show that

$$\int_{\mathbb{T}^2}\phi^{(n)}(x_1,x_2)dx_1dx_2\to 0.$$

Next note that

$$\phi^{(n)}(x_1,x_2)=\exp 2\pi i(\tilde{\phi}^{(n)}(x_1,x_2)+(a_1S_1^n(x_2)+a_2S_2^n(x_2))x_1+c_n(x_2)),$$

where $S_i^n(x)=\sum_{k=0}^{n-1}\mathbf{1}_{I_i}(x+k\beta)$ and $c_n(x)=\sum_{k=0}^{n-1}k\alpha(a_1\mathbf{1}_{I_1}+a_2\mathbf{1}_{I_2})(x+k\beta)$. Since the rotation by β is uniquely ergodic,

$$\frac{1}{n}(a_1S_1^n+a_2S_2^n)\to a_1\gamma+a_2(1-\gamma)$$

uniformly. Since $a_1\neq a_2$ and γ is irrational, there exists $S>0$ and $n_0\in\mathbb{N}$ such that $|a_1S_1^n(x)+a_2S_2^n(x)|\geqslant nS$ for all $x\in\mathbb{T}$ and $n\geqslant n_0$. Applying integration by parts, we get

$$\begin{aligned}
&\left|\int_{\mathbb{T}^2}\phi^{(n)}(x_1,x_2)dx_1dx_2\right|\\
&\quad\leqslant\int_0^1\left|\int_0^1 e^{2\pi i(\tilde{\phi}^{(n)}(x_1,x_2)+(a_1S_1^n(x_2)+a_2S_2^n(x_2))x_1)}dx_1\right|dx_2\\
&\quad=\int_0^1\frac{1}{2\pi|a_1S_1^n(x_2)+a_2S_2^n(x_2)|}\left|\int_0^1 e^{2\pi i\tilde{\phi}^{(n)}(x_1,x_2)}de^{2\pi i(a_1S_1^n(x_2)+a_2S_2^n(x_2))x_1}\right|dx_2\\
&\quad=\int_0^1\frac{1}{2\pi|a_1S_1^n(x_2)+a_2S_2^n(x_2)|}\left|\int_0^1 e^{2\pi i(a_1S_1^n(x_2)+a_2S_2^n(x_2))x_1}de^{2\pi i\tilde{\phi}^{(n)}(x_1,x_2)}\right|dx_2\\
&\quad\leqslant\int_0^1\frac{1}{nS}\left|\int_0^1 e^{2\pi i(\tilde{\phi}^{(n)}(x_1,x_2)+(a_1S_1^n(x_2)+a_2S_2^n(x_2))x_1)}\frac{\partial}{\partial x_1}\tilde{\phi}^{(n)}(x_1,x_2)dx_1\right|dx_2\\
&\quad\leqslant\frac{1}{nS}\int_{\mathbb{T}^2}\left|\frac{\partial}{\partial x_1}\tilde{\phi}^{(n)}(x_1,x_2)\right|dx_1dx_2.
\end{aligned}$$

Since $\frac{\partial}{\partial x_1}\tilde{\phi}\in L^1(\mathbb{T}^2,\mathbb{C})$,

$$\frac{1}{n}\frac{\partial}{\partial x_1}\tilde{\phi}^{(n)}\to\int_{\mathbb{T}^2}\frac{\partial}{\partial x_1}\tilde{\phi}(x_1,x_2)dx_1dx_2=0$$

in $L^1(\mathbb{T}^2,\mathbb{C})$, by the Birkhoff ergodic theorem, and the proof is complete. □

This leads to the following conclusion.

Corollary 9.5. *For every ergodic rotation T on $\mathbb{T}^2$ there exists a C^2-cocycle ψ with nonzero degree such that the Lebesgue component in the spectrum of T_ψ has countable multiplicity and ψ is not cohomologous to any diagonal C^1-cocycle.*

Proof. Let $\check{\varphi} : \mathbb{T} \to \mathbb{T}$ be a C^2-function with nonzero topological degree. Let $\varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2)$ be a C^2-cocycle over S such that $\hat{\varphi} = \begin{bmatrix} \check{\varphi} & 0 \\ 0 & (\check{\varphi})^{-1} \end{bmatrix}$. Define $\psi = \varphi_\beta$. Then $d(\psi) = 2\pi(|\beta|, |\alpha|)|d(\check{\varphi})| \neq 0$. Moreover, ψ and the diagonal cocycle $\delta : \mathbb{T}^2 \to SU(2)$ given by

$$\delta(x_1, x_2) = \begin{cases} \mathrm{Id} & \text{for } x_2 \in [0, 1-\beta) \\ \hat{\varphi}(x_1 - x_2\omega) & \text{for } x_2 \in [1-\beta, 1) \end{cases}$$

are cohomologous with a transfer function in $BV^{\mathscr{R}}(\mathbb{T}^2, SU(2))$. Applying Theorem 8.2 and Lemma 8.3, we get the first part of our claim.

Next suppose that ψ is cohomologous to a diagonal C^1-cocycle. Then it is easy to see that the cocycle $\eta : \mathbb{T}^2 \to \mathbb{T}$ given by

$$\eta(x_1, x_2) = \begin{cases} \mathrm{Id} & \text{for } x_2 \in [0, 1-\beta) \\ \check{\varphi}(x_1 - x_2\omega) & \text{for } x_2 \in [1-\beta, 1). \end{cases}$$

is cohomologous to a C^1-cocycle $g : \mathbb{T}^2 \to \mathbb{T}$. Applying Lemma 9.4 for $\phi = \eta g^{-1}$ and $\gamma = 1 - \beta$ we find that ηg^{-1} is not a coboundary, which completes the proof. □

References

[1] Cornfeld IP, Fomin SW, Sinai JG (1982) Ergodic Theory. Berlin: Springer
[2] Furstenberg H (1961) Strict ergodicity and transformations on the torus. Amer J Math **83**: 573–601
[3] Gabriel P, Lemańczyk M, Liardet P (1991) Esemble d'invariants pour les produits croisés de Anzai. Mém Soc Math France **47**
[4] Helson H (1986) Cocycles on the circle. J Operator Th **16**: 189–199
[5] Iwanik A, Lemańczyk M, Rudolph D (1993) Absolutely continuous cocycles over irrational rotations. Israel J Math **83**: 73–95
[6] Kuipers L, Niederreiter H (1974) Uniform Distribution of Sequences. New York: Wiley
[7] Parry W (1981) Topics in Ergodic Theory. Cambridge: Univ Press
[8] Rychlik M (1992) Renormalization of cocycles and linear ODE with almost-periodic coefficients. Invent Math **110**: 173–206

Author's address: Faculty of Mathematics and Computer Science, Nicholas Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland, e-mail: fraczek@mat.uni.torun.pl