On Cocycles with Values in the Group SU(2)

By

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Abstract. In this paper we introduce the notion of degree for $C^1$-cocycles over irrational rotations on the circle with values in the group SU(2). It is shown that if a $C^1$-cocycle $\varphi : \mathbb{T} \to SU(2)$ over an irrational rotation by $\alpha$ has nonzero degree, then the skew product

$$
\mathbb{T} \times SU(2) \ni (x, g) \mapsto (x + \alpha, g \varphi(x)) \in \mathbb{T} \times SU(2)
$$

is not ergodic and the group of essential values of $\varphi$ is equal to the maximal Abelian subgroup of $SU(2)$. Moreover, if $\varphi$ is of class $C^2$ (with some additional assumptions) the Lebesgue component in the spectrum of the skew product has countable multiplicity. Possible values of degree are discussed, too.

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1. Introduction

Assume that $T : (X, \mathcal{B}, \lambda) \to (X, \mathcal{B}, \lambda)$ is an ergodic measure-preserving automorphism of standard Borel space. Let $G$ be a compact Lie group, $\mu$ its Haar measure. For a given measurable function $\varphi : X \to G$ we study spectral properties of the measure-preserving automorphism of $X \times G$ (called skew product) defined by

$$
T_\varphi : (X \times G, \lambda \otimes \mu) \to (X \times G, \lambda \otimes \mu), T_\varphi(x, g) = (Tx, g\varphi(x)).
$$

A measurable function $\varphi : X \to G$ determines a measurable cocycle over the automorphism $T$ given by

$$
\varphi^{(n)}(x) = \begin{cases} 
\varphi(x)\varphi(Tx)\ldots\varphi(T^{n-1}x) & \text{for } n > 0 \\
e & \text{for } n = 0 \\
(\varphi(T^n x)\varphi(T^{n+1} x)\ldots\varphi(T^{-1} x))^{-1} & \text{for } n < 0,
\end{cases}
$$

which we will identify with the function $\varphi$. Then $T_\varphi^n(x, g) = (Tx, g\varphi^{(n)}(x))$ for any integer $n$. Two cocycles $\varphi, \psi : X \to G$ are cohomologous if there exists a measurable map $p : X \to G$ such that

$$
\varphi(x) = p(x)^{-1}\psi(x)p(Tx).
$$

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In this case, $p$ will be called a transfer function. If $\varphi$ and $\psi$ are cohomologous, then the map $(x, g) \mapsto (x, p(x)g)$ establishes a metrical isomorphism of $T_\varphi$ and $T_\psi$.

By $\mathbb{T}$ we will mean the circle group $\{z \in \mathbb{C}; |z| = 1\}$ which most often will be treated as the group $\mathbb{R}/\mathbb{Z}; \lambda$ will denote Lebesgue measure on $\mathbb{T}$. We will identify functions on $\mathbb{T}$ with periodic functions of period 1 on $\mathbb{R}$. Assume that $\alpha \in \mathbb{T}$ is irrational. We will deal with the case where $T$ is the ergodic rotation on $\mathbb{T}$ given by $Tx = x + \alpha$.

In the case where $G$ is the circle and $\varphi$ is a smooth cocycle, spectral properties of $T_\varphi$ depend on the topological degree $d(\varphi)$ of $\varphi$. For example, in [5], Iwanik, Lemańczyk, Rudolph have proved that if $\varphi$ is a $C^2$-cocycle with $d(\varphi) \neq 0$, then $T_\varphi$ is ergodic and it has countable Lebesgue spectrum on the orthocomplement of the space of functions depending only on the first variable. On the other hand, in [3], Gabriel, Lemańczyk and Liardet have proved that if $\varphi$ is absolutely continuous with $d(\varphi) = 0$, then $T_\varphi$ has singular spectrum.

The aim of this paper is to find a spectral equivalent of topological degree in case $G = SU(2)$.

2. Degree of Cocycle

In this section we introduce the notion of degree in case $G = SU(2)$. For a given matrix $A = [a_{ij}]_{i,j=1,\ldots,d} \in M_d(\mathbb{C})$ define $\|A\| = \sqrt{\frac{1}{d} \sum_{i,j=1}^d |a_{ij}|^2}$. Observe that if $A$ is an element of the Lie algebra $su(2)$, i.e.

$$A = \begin{bmatrix} ia & b + ic \\ -b + ic & -ia \end{bmatrix},$$

where $a, b, c \in \mathbb{R}$, then $\|A\| = \sqrt{\det A}$. Moreover, if $B$ is an element of the group $SU(2)$, i.e.

$$B = \begin{bmatrix} z_1 & z_2 \\ -z_2 & z_1 \end{bmatrix},$$

where $z_1, z_2 \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1$, then $\text{Ad}_B A = BAB^{-1} \in su(2)$ and $\|\text{Ad}_B A\| = \|A\|$.

Consider the scalar product of $su(2)$ given by

$$\langle A, B \rangle = -\frac{1}{8} \text{tr}(\text{ad}A \circ \text{ad}B).$$

Then $\|A\| = \sqrt{\langle A, A \rangle}$. By $L^2(X, su(2))$ we mean the space of all functions $f : X \to su(2)$ such that

$$\|f\|_{L^2} = \sqrt{\int_X \|f(x)\|^2 dx} < \infty.$$ 

For two $f_1, f_2 \in L^2(X, su(2))$ set

$$\langle f_1, f_2 \rangle_{L^2} = \int_X \langle f_1(x), f_2(x) \rangle dx.$$ 

The space $L^2(X, su(2))$ endowed with the above scalar product is a Hilbert space.
By $L^1(X, \mathfrak{su}(2))$ we mean the space of all functions $f : X \to \mathfrak{su}(2)$ such that

$$\|f\|_{L^1} = \int_X \|f(x)\|dx < \infty.$$  

The space $L^1(X, \mathfrak{su}(2))$ endowed with the norm $\| \cdot \|_{L^1}$ is a Banach space.

For a given measurable cocycle $\varphi : \mathbb{T} \to SU(2)$ consider the unitary operator

$$U : L^2(\mathbb{T}, \mathfrak{su}(2)) \to L^2(\mathbb{T}, \mathfrak{su}(2)), \quad Uf(x) = \text{Ad}_{\varphi(x)}f(Tx). \quad (1)$$

Then $U^nf(x) = \text{Ad}_{\varphi^n(x)}f(T^n x)$ for any integer $n$.

**Lemma 2.1.** There exists an operator $P : L^2(\mathbb{T}, \mathfrak{su}(2)) \to L^2(\mathbb{T}, \mathfrak{su}(2))$ such that

$$\frac{1}{n} \sum_{j=0}^{n-1} U^j f \to Pf \quad \text{in} \quad L^2(\mathbb{T}, \mathfrak{su}(2))$$

for any $f \in L^2(\mathbb{T}, \mathfrak{su}(2))$ and $U \circ P = P$. Moreover, $\| Pf \|$ is constant $\lambda$-a.e..

**Proof.** The first claim of the lemma follows from the von Neumann ergodic theorem. Since $U \circ P = P$, we have $\text{Ad}_{\varphi(x)} Pf(Tx) = Pf(x)$, for $\lambda$-a.e. $x \in \mathbb{T}$. It follows that $\| Pf(Tx) \| = \| Pf(x) \|$, for $\lambda$-a.e. $x \in \mathbb{T}$. Hence $\| Pf(x) \| = c$, for $\lambda$-a.e. $x \in \mathbb{T}$, by the ergodicity of $T$.

**Lemma 2.2.** For every $f \in L^2(\mathbb{T}, \mathfrak{su}(2))$ the sequence $\frac{1}{n} \sum_{j=0}^{n-1} U^j f$ converges $\lambda$-almost everywhere.

**Proof.** Let $\tilde{f} \in L^2(\mathbb{T} \times SU(2), \mathfrak{su}(2))$ be given by $\tilde{f}(x, g) = \text{Ad}_g f(x)$. Then

$$\tilde{f}(T^\varphi(x, g)) = \text{Ad}_g(U^nf(x))$$

for any integer $n$. By the Birkhoff ergodic theorem, the sequence

$$\frac{1}{n} \sum_{j=0}^{n-1} \tilde{f}(T^\varphi(x, g)) = \text{Ad}_g \left( \frac{1}{n} \sum_{j=0}^{n-1} U^j f(x) \right)$$

converges for $\lambda \otimes \mu$-a.e. $(x, g) \in \mathbb{T} \times SU(2)$. Hence there exists $g \in SU(2)$ such that $\text{Ad}_g(\frac{1}{n} \sum_{j=0}^{n-1} U^j f(x))$ converges for $\lambda$-a.e. $x \in \mathbb{T}$, and the proof is complete.

Recall that, if a function $\varphi : \mathbb{T} \to SU(2)$ is of class $C^1$, then $D\varphi(x)\varphi(x)^{-1} \in \mathfrak{su}(2)$ for every $x \in \mathbb{T}$.

**Lemma 2.3.** For every $C^1$-cocycle $\varphi : \mathbb{T} \to SU(2)$, there exists $\psi \in L^2(\mathbb{T}, \mathfrak{su}(2))$ such that

$$\frac{1}{n} D\varphi^{(n)}(\varphi^{(n)})^{-1} \to \psi \quad \text{in} \quad L^2(\mathbb{T}, \mathfrak{su}(2))$$

and $\lambda$-almost everywhere.

Moreover, $\| \psi \|$ is a constant function $\lambda$-a.e. and $\varphi(x)\psi(Tx)\varphi(x)^{-1} = \psi(x)$ for $\lambda$-a.e. $x \in \mathbb{T}$.

**Proof.** Since

$$D\varphi^{(n)}(x) = \sum_{j=0}^{n-1} \varphi(x) \ldots \varphi(T^{j-1}x) D\varphi(T^jx) \varphi(T^{j+1}x) \ldots \varphi(T^{n-1}x),$$
we have
\[ D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1} \]
\[ = \sum_{j=0}^{n-1} \varphi(x) \ldots \varphi(T^{j-1}x)D\varphi(T^jx)\varphi(T^jx)^{-1} \varphi(T^{j-1}x)^{-1} \ldots \varphi(x)^{-1} \]
\[ = \sum_{j=0}^{n-1} \varphi^{(j)}(x)D\varphi(T^jx)\varphi(T^jx)^{-1}(\varphi^{(j)}(x))^{-1} \]
\[ = \sum_{j=0}^{n-1} U^j(D\varphi\varphi^{-1})(x), \]
where \( U \) is the unitary operator given by (1). Put \( \hat{\psi} = P(D\varphi\varphi^{-1}) \). Applying Lemmas 2.1 and 2.2, we conclude that
\[ \frac{1}{n} D\varphi^{(n)}(\varphi^{(n)})^{-1} \rightarrow \psi \text{ in } L^2(\mathbb{T}, \mathfrak{su}(2)) \text{ and } \lambda\text{-almost everywhere.} \]
Moreover,
\[ U\psi = U \circ P(D\varphi\varphi^{-1}) = P(D\varphi\varphi^{-1}) = \psi \]
and \( \|\psi\| = \|P(D\varphi\varphi^{-1})\| \) is a constant function \( \lambda\text{-a.e., by Lemma 2.1, which completes the proof.} \)

**Definition 1.** The number \( \|\psi\| \) will be called the **degree** of the cocycle \( \varphi \) and denoted by \( d(\varphi) \).

Lemma 2.3 shows that
\[ \frac{1}{n} \|D\varphi^{(n)}(\varphi^{(n)})^{-1}\|_{L^1} \rightarrow d(\varphi). \]
On the other hand, \( \|D\varphi^{(n)}(\varphi^{(n)})^{-1}\|_{L^1} \) is the length of the curve \( \varphi^{(n)} \). Geometrically speaking, the degree of \( \varphi \) is the limit of length \( (\varphi^{(n)})/n \).

A measurable cocycle \( \delta : \mathbb{T} \rightarrow SU(2) \) is said to be **diagonal** if there exists a measurable function \( \gamma : \mathbb{T} \rightarrow \mathbb{T} \) such that
\[ \delta(x) = \begin{bmatrix} \gamma(x) & 0 \\ 0 & \gamma(x) \end{bmatrix}. \]

**Theorem 2.4.** Suppose that \( \varphi : \mathbb{T} \rightarrow SU(2) \) is a \( C^1 \)-cocycle with \( d(\varphi) \neq 0 \). Then \( \varphi \) is cohomologous to a diagonal cocycle.

**Proof.** For every nonzero \( A \in \mathfrak{su}(2) \) there exists \( B_A \in SU(2) \) such that
\[ B_A A(B_A)^{-1} = \begin{bmatrix} i\|A\| & 0 \\ 0 & -i\|A\| \end{bmatrix}. \]
Indeed, if \( A = \begin{bmatrix} ia & b + ic \\ -b + ic & -ia \end{bmatrix} \), then we can take
Set $p(x) = B_{\psi(x)}$. Then $p : \mathbb{T} \to SU(2)$ is a measurable function and

$$\psi(x) = p(x)^{-1} \begin{bmatrix} i d(\varphi) & 0 \\ 0 & -i d(\varphi) \end{bmatrix} p(x).$$

Since $\varphi(x)\psi(Tx)\varphi(x)^{-1} = \psi(x)$, we have

$$\varphi(x)p(Tx)^{-1} \begin{bmatrix} i d(\varphi) & 0 \\ 0 & -i d(\varphi) \end{bmatrix} p(Tx)\varphi(x)^{-1} = p(x)^{-1} \begin{bmatrix} i d(\varphi) & 0 \\ 0 & -i d(\varphi) \end{bmatrix} p(x).$$

Hence

$$p(x)\varphi(x)p(Tx)^{-1} \begin{bmatrix} i d(\varphi) & 0 \\ 0 & -i d(\varphi) \end{bmatrix} = \begin{bmatrix} i d(\varphi) & 0 \\ 0 & -i d(\varphi) \end{bmatrix} p(x)\varphi(x)p(Tx)^{-1}.$$ 

Since $d(\varphi) \neq 0$, we see that the cocycle $\delta : \mathbb{T} \to SU(2)$ defined by $\delta(x) = p(x)\varphi(x)p(Tx)^{-1}$ is diagonal. \qed

For a given $C^1$-cocycle $\varphi : \mathbb{T} \to SU(2)$ with nonzero degree let $\gamma = \gamma(\varphi) : \mathbb{T} \to \mathbb{T}$ be a measurable cocycle such that the cocycles $\varphi$ and $\begin{bmatrix} \gamma & 0 \\ 0 & \bar{\gamma} \end{bmatrix}$ are cohomologous. It is easy to check that the choice of $\gamma$ is unique up to a measurable cohomology with values in the circle and up to the complex conjugacy.

Theorem 2.4 shows that if $d(\varphi) \neq 0$, then the skew product $T_\varphi$ is metrically isomorphic to a skew product of an irrational rotation on the circle and a diagonal cocycle. It follows that $T_\varphi$ is not ergodic. However, in the next sections we show that if $d(\varphi) \neq 0$, then $\varphi$ is not cohomologous to a constant cocycle. Moreover, the skew product $T_\gamma : \mathbb{T} \times \mathbb{T} \to \mathbb{T} \times \mathbb{T}$ is ergodic and it is mixing on the orthocomplement of the space of functions depending only on the first variable. We will prove also that (with some additional assumptions on $\varphi$) the Lebesgue component in the spectrum of $T_\gamma$ has countable multiplicity. It follows that if $d(\varphi) \neq 0$, then:

- all ergodic components of $T_\varphi$ are metrically isomorphic to $T_\gamma$,
- the spectrum of $T_\varphi$ consists of two parts: discrete and mixing,
- (with some additional assumptions on $\varphi$) the Lebesgue component in the spectrum of $T_\varphi$ has countable multiplicity.

In case $G = \mathbb{T}$, the topological degree of each $C^1$-cocycle is an integer number. An important question is: what can one say on values of degree in case $G = SU(2)$?
If a cocycle $\varphi$ is cohomologous to a diagonal cocycle via a smooth transfer function, then $d(\varphi) \in 2\pi\mathbb{N}_0 = 2\pi(\mathbb{N} \cup \{0\})$.

We call a function $f : \mathbb{T} \to SU(2)$ absolutely continuous if $f_{ij} : \mathbb{T} \to \mathbb{C}$ is absolutely continuous for $i,j = 1,2$. Suppose that $\varphi$ is cohomologous to a diagonal cocycle via an absolutely continuous transfer function. Then $\varphi$ can be represented as $\varphi(x) = p(x)^{-1}\delta(x)p(Tx)$, where $\delta, p : \mathbb{T} \to SU(2)$ are absolutely continuous and $\delta$ is diagonal. Since $\varphi^{(n)}(x) = p(x)^{-1}\delta^{(n)}(x)p(T^n x)$, we have

$$\frac{1}{n} D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1} = \frac{1}{n} \left( -p(x)^{-1}Dp(x) + \varphi^{(n)}(x)p(T^n x)^{-1}Dp(T^n x)(\varphi^{(n)}(x))^{-1} \ight. + p(x)^{-1}D\delta^{(n)}(x)(\delta^{(n)}(x))^{-1}p(x) ).$$

On the other hand, $\delta(x) = \begin{bmatrix} \gamma(x) & 0 \\ 0 & \gamma(x) \end{bmatrix}$, where $\gamma : \mathbb{T} \to \mathbb{T}$ is an absolutely continuous cocycle of the form $\gamma(x) = \exp 2\pi i(\hat{\gamma}(x) + kx)$, where $k$ is the topological degree of $\gamma$ and $\hat{\gamma} : \mathbb{T} \to \mathbb{R}$ is an absolutely continuous function. Then

$$\frac{1}{n} D\gamma^{(n)}(x)(\gamma^{(n)}(x))^{-1} = 2\pi i \left( \frac{1}{n} \sum_{j=0}^{n-1} D\hat{\gamma}(T^j x) + k \right) \to 2\pi ik$$

in $L^1(\mathbb{T}, \mathbb{R})$, by the Birkhoff ergodic theorem. It follows that

$$\frac{1}{n} D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1} \to p(x)^{-1} \begin{bmatrix} 2\pi ik & 0 \\ 0 & -2\pi ik \end{bmatrix} p(x)$$

in $L^1(\mathbb{T}, \mathfrak{su}(2))$. Hence $d(\varphi) = 2\pi|d(\gamma)| \in 2\pi\mathbb{N}_0$.

In Section 7, it is shown that if $\alpha$ is the golden ratio, then the degree of every $C^2$-cocycle belongs to $2\pi\mathbb{N}_0$, too.

3. Notation and Facts From Spectral Theory

Let $U$ be a unitary operator on a separable Hilbert space $\mathcal{H}$. By the cyclic space generated by $f \in \mathcal{H}$ we mean the space $\mathcal{Z}(f) = \text{span}\{U^n f ; n \in \mathbb{Z}\}$. By the spectral measure $\sigma_f$ of $f$ we mean a Borel measure on $\mathbb{T}$ determined by the equalities

$$\hat{\sigma}_f(n) = \int_{\mathbb{T}} e^{2\pi inx} d\sigma_f(x) = \langle U^nf, f \rangle$$

for $n \in \mathbb{Z}$. Recall that there exists a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $\mathcal{H}$ such that

$$\mathcal{H} = \bigoplus_{n=-\infty}^{\infty} \mathcal{Z}(f_n) \quad \text{and} \quad \sigma_{f_1} \gg \sigma_{f_2} \gg \ldots.$$ (2)

Moreover, for any sequence $\{f'_n\}_{n \in \mathbb{N}}$ in $\mathcal{H}$ satisfying (2) we have $\sigma_{f_1} \equiv \sigma_{f'_1}$, $\sigma_{f_2} \equiv \sigma_{f'_2}$, \ldots. The above decompositions of $\mathcal{H}$ are called spectral decompositions of $U$.

The spectral type of $\sigma_{f_1}$ (the equivalence class of measures) will be called the maximal spectral type of $U$. We say that $U$ has Lebesgue (continuous singular, discrete) spectrum if $\sigma_{f_1}$ is equivalent to Lebesgue (continuous singular, discrete) measure on the circle. An operator $U$ is called mixing if

$$\sigma_f(n) = \langle U^nf, f \rangle \to 0$$
for any \( f \in \mathcal{H} \). We say that the spectrum of \( U \) has uniform multiplicity if either 
\[ \sigma_{f_n} \equiv \sigma_{f_1} \] or 
\[ \sigma_{f_n} \equiv 0 \] for all natural \( n \). We say that the Lebesgue component in the spectrum of \( U \) has countable multiplicity if 
\[ \lambda \ll \sigma_{f_n} \] for every natural \( n \) or equivalently if there exists a sequence \( \{g_n\}_{n \in \mathbb{N}} \) in \( \mathcal{H} \) such that the cyclic spaces 
\( Z(g_n) \) are pairwise orthogonal and \( \sigma_{g_n} \equiv \lambda \) for every natural \( n \).

For a skew product \( T_\varphi \) consider its Koopman operator
\[ U_{T_\varphi} : L^2(\mathbb{T} \times G, \lambda \otimes \mu) \rightarrow L^2(\mathbb{T} \times G, \lambda \otimes \mu), \]
\[ U_{T_\varphi}f(x, g) = f(Tx, g\varphi(x)). \]

Denote by \( \mathcal{G} \) the set of all equivalence classes of unitary irreducible representations of the group \( G \). For any unitary irreducible representation \( \Pi : G \to \mathcal{U}(\mathcal{H}_\Pi) \) by \( \{\Pi_{ij}\}_{i, j = 1}^{d_\Pi} \) we mean the matrix elements of \( \Pi \), where \( d_\Pi = \dim \mathcal{H}_\Pi \). Let us decompose
\[ L^2(\mathbb{T} \times G) = \bigoplus_{\Pi \in \mathcal{G}} \bigoplus_{i=1}^{d_\Pi} \mathcal{H}_i^\Pi, \]
where
\[ \mathcal{H}_i^\Pi = \left\{ \sum_{j=1}^{d_\Pi} \Pi_{ij}(g) f_j(x) ; f_j \in L^2(\mathbb{T}, \lambda), j = 1, \ldots, d_\Pi \right\} \]
\[ \simeq L^2(\mathbb{T}, \lambda) \oplus \ldots \oplus L^2(\mathbb{T}, \lambda). \]

Observe that \( \mathcal{H}_i^\Pi \) is a closed \( U_{T_\varphi} \)-invariant subspace of \( L^2(\mathbb{T} \times G) \) and
\[ U^n_{T_\varphi} \left( \sum_{j=1}^{d_\Pi} \Pi_{ij}(g) f_j(x) \right) = \sum_{j=1}^{d_\Pi} \Pi_{ik}(g) \Pi_{kj}(\varphi^n(x)) f_j(T^n x). \]

Consider the unitary operator \( M_i^\Pi : \mathcal{H}_i^\Pi \to \mathcal{H}_i^\Pi \) given by
\[ M_i^\Pi \left( \sum_{j=1}^{d_\Pi} \Pi_{ij}(g) f_j(x) \right) = \sum_{j=1}^{d_\Pi} e^{2\pi i x} \Pi_{ij}(g) f_j(x). \]

Then
\[ U^n_{T_\varphi} M_i^\Pi f = e^{2\pi i n x} M_i^\Pi U^n_{T_\varphi} f \quad (3) \]
for any \( f \in \mathcal{H}_i^\Pi \). It follows that
\[ \int_{\mathbb{T}} e^{2\pi i n x} d\sigma_{M_i^\Pi f}(x) = \langle U^n_{T_\varphi} M_i^\Pi f, M_i^\Pi f \rangle = e^{2\pi i n x} \langle U^n_{T_\varphi} f, f \rangle = \int_{\mathbb{T}} e^{2\pi i n x} d(T^* \sigma_f)(x) \]
for any \( f \in \mathcal{H}_i^\Pi \). Hence \( \sigma_{M_i^\Pi f} = T^* \sigma_f \).

**Lemma 3.1.** For every \( \Pi \in \mathcal{G} \) and \( i = 1, \ldots, d_\pi \) if the operator \( U_{T_\varphi} : \mathcal{H}_i^\Pi \to \mathcal{H}_i^\Pi \) has absolutely continuous spectrum, then it has Lebesgue spectrum of uniform multiplicity.
Proof. Let $\mathcal{H}_i^\Pi = \bigoplus_{n=1}^{\infty} \mathbb{Z}(f_n)$ be a spectral decomposition. Then

$$\mathcal{H}_i^\Pi = (M_i^\Pi)^m \mathcal{H}_i^\Pi = \bigoplus_{n=1}^{\infty} \mathbb{Z}((M_i^\Pi)^m f_n)$$

is a spectral decomposition for any integer $m$. Therefore $\sigma_{f_n} \equiv \sigma((M_i^\Pi)^m f_n) \ll \lambda$ for every natural $n$ and integer $m$. Suppose that there exists a Borel set $A \subset \mathbb{T}$ such that $\sigma_{f_n}(A) = 0$ and $\lambda(A) > 0$. Then

$$\sigma_{f_n} \left( \bigcup_{m \in \mathbb{Z}} T^m A \right) = 0 \quad \text{and} \quad \lambda \left( \bigcup_{m \in \mathbb{Z}} T^m A \right) = 1,$$

by the ergodicity of $T$. It follows that $\sigma_{f_n} \equiv \lambda$ or $\sigma_{f_n} = 0$ for every natural $n$. □

Lemma 3.2. If

$$\sum_{n \in \mathbb{Z}} \left| \int \Pi g_j(\varphi^{(n)}(x))dx \right|^2 < \infty$$

for $j = 1, \ldots, d_\Pi$, then $U_{T_v}$ has Lebesgue spectrum of uniform multiplicity on $\mathcal{H}_i^\Pi$ for $i = 1, \ldots, d_\Pi$.

Proof. Fix $1 \leq i \leq d_\Pi$. Note that

$$\langle U_{T_v}^n \Pi_{ij}, \Pi_{ij} \rangle = \sum_{k=1}^{d_\Pi} \int_G \langle \Pi_k(g) \Pi_{kj}(\varphi^{(n)}(x)), \Pi_{ij}(g) \rangle dg dx = \frac{1}{d_\Pi} \int_G \Pi_{ij}(\varphi^{(n)}(x)) dx.$$

Since

$$\sum_{n \in \mathbb{Z}} \left| \langle U_{T_v}^n \Pi_{ij}, \Pi_{ij} \rangle \right|^2 < \infty,$$

we have $\sigma_{ij} \ll \lambda$ for $j = 1, \ldots, d_\Pi$. From (3) we get $\sigma((M_i^\Pi)^m \Pi_{ij}) \ll \lambda$ for any integer $m$. Since $\{f \in \mathcal{H}_i^\Pi; \sigma_f \ll \lambda\}$ is a closed linear subspace of $L^2(\mathbb{T} \times G)$ and the set $\{(M_i^\Pi)^m \Pi_{ij}; j = 1, \ldots, d_\Pi, m \in \mathbb{Z}\}$ generates the space $\mathcal{H}_i^\Pi$, it follows that $U_{T_v}$ has absolutely continuous spectrum on $\mathcal{H}_i^\Pi$. By Lemma 3.1, $U_{T_v}$ has Lebesgue spectrum of uniform multiplicity on $\mathcal{H}_i^\Pi$. □

Corollary 3.3. For any $\Pi \in \hat{G}$, if

$$\sum_{n \in \mathbb{Z}} \left\| \int \Pi(\varphi^{(n)}(x)) dx \right\|^2 < \infty,$$

then $U_{T_v}$ has Lebesgue spectrum of uniform multiplicity on $\bigoplus_{i=1}^{d_\Pi} \mathcal{H}_i^\Pi$.

Similarly one can prove the following result.

Theorem 3.4. For any $\Pi \in \hat{G}$, if

$$\lim_{n \to \infty} \int T \Pi(\varphi^{(n)}(x)) dx = 0,$$

then $U_{T_v}$ is mixing on $\bigoplus_{i=1}^{d_\Pi} \mathcal{H}_i^\Pi$.
4. Representations of $SU(2)$

In this section, some basic information about the theory of representations of the group $SU(2)$ are presented. By $\mathcal{P}_k$ we mean the linear space of all homogeneous polynomials of degree $k \in \mathbb{N}_0$ in two variables $u$ and $v$. Denote by $\Pi_k$ the representation of the group $SU(2)$ in $\mathcal{P}_k$ given by

$$\Pi_k \left( \begin{bmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{bmatrix} \right) f(u, v) = f(z_1 u - \overline{z_2} v, z_2 u + \overline{z_1} v).$$

Of course, all $\Pi_k$ are unitary (under an appropriate inner product on $\mathcal{P}_k$) and the family $\{\Pi_0, \Pi_1, \Pi_2, \ldots\}$ is a complete family of continuous unitary irreducible representations of $SU(2)$. In the Lie algebra $\mathfrak{su}(2)$, we choose the following basis:

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $V_k$ be a $k + 1$-dimension linear space. For every natural $k$ there exists a basis $v_0, \ldots, v_k$ of $V_k$ such that the corresponding representation $\Pi_k^*$ of $\mathfrak{su}(2)$ in $V_k$ has the following form:

$$\Pi_k^*(e)v_i = i(k - i + 1)v_{i-1},$$
$$\Pi_k^*(f)v_i = v_{i+1},$$
$$\Pi_k^*(h)v_i = (k - 2i)v_i$$

for $i = 0, \ldots, k$. Then

$$\|A\| \leq \|\Pi_k^*(A)\| \leq k^2 \|A\| \quad (4)$$

for any $A \in \mathfrak{su}(2)$.

For abbreviation, we will write $(2k - 1)!!$ instead of $1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k - 3) \cdot (2k - 1)$ for any natural $k$.

**Lemma 4.1.**

$$\det \Pi_{2k-1}^*(A) = ((2k - 1)!!)^2 (\det A)^k$$

for any $A \in \mathfrak{su}(2)$ and $k \in \mathbb{N}$.

**Proof.** For every $A \in \mathfrak{su}(2)$ there exists $B \in SU(2)$ and $d \in \mathbb{R}$ such that $A = \text{Ad}_B \begin{bmatrix} id & 0 \\ 0 & -id \end{bmatrix}$. Then

$$\Pi_{2k-1}^*(A) = \Pi_{2k-1}^*(\text{Ad}_B \begin{bmatrix} id & 0 \\ 0 & -id \end{bmatrix}) = \text{Ad}_{\Pi_{2k-1}^*(B)} \Pi_{2k-1}^*(\begin{bmatrix} id & 0 \\ 0 & -id \end{bmatrix}).$$

It follows that

$$\det \Pi_{2k-1}^*(A) = \det \Pi_{2k-1}^*(\begin{bmatrix} id & 0 \\ 0 & -id \end{bmatrix}) = ((2k - 1)!!)^2 d^{2k} = ((2k - 1)!!)^2 (\det A)^k.$$

□
Lemma 4.2. For any nonzero \( A \in \text{su}(2) \) the matrix \( \Pi_{2k-1}^*(A) \) is invertible. Moreover, for every natural \( k \) there exists a real constant \( K_k > 0 \) such that
\[
\| \Pi_{2k-1}^*(A)^{-1} \| \leq K_k \| A \|^{-1}
\]
for every nonzero \( A \in \text{su}(2) \).

Proof. The first claim of the lemma follows from Lemma 4.1. Set \( C = \Pi_{2k-1}^*(A) \). Then
\[
|C|_{ij} \leq (2k)^{4k}(2k-1)!! \| A \|^{2k-1}
\]
for \( i, j = 1, \ldots, 2k \). It follows that
\[
|(C^{-1})|_{ij} = \frac{|C|_{ij}}{\det \Pi_{2k-1}^*(A)} \leq \frac{(2k)^{4k}(2k-1)!! \| A \|^{2k-1}}{((2k-1)!!)^2 \| A \|^{2k}} = \frac{(2k)^{4k}(2k-1)!! \| A \|^{-1}}{((2k-1)!!)^2}.
\]
Hence
\[
\| C^{-1} \| \leq \frac{(2k)^{4k+1}(2k-1)!}{((2k-1)!!)^2} \| A \|^{-1}.
\]

5. Ergodicity and Mixing of \( T_\gamma \)

Lemma 5.1. Suppose that \( \{f_n\}_{n \in \mathbb{N}} \) is a sequence in \( L^2(\mathbb{T}, \mathbb{C}) \) such that \( \int_0^1 f_n(y)dy \to 0 \) for any \( x \in \mathbb{T} \). Let \( g : \mathbb{T} \to \mathbb{C} \) be a bounded measurable function. Then
\[
\lim_{n \to \infty} \int_{\mathbb{T}} f_n(y)g(T^n y)dy = 0 \quad \text{and} \quad \lim_{n \to \infty} \int_0^x f_n(y)g(y)dy = 0
\]
for any \( x \in \mathbb{T} \).

Proof. By assumption, the sequence \( \{f_n\}_{n \in \mathbb{N}} \) tends to zero in the weak topology in \( L^2(\mathbb{T}, \mathbb{C}) \), which implies immediately the second claim of the lemma. Since \( \{f_n\}_{n \in \mathbb{N}} \) converges weakly to zero, for every integer \( m \) we have
\[
\lim_{n \to \infty} \int_{\mathbb{T}} f_n(T^{-n} y) \exp 2\pi imy dy = \lim_{n \to \infty} \int_{\mathbb{T}} f_n(y) \exp 2\pi im(y + n\alpha) dy = 0.
\]
It follows that the sequence \( \{f_n \circ T^{-n}\}_{n \in \mathbb{N}} \) converges weakly to zero. Therefore
\[
\lim_{h \to \infty} \int_{\mathbb{T}} f_n(y)g(T^h y)dy = \lim_{h \to \infty} \int_{\mathbb{T}} f_n(T^{-h} y)g(y)dy = 0.
\]
This gives immediately the following conclusion.

Corollary 5.2. Suppose that \( \{f_n\}_{n \in \mathbb{N}} \) is a sequence in \( L^2(\mathbb{T}, \mathbb{M}_k(\mathbb{C})) \) (\( k \) is a natural number) such that \( \int_0^x f_n(y)dy \to 0 \) for any \( x \in \mathbb{T} \). Let \( g : \mathbb{T} \to \mathbb{M}_k(\mathbb{C}) \) be a bounded measurable function. Then
\[
\lim_{n \to \infty} \int_{\mathbb{T}} f_n(y)g(T^n y)dy = 0 \quad \text{and} \quad \lim_{n \to \infty} \int_0^x f_n(y)g(y)dy = 0
\]
for any \( x \in \mathbb{T} \).
Theorem 5.3. Let $\varphi : \mathbb{T} \to SU(2)$ be a $C^1$-cocycle with nonzero degree. Then the skew product $T_{\gamma(\varphi)} : \mathbb{T} \times \mathbb{T} \to \mathbb{T} \times \mathbb{T}$ is ergodic and it is mixing on the orthocomplement of the space of functions depending only on the first variable.

Proof. By Theorem 3.4, it suffices to show that

$$\lim_{n \to \infty} \int_{\mathbb{T}} (\gamma(n)(x))^k dx = 0$$

for every nonzero integer $k$. Fix $k \in \mathbb{N}$. Denote by $\psi : \mathbb{T} \to \mathfrak{su}(2)$ the limit (in $L^2(\mathbb{T}, \mathfrak{su}(2))$) of the sequence $\{\frac{1}{n} D \varphi(n)(\varphi(n)^{-1})\}_{n \in \mathbb{N}}$. Let $\rho : \mathbb{T} \to SU(2)$ be a measurable function such that

$$\left[ \begin{array}{cc} 0 & \gamma(x) \\ \gamma(x) & 0 \end{array} \right] \rho(x) = \rho(T\rho(x)^{-1}) \quad \text{and} \quad \text{Ad}_{\rho(x)}(\psi(x)) = \left[ \begin{array}{cc} \text{id} & 0 \\ 0 & -\text{id} \end{array} \right],$$

where $d$ is the degree of $\varphi$ (see the proof of Theorem 2.4). Then

$$\left[ \begin{array}{cccc} (\gamma(n))^k & 0 & \cdots & 0 \\ 0 & (\gamma(n))^{k-2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & (\gamma(n))^{-(k+2)} \\ 0 & \cdots & 0 & (\gamma(n))^{-k} \end{array} \right] = \Pi_k(\rho) \Pi_k(\varphi(n)) \Pi_k(\rho \circ T^n)^{-1}$$

for any natural $n$ and

$$\text{Ad}_{\Pi_k(\rho(x))} \Pi_k^*(\psi(x)) = \Pi_k^*(\text{Ad}_{\rho(x)} \psi(x)) = \Pi_k^* \left( \left[ \begin{array}{cc} \text{id} & 0 \\ 0 & -\text{id} \end{array} \right] \right)$$

$$= \left[ \begin{array}{cccc} \text{id} & 0 & \cdots & 0 \\ 0 & (k-2)\text{id} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & (-k+2)\text{id} \end{array} \right].$$

Recall that for any differentiable function $\xi : \mathbb{T} \to SU(2)$ and for any representation $\Pi$ of $SU(2)$ we have

$$D(\Pi \xi(x))(\Pi \xi(x))^{-1} = \Pi^*(D \xi(x) \xi(x)^{-1}).$$

Therefore

$$\int_0^x \frac{1}{n} \Pi_k^* D \varphi(n)(y)(\varphi(n)(y))^{-1}) \Pi_k(\varphi(n)(y)) dy = \int_0^x \frac{1}{n} D(\Pi_k \varphi(n)(y)) dy$$

$$= \frac{1}{n} (\Pi_k(\varphi(n)(x)) - \Pi_k(\varphi(n)(0))).$$
tends to zero for any \( x \in \mathbb{T} \). Since
\[
\frac{1}{n} \Pi_k^*(D\varphi^{(n)}(\varphi^{(n)})^{-1}) \to \Pi_k^*\psi
\]
in \( L^2(\mathbb{T}, M_{k+1}(\mathbb{C})) \), it follows that
\[
\int_0^x \Pi_k^*(\psi(y))\Pi_k(\varphi^{(n)}(y))dy \to 0
\]
for any \( x \in \mathbb{T} \). By Corollary 5.2,
\[
\int_\mathbb{T} \Pi_k(p(y))\Pi_k^*(\psi(y))\Pi_k(\varphi^{(n)}(y))\Pi_k(p(T^ny))^{-1}dy \to 0.
\]
On the other hand,
\[
\Pi_k(p(y))\Pi_k^*(\psi(y))\Pi_k(\varphi^{(n)}(y))\Pi_k(p(T^ny))^{-1}
= \begin{bmatrix}
    ikd(\gamma^{(n)}(y))^k & 0 \\
    \vdots & \ddots & \ddots \\
    0 & \cdots & -ikd(\gamma^{(n)}(y))^{-k}
\end{bmatrix},
\]
by (5) and (6). Therefore
\[
\lim_{n \to \infty} \int_\mathbb{T} (\gamma^{(n)}(y))^m dy = 0
\]
for any nonzero \( m \in \{-k, -k+2, \ldots, k-2, k\} \), which completes the proof. \( \square \)

6. Spectral Analysis of Cocycles with Nonzero Degree

In this section, it is shown that for every cocycle \( \varphi : \mathbb{T} \to SU(2) \) if \( d(\varphi) \neq 0 \) and if it satisfies some additional assumptions, then the Lebesgue component in the spectrum of \( T_\varphi \) has countable multiplicity.

Now we introduce a notation that is necessary to prove the main theory. Let \( f, g : \mathbb{T} \to M_k(\mathbb{C}) \) be functions of bounded variation (i.e. \( f_{ij}, g_{ij} : \mathbb{T} \to \mathbb{C} \) have bounded variation for \( i, j = 1, \ldots, k \)) and let one of them be continuous. We will use the symbol \( \int f dg \) to denote the \( k \times k \)-matrix given by
\[
\left( \int f dg \right)_{ij} = \sum_{l=1}^k \int f_{il}dg_{lj}
\]
for \( i, j = 1, \ldots, d \). It is clear that if \( g \) is absolutely continuous, then
\[
\int f dg = \int f(x)Dg(x)dx.
\]
Moreover, applying integration by parts, we have
\[
\int f dg = -\left( \int g^T df^T \right)^T.
\]
Theorem 6.1. Let $\varphi : \mathbb{T} \to SU(2)$ be a $C^2$-cocycle with $d(\varphi) \neq 0$. Suppose that the sequence $\{\frac{1}{n} D\varphi(n)(\varphi(n))^{-1}\}_{n \in \mathbb{N}}$ is uniformly convergent and $\{D(\frac{1}{n} D\varphi(n)(\varphi(n))^{-1})\}_{n \in \mathbb{N}}$ is bounded in $L^1(\mathbb{T}, su(2))$. Then the Lebesgue component in the spectrum of $T_{\varphi}$ has countable multiplicity. Moreover, the Lebesgue component in the spectrum of $T_{\gamma(\varphi)}$ has countable multiplicity, too.

Proof. First, observe that it suffices to show that for every natural $k$ there exists a real constant $C_k > 0$ such that

$$\left\| \int_{\mathbb{T}} \Pi_{2k-1}(\varphi(n)(x)) dx \right\| \leq \frac{C_k}{n}$$

for large enough natural $n$. Indeed, let $p : \mathbb{T} \to SU(2)$ be a measurable function such that

$$p(x)\varphi(x)p(Tx)^{-1} = \delta(x) = \begin{bmatrix} \gamma(x) & 0 \\ 0 & \gamma(x) \end{bmatrix}.$$ 

Consider the unitary operator $V : \mathcal{H}_1^{\Pi_{2k-1}} \to \mathcal{H}_1^{\Pi_{2k-1}}$ given by

$$V \left( \sum_{i=1}^{d_{\Pi_{2k-1}}} \Pi_{1i}(g) f_i(x) \right) = \sum_{i,j=1}^{d_{\Pi_{2k-1}}} \Pi_{1i}(g) \Pi_{ji}(p(x)^{-1}) f_i(x).$$

Then

$$V^{-1} U_{T_{\varphi}} V \left( \sum_{i=1}^{d_{\Pi_{2k-1}}} \Pi_{1i}(g) f_i(x) \right)$$

$$= \sum_{i,j,l,m=1}^{d_{\Pi_{2k-1}}} \Pi_{1m}(g) \Pi_{ml}(p(x)) \Pi_{ij}(\varphi(x)) \Pi_{ji}(p(Tx)^{-1}) f_i(Tx)$$

$$= \sum_{i=1}^{d_{\Pi_{2k-1}}} \Pi_{1i}(g) \Pi_{ii}(\delta(x)) f_i(Tx).$$

From (9), $U_{T_{\varphi}} : \mathcal{H}_1^{\Pi_{2k-1}} \to \mathcal{H}_1^{\Pi_{2k-1}}$ has Lebesgue spectrum of uniform multiplicity, by Corollary 3.3. Hence $V^{-1} U_{T_{\varphi}} V$ has Lebesgue spectrum of uniform multiplicity and it is the product of the operators $U_j : L^2(\mathbb{T}, \mathbb{C}) \to L^2(\mathbb{T}, \mathbb{C})$ given by $U_j f(x) = (\gamma(x))^{2k-2j+1} f(Tx)$ for $j = 1, \ldots, 2k$. Therefore $U_j$ has absolutely continuous spectrum for $j = 1, \ldots, 2k$. By Lemma 3.1, $U_j$ has Lebesgue spectrum for all $j = 1, \ldots, 2k$ and $k \in \mathbb{N}$. It follows that the Lebesgue component in the spectrum of $T_{\gamma(\varphi)}$ has countable multiplicity.

By assumption,

$$\left\| \frac{1}{n} D\varphi(n)(\varphi(n))^{-1} \right\| \to d(\varphi)$$

uniformly. Therefore

$$\left\| \frac{1}{n} D\varphi(n)(\varphi(n))^{-1} \right\| \geq d(\varphi)/2$$ (10)
for large enough natural $n$. For all $A, B \in M_k(\mathbb{C})$ we have $\|AB\| \leq \sqrt{k}\|A\|\|B\|$. Applying these facts, (7) and (8) we get

\[
\|\int_{\mathbb{T}} \Pi_{2k-1}(\varphi^{(n)}(x))dx\| = \left\| \int_{\mathbb{T}} \Pi_{2k-1}(\varphi^{(n)}(x))(D\Pi_{2k-1}(\varphi^{(n)}(x)))^{-1} d\Pi_{2k-1}(\varphi^{(n)}(x)) \right\|
\]

\[
= \left\| \int_{\mathbb{T}} (\Pi_{2k-1}^*(D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}))^{-1} d\Pi_{2k-1}(\varphi^{(n)}(x)) \right\|
\]

\[
= \left\| \int_{\mathbb{T}} (\Pi_{2k-1}(\varphi^{(n)}(x)))^T d((\Pi_{2k-1}^*(D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}))^{-1})^T \right\|
\]

\[
= \left\| \int_{\mathbb{T}} (\Pi_{2k-1}^*(D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}))^T ((\Pi_{2k-1}^*(D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}))^{-1})^T \right\| dx
\]

\[
\leq 2k \int_{\mathbb{T}} \left\| (\Pi_{2k-1}^*(D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}))^{-1} \right\|^2 \left\| \Pi_{2k-1}^*(D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}) \right\| dx.
\]

By Lemma 4.2, we have

\[
\| (\Pi_{2k-1}^*(D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}))^{-1} \| \leq K_k \| D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1} \|^{-1}.
\]

From this, (4) and (10) we obtain

\[
\| \int_{\mathbb{T}} \Pi_{2k-1}(\varphi^{(n)}(x))dx \|
\]

\[
\leq \frac{K_k^2(2k)^3}{n} \int_{\mathbb{T}} \left\| \frac{1}{n} D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1} \right\|^2 \left\| D\left(\frac{1}{n} D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}\right) \right\| dx
\]

\[
\leq \frac{1}{n} \left( \frac{8K_k^2}{d(\varphi)} \right)^2 \left\| D\left(\frac{1}{n} D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}\right) \right\|_{L^1}
\]

for large enough natural $n$. By assumption, there exists a real constant $M > 0$ such that $\| D\left(\frac{1}{n} D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}\right) \|_{L^1} \leq M$. Then

\[
\| \int_{\mathbb{T}} \Pi_{2k-1}(\varphi^{(n)}(x))dx \| \leq \frac{C_k}{n}
\]

for large enough natural $n$, where $C_k = \left( \frac{8K_k^2}{d(\varphi)} \right)^2 M$.

In this section we also present a class of cocycles satisfying the assumptions of Theorem 6.1. For $r = 1, 2$ let $L'_r(\mathbb{T}, \mathbb{R}) = \{ f \in L'_r(\mathbb{T}, \mathbb{R}); f \geq 0 \}$. We will need the following lemma.

**Lemma 6.2.** Let $\{f_n : \mathbb{T} \to \mathbb{C}^d; n \in \mathbb{N}\}$ be a sequence of absolutely continuous functions. Assume that the sequence $\{f_n\}_{n \in \mathbb{N}}$ converges in $L^1(\mathbb{T}, \mathbb{C}^d)$ to a function
$f$ and it is bounded for the sup norm. Suppose that there is a sequence $\{h_n\}_{n \in \mathbb{N}}$ convergent in $L^2_+ (\mathbb{T}, \mathbb{R})$ and a sequence $\{k_n\}_{n \in \mathbb{N}}$ bounded in $L^2_+ (\mathbb{T}, \mathbb{R})$ such that
\[
\|Df_n(x)\| \leq h_n(x)k_n(x) \text{ for } \lambda\text{-a.e. } x \in \mathbb{T}
\]
and for any natural $n$. Then $\{f_n\}_{n \in \mathbb{N}}$ converges to $f$ uniformly.

**Proof.** Denote by $h \in L^2_+ (\mathbb{T}, \mathbb{R})$ the limit of the sequence $\{h_n\}_{n \in \mathbb{N}}$. Let $M > 0$ be a real number such that $\|k_n\|_{L^2} \leq M$ for all natural $n$. First, observe that the sequence $\{f_n\}_{n \in \mathbb{N}}$ is equicontinuous. Fix $\varepsilon > 0$. Take $n_0 \in \mathbb{N}$ such that $\|h_n - h\|_{L^2} < \varepsilon/2M$ for any $n \geq n_0$. Then for all $x, y \in \mathbb{T}$ and $n \geq n_0$ we have
\[
\|f_n(x) - f_n(y)\| = \left| \int_x^y Df_n(t) dt \right| \leq \int_x^y \|Df_n(t)\| dt \\
\leq \int_x^y h_n(t)k_n(t) dt \leq \|k_n\|_{L^2} \sqrt{\int_x^y h^2_n(t) dt} \\
\leq M \left( \sqrt{\int_x^y h^2(t) dt} + \|h_n - h\|_{L^2} \right) \leq M \left( \sqrt{\int_x^y h^2(t) dt} + \frac{\varepsilon}{2M} \right).
\]
Choose $\delta_1 > 0$ such that $|x - y| < \delta_1$ implies $\int_x^y h^2(t) dt < (\varepsilon/2M)^2$. Hence if $|x - y| < \delta_1$, then $\|f_n(x) - f_n(y)\| < \varepsilon$ for any $n \geq n_0$. Next choose $0 < \delta \leq \delta_1$ such that $|x - y| < \delta$ implies $\|f_n(x) - f_n(y)\| < \varepsilon$ for any $n \leq n_0$. It follows that if $|x - y| < \delta$, then $\|f_n(x) - f_n(y)\| < \varepsilon$ for every natural $n$.

By the Arzela-Ascoli theorem, for any subsequence of $\{f_n\}_{n \in \mathbb{N}}$ there exists a subsequence convergent to $f$ uniformly. Consequently, the sequence $\{f_n\}_{n \in \mathbb{N}}$ converges to $f$ uniformly.

This gives the following corollary.

**Corollary 6.3.** Let $\{f_n : \mathbb{T} \to \mathbb{C}^d ; n \in \mathbb{N}\}$ be a sequence of absolutely continuous functions. Assume that the sequence $\{f_n\}_{n \in \mathbb{N}}$ converges in $L^1 (\mathbb{T}, \mathbb{C}^d)$ to a function $f$ and it is bounded in the sup norm. Suppose that there is a sequence $\{h_n\}_{n \in \mathbb{N}}$ convergent in $L^2_+ (\mathbb{T}, \mathbb{R})$, a sequence $\{k_n\}_{n \in \mathbb{N}}$ bounded in $L^2_+ (\mathbb{T}, \mathbb{R})$ and a sequence $\{l_n\}_{n \in \mathbb{N}}$ convergent in $L^1_+ (\mathbb{T}, \mathbb{R})$ such that
\[
\|Df_n(x)\| \leq l_n(x) + h_n(x)k_n(x) \text{ for } \lambda\text{-a.e. } x \in \mathbb{T}
\]
and for any natural $n$. Then $\{f_n\}_{n \in \mathbb{N}}$ converges to $f$ uniformly.

We will denote by $BV^2 (\mathbb{T}, SU(2))$ the set of all functions $f : \mathbb{T} \to SU(2)$ of bounded variation such that $Df(f)^{-1} \in L^2 (\mathbb{T}, su(2))$.

**Lemma 6.4.** Let $\varphi : \mathbb{T} \to SU(2)$ be a $C^2$-cocycle. Suppose that $\varphi$ is cohomologous to a diagonal cocycle with a transfer function in $BV^2 (\mathbb{T}, SU(2))$. Then the sequence $\{D^n \varphi^{(n)}(\varphi^{(n)}(\varphi^{(n)})^{-1})\}_{n \in \mathbb{N}}$ is uniformly convergent and $\{D^n_0 D_0 \varphi^{(n)}(\varphi^{(n)}(\varphi^{(n)})^{-1})\}_{n \in \mathbb{N}}$ is bounded in $L^1 (\mathbb{T}, su(2))$.

**Proof.** By Corollary 6.3, it suffices to show that there exist a sequence $\{h_n\}_{n \in \mathbb{N}}$ convergent in $L^2_+ (\mathbb{T}, \mathbb{R})$, a sequence $\{k_n\}_{n \in \mathbb{N}}$ bounded in $L^2_+ (\mathbb{T}, \mathbb{R})$ and a sequence
\( \{l_n\}_{n \in \mathbb{N}} \) convergent in \( L^1_+(\mathbb{T}, \mathbb{R}) \) such that
\[
\left\| D\left( \frac{1}{n} D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1} \right) \right\| \leq l_n(x) + h_n(x)k_n(x) \text{ for } \lambda\text{-a.e. } x \in \mathbb{T}.
\]

By assumption, there exist \( \delta, p \in BV^2(\mathbb{T}, SU(2)) \) such that \( \varphi(x) = p(x)^{-1}\delta(x)p(Tx) \), where \( \delta \) is a diagonal cocycle. Then
\[
D\varphi(x)\varphi(x)^{-1} = -p(x)^{-1}Dp(x) + p(x)^{-1}D\delta(x)\delta(x)^{-1}p(x)
\]
\[
+ \varphi(x)p(Tx)^{-1}Dp(Tx)\varphi(x)^{-1}
\]
for \( \lambda\text{-a.e. } x \in \mathbb{T} \). Set
\[
\tilde{\varphi}(x) = D\varphi(x)\varphi(x)^{-1}, \tilde{p}(x) = p(x)^{-1}Dp(x) \quad \text{and} \quad \tilde{\delta}(x) = p(x)^{-1}D\delta(x)\delta(x)^{-1}p(x).
\]
Then \( \tilde{\varphi}(x) = -\tilde{p}(x) + U\tilde{p}(x) + \tilde{\delta}(x) \), where \( \tilde{p}, \tilde{\delta} \in L^2(\mathbb{T}, su(2)) \). We adopt the convention that \( \sum_{j=0}^{-1} = 0 \). Since
\[
\frac{1}{n} D\varphi^{(n)}(\varphi^{(n)})^{-1} = \frac{1}{n} \sum_{k=0}^{n-1} \varphi^{(k)} \tilde{\varphi} \circ T^k (\varphi^{(k)})^{-1},
\]
we have
\[
D\left( \frac{1}{n} D\varphi^{(n)}(\varphi^{(n)})^{-1} \right)
\]
\[
= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} (\text{Ad}_{\varphi^{(j)}}(\tilde{\varphi} \circ T^j) \text{Ad}_{\varphi^{(k)}}(\tilde{\varphi} \circ T^k) - \text{Ad}_{\varphi^{(k)}}(\tilde{\varphi} \circ T^k) \text{Ad}_{\varphi^{(j)}}(\tilde{\varphi} \circ T^j))
\]
\[
+ \frac{1}{n} \sum_{k=0}^{n-1} \text{Ad}_{\varphi^{(k)}}(D\tilde{\varphi} \circ T^k)
\]
\[
= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} [U^j \tilde{\varphi}, U^k \tilde{\varphi}] + \frac{1}{n} \sum_{k=0}^{n-1} U^k(D\tilde{\varphi}).
\]

However,
\[
\sum_{k=0}^{n-1} \sum_{j=0}^{k-1} [U^j \tilde{\varphi}, U^k \tilde{\varphi}] = \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} [U^{j+1} \tilde{p} - U^j \tilde{p} + U^j \tilde{\delta}, U^k \tilde{\varphi}]
\]
\[
= \sum_{k=0}^{n-1} [U^k \tilde{p} - \tilde{p}, U^k \tilde{\varphi}] + \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} [U^j \tilde{\delta}, U^{k+1} \tilde{p} - U^k \tilde{p} + U^k \tilde{\delta}]
\]
\[
= \sum_{k=0}^{n-1} [U^k \tilde{p} - \tilde{p}, U^k \tilde{\varphi}] + \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} [U^j \tilde{\delta}, U^k \tilde{\varphi}]
\]
\[
+ \sum_{j=0}^{n-2} [U^j \tilde{\delta}, U^j \tilde{\varphi} - U^{j+1} \tilde{p}] .
\]
Since

\[ U^j\tilde{\delta}(x) = \text{Ad}_{\varphi^{(n)}(x)p(T^nx)^{-1}}(D\delta(T^nx)\delta(T^nx)^{-1}) \]
\[ = \text{Ad}_{p(x)^{-1}\delta^{(n)}(x)}(D\delta(T^nx)\delta(T^nx)^{-1}) \]
\[ = \text{Ad}_{p(x)^{-1}}(D\delta(T^nx)\delta(T^nx)^{-1}), \]

we have \([U^j\tilde{\delta}, U^k\tilde{\delta}] = 0\) for any integers \(j, k\). Observe that \(k \leq A, B \leq k + 2\) for any \(A, B \in \text{su}(2)\). It follows that

\[ \left\| \frac{1}{n} D\left(\frac{1}{n} D\varphi^{(n)}(\varphi^{(n)})^{-1}\right) \right\| \]
\[ \leq \frac{2}{n} \sum_{k=0}^{n-1} (\|D\tilde{\varphi} \circ T^k\| + \|\tilde{p} \circ T^k\| \|\tilde{\varphi} \circ T^k\| + \|\tilde{p}\| \|\tilde{\varphi} \circ T^k\| \]
\[ + \|\tilde{\delta} \circ T^k\| \|\tilde{p} \circ T^{k+1}\|) + \|\tilde{p} \circ T^n\| \frac{2}{n} \sum_{k=0}^{n-1} \|\tilde{\delta} \circ T^k\| \]

Set

\[ h_n = \frac{2}{n} \sum_{k=0}^{n-1} \|\tilde{\delta} \circ T^k\| \]
\[ k_n = \|\tilde{p} \circ T^n\| \]
\[ l_n = \frac{2}{n} \sum_{k=0}^{n-1} (\|D\tilde{\varphi} \circ T^k\| + \|\tilde{p} \circ T^k\| \|\tilde{\varphi} \circ T^k\| + \|\tilde{p}\| \|\tilde{\varphi} \circ T^k\| \]
\[ + \|\tilde{\delta} \circ T^k\| \|\tilde{p} \circ T^{k+1}\|) \]

By the Birkhoff ergodic theorem, the sequence \(\{h_n\}_{n \in \mathbb{N}}\) converges in \(L^2(\mathbb{T}, \mathbb{R})\) and the sequence \(\{l_n\}_{n \in \mathbb{N}}\) converges in \(L^1(\mathbb{T}, \mathbb{R})\). This completes the proof. \(\square\)

Theorem 6.1 and Lemma 6.4 lead to the following conclusion.

**Corollary 6.5.** Let \(\varphi : \mathbb{T} \to SU(2)\) be a \(C^2\)-cocycle with \(d(\varphi) \neq 0\). Suppose that \(\varphi\) is cohomologous to a diagonal cocycle with a transfer function in \(BV^2(\mathbb{T}, SU(2))\). Then the Lebesgue component in the spectrum of \(T_\varphi\) has countable multiplicity. Moreover, the Lebesgue component in the spectrum of \(T_{\gamma(\varphi)}\) has countable multiplicity, too.

The following result will be useful in the next section of the paper.

**Proposition 6.6.** For every \(C^2\)-cocycle \(\varphi : \mathbb{T} \to SU(2)\), the sequence

\[ \frac{1}{n^2} D\varphi^{(n)}(\varphi^{(n)})^{-1} \]

converges to zero in \(L^1(\mathbb{T}, \text{su}(2))\).

The following lemmas are some simple generalizations of some classical results. Their proofs are left to the reader.
Lemma 6.7. Let \( \{a_n\}_{n \in \mathbb{N}} \) be an increasing sequence of natural numbers and let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence in the Banach space \( L^2(\mathbb{T}, M_2(\mathbb{C})) \). Then
\[
\frac{f_{n+1} - f_n}{a_{n+1} - a_n} \to g \text{ in } L^2(\mathbb{T}, M_2(\mathbb{C})) \implies \frac{f_n}{a_n} \to g \text{ in } L^2(\mathbb{T}, M_2(\mathbb{C})).
\]

Lemma 6.8. Let \( \{g^n_k; n \in \mathbb{N}, 0 \leq k < n\} \) be a triangular matrix of elements from \( L^2(\mathbb{T}, M_2(\mathbb{C})) \) such that \( \|g^n_k\| = O(1/n) \) and
\[
g^n_0 + g^n_1 + \cdots + g^n_{n-1} \to g \text{ in } L^2(\mathbb{T}, M_2(\mathbb{C})).
\]
Then \( f_n \to f \) in \( L^2(\mathbb{T}, M_2(\mathbb{C})) \) implies
\[
\sum_{k=0}^{n-1} g^n_k f_k \to gf \text{ and } \sum_{k=0}^{n-1} f_k g^n_k \to fg \text{ in } L^1(\mathbb{T}, M_2(\mathbb{C})).
\]

Proof of Proposition 6.6. First, recall that
\[
\frac{1}{n^2} D(D\varphi^{(n)}(\varphi^{(n)})^{-1}) = \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} [U^j\varphi, U^k\varphi] + \frac{1}{n^2} \sum_{k=0}^{n-1} U^k(D\varphi),
\]
where \( \hat{\varphi} = D\varphi(\varphi)^{-1} \) and
\[
\frac{1}{n^2} \sum_{k=0}^{n-1} U^k\hat{\varphi} \to \psi \text{ in } L^2(\mathbb{T}, su(2)).
\]
Since \( \frac{1}{n^2} \sum_{k=0}^{n-1} U^k(D\hat{\varphi}) \) uniformly converges to zero, it suffices to show that
\[
\lim_{n \to \infty} \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} U^j\varphi U^k\varphi = \lim_{n \to \infty} \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} U^k\varphi U^j\varphi = \frac{1}{2} \psi \psi \text{ in } L^2(\mathbb{T}, M_2(\mathbb{C})).
\]
Set \( f_n = \sum_{k=0}^{n-1} (n-k)U^k\varphi \) and \( a_n = n^2 \). Then
\[
\frac{f_{n+1} - f_n}{a_{n+1} - a_n} = \frac{\sum_{k=0}^{n} (n+1-k)U^k\varphi - \sum_{k=0}^{n-1} (n-k)U^k\varphi}{(n+1)^2 - n^2} = \frac{\sum_{k=0}^{n} U^k\varphi}{2n+1} \to \frac{1}{2} \psi
\]
in \( L^2(\mathbb{T}, M_2(\mathbb{C})) \). Applying Lemma 6.7, we get
\[
\frac{1}{n^2} \sum_{k=0}^{n-1} (n-k)U^k\varphi \to \frac{1}{2} \psi \text{ in } L^2(\mathbb{T}, M_2(\mathbb{C})).
\]
Therefore
\[
\frac{1}{n^2} \sum_{k=0}^{n-1} kU^k\varphi = \frac{1}{n} \sum_{k=0}^{n-1} U^k\varphi - \frac{1}{n^2} \sum_{k=0}^{n-1} (n-k)U^k\varphi \to \psi - \frac{1}{2} \psi = \frac{1}{2} \psi
\]
in \( L^2(\mathbb{T}, M_2(\mathbb{C})) \). Applying Lemma 6.8 with \( g^n_k = \frac{k}{n} U^k\varphi \) and \( f_k = \frac{1}{k} \sum_{j=0}^{k-1} U^j\varphi \), we conclude that
\[
\sum_{k=0}^{n-1} g^n_k f_k = \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} U^k\varphi U^j\varphi \to \frac{1}{2} \psi \psi
\]
and
\[ \sum_{k=0}^{n-1} f_k g_k^n = \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} U^j \tilde{\varphi} U^k \tilde{\varphi} \to \frac{1}{2} \psi \psi \]
in $L^2(\mathbb{T}, M_2(\mathbb{C}))$, which completes the proof. \qed

7. Possible Values of Degree

One may ask what we know about the set of possible values of degree. For $G = \mathbb{T}$ the degree of each smooth cocycle is an integer number. Probably, in the case of cocycles with values in $SU(2)$ the set of possible values of degree is more complicated. However, in this section, we show that if $\alpha$ is the golden ratio, then the degree of each smooth cocycle belongs to $2\pi \mathbb{N}_0$. The idea of renormalization, which is used to prove this result is due to Rychlik [8].

Let $\alpha$ be the golden ratio (i.e. the positive root of the equation $\alpha^2 + \alpha = 1$). It will be advantageous for our notation to consider the interval $[-\alpha^2, \alpha)$ to be the model of the circle. Then the map $T : [-\alpha^2, \alpha) \to [-\alpha^2, \alpha)$ given by

\[ T(x) = \begin{cases} x + \alpha & \text{for } x \in [-\alpha^2, 0) \\ x - \alpha^2 & \text{for } x \in [0, \alpha) \end{cases} \]
is the rotation by $\alpha$. Let $X = [-\alpha^2, \alpha^3)$. Then the first return time to $X$, which we call $\tau$, satisfies the following formula

\[ \tau(x) = \begin{cases} 1 & \text{for } x \in [0, \alpha^3) \\ 2 & \text{for } x \in [-\alpha^2, 0) \end{cases} \]

and the first return map $T_X : X \to X$ is equal to $T$ up to a linear scaling. Indeed, if $M : \mathbb{T} \to X$ is the map given by $M(x) = -\alpha x$, then $T_X \circ M = M \circ T$.

By $W^1$ we mean the space of all cocycles $\varphi : \mathbb{T} \to SU(2)$ such that the functions $\varphi : [-\alpha^2, 0) \to SU(2)$, $\varphi : [0, \alpha) \to SU(2)$ are both of class $C^1$ and

\[ \lim_{x \to 0^-} D\varphi(x) \varphi(x)^{-1} \quad \text{and} \quad \lim_{x \to \alpha} D\varphi(x) \varphi(x)^{-1} \]
exist. The topology of $W^1$ is induced from $C^1((-\alpha^2, 0) \cup (0, \alpha))$. Consider the renormalization operator $\Phi : W^1 \to W^1$ defined by

\[ \Phi \varphi(x) = \varphi(\tau(Mx))(Mx). \]

Then

\[ \Phi^n \varphi(x) = \begin{cases} \varphi^{(q_{n+1})}(M^n x) & \text{for } x \in [-\alpha^2, 0) \\ \varphi^{(q_{n+2})}(M^n x) & \text{for } x \in [0, \alpha) \end{cases} \]

for any natural $n$, where $\{q_n\}_{n \in \mathbb{N}}$ is the Fibonacci sequence. By $W^1_0$ we mean the set of all cocycles $\varphi \in W^1$ such that $\varphi^{(2)}$ is continuous at $0$. The set $W^1_0$ is a closed subset of $W^1$ and

\[ \Phi(W^1_0) \subset W^1_0 \] (11)
It is easy to check that \( \varphi \mapsto \|D\varphi(\varphi)^{-1}\|_{L^1} \) is a Lyapunov function for the renormalization map \( \Phi \), i.e. \( \|D(\Phi\varphi)(\Phi\varphi)^{-1}\|_{L^1} \leq \|D\varphi(\varphi)^{-1}\|_{L^1} \) for any \( \varphi \in W^1 \).

The following result is due to Rychlik [8].

**Proposition 7.1.** If \( \|D(\Phi^k\varphi)(\Phi^k\varphi)^{-1}\|_{L^1} = \|D\varphi(\varphi)^{-1}\|_{L^1} \) for all natural \( k \), then
\[
D\varphi(x)(\varphi(x))^{-1} = \alpha \text{Ad}_{\varphi(x)}[D\varphi(Tx)(\varphi(Tx))^{-1}]
\]
for every \( x \in [-\alpha^2, 0) \).

**Lemma 7.2.** Let \( \varphi : \mathbb{T} \to SU(2) \) be a \( C^2 \)-cocycle. Assume that
\[
\frac{1}{n} \|D\varphi^{(n)}(0)(\varphi^{(n)}(0))^{-1}\|_{L^1} \to H \in \text{su}(2)
\]
and there is an increasing sequence \( \{n_k\}_{k \in \mathbb{N}} \) of even numbers such that
\[
\lim_{k \to \infty} \alpha^n \int_0^{\alpha x_k} |D(D\varphi^{(q_{n+1})(M^n\varphi)(\varphi^{(q_{n+1})(M^n\varphi)})^{-1}}) - \frac{1}{q_{n+1}} D\varphi^{(q_{n+1})(\varphi^{(q_{n+1})(\varphi^{(q_{n+1})(0)}))^{-1}}}|dx = 0
\]
for \( i = 1, 2 \). Then \( \|H\| \in 2\pi\mathbb{N}_0 \).

**Proof.** First, note that
\[
D\Phi^n\varphi(x)(\Phi^n\varphi(x))^{-1} = \begin{cases} \alpha^n D\varphi^{(q_{n+1})(M^n\varphi)(\varphi^{(q_{n+1})(M^n\varphi)})^{-1}} & \text{for } x \in [-\alpha^2, 0) \\ \alpha^n D\varphi^{(q_{n+2})(M^n\varphi)(\varphi^{(q_{n+2})(M^n\varphi)})^{-1}} & \text{for } x \in [0, \alpha) \end{cases}
\]
for any even \( n \). Since
\[
\left| \frac{1}{q_{n+1}} D\varphi^{(q_{n+1})(M^n\varphi)(\varphi^{(q_{n+1})(M^n\varphi)})^{-1}} - \frac{1}{q_{n+1}} D\varphi^{(q_{n+1})(\varphi^{(q_{n+1})(\varphi^{(q_{n+1})(0)}))^{-1}}}|d\lambda
\]
\[
\leq \frac{1}{q_{n+1}} \int_0^{\alpha x_k} |D(D\varphi^{(q_{n+1})(\varphi^{(q_{n+1})(\varphi^{(q_{n+1})(0)}))^{-1}})}|d\lambda
\]
\[
\leq \frac{1}{q_{n+1} \alpha^n} \int_0^{\alpha^n} |D(D\varphi^{(q_{n+1})(\varphi^{(q_{n+1})(\varphi^{(q_{n+1})(0)}))^{-1}})}|d\lambda
\]
for all even \( n, i = 1, 2 \) and
\[
\lim_{n \to \infty} q_{n+1} = 1/(1 + \alpha^2), \quad \lim_{n \to \infty} q_{n+2} = 1/(\alpha + \alpha^3),
\]
it follows that
\[
\lim_{k \to \infty} \alpha^n q_{n+1} = \frac{1}{1 + \alpha^2} H
\]
uniformly on \([-\alpha^2, 0)\) and
\[
\lim_{k \to \infty} \alpha^n q_{n+2} = \frac{1}{\alpha + \alpha^3} H
\]
uniformly on $[0, \alpha)$. Therefore we can assume that there exists $v \in W^1$ such that
\[ \Phi^n \varphi \to v \quad \text{and} \quad D\Phi^n \varphi(\Phi^n \varphi)^{-1} \to Dv v^{-1} \]
uniformly. Then
\[ Dv(x)(v(x))^{-1} = \begin{cases} \alpha A & \text{for } x \in [-\alpha^2, 0) \\ A & \text{for } x \in [0, \alpha) \end{cases}, \]
where $A = 1/(\alpha + \alpha^3)H \in \mathfrak{su}(2)$. Therefore
\[ v(x) = \begin{cases} e^{\alpha A} B & \text{for } x \in [-\alpha^2, 0) \\ e^{\alpha A} C & \text{for } x \in [0, \alpha) \end{cases}, \]
where $B = v_-(0)$ and $C = v_+(0)$. Since the set $W^1_0 \subset W^1$ is closed and $\Phi$-invariant, $v \in W^1_0$. It follows that
\[ Ce^{-\alpha A} B = Be^{\alpha A} C. \tag{12} \]
Since $v$ is a limit point of the sequence $\{\Phi^n \varphi\}_{n \in \mathbb{N}}$ and $\varphi \mapsto \|D\Phi(\varphi)^{-1}\|_{L^1}$ is a Lyapunov function for the renormalization map $\Phi$, we have $\|D\Phi^k v(\Phi^k v)^{-1}\|_{L^1} = \|Dv v^{-1}\|_{L^1}$ for any natural $k$. By Proposition 7.1,
\[ \lim_{x \to 0} Dv(x)(v(x))^{-1} = \alpha \text{Ad}_{v_-(0)} \lim_{x \to 0} Dv(x)(v(x))^{-1}. \]
Hence
\[ \alpha A = \alpha \text{Ad}_B(A) \]
and finally $AB = BA$. Therefore
\[ \Phi v(x) = \begin{cases} e^{-\alpha A} C & \text{for } x \in [-\alpha^2, 0) \\ e^{-x A + \alpha A} BC & \text{for } x \in [0, \alpha) \end{cases}. \]
By Proposition 7.1,
\[ \lim_{x \to 0} D\Phi v(x)(\Phi v(x))^{-1} = \alpha \text{Ad}_{v_-(0)} \lim_{x \to 0} D\Phi v(x)(\Phi v(x))^{-1}. \]
Hence
\[ -\alpha A = \alpha \text{Ad}_C(-A) \]
and finally $AC = CA$. It follows that $B$ and $C$ commute, by (12). From (12), we obtain $e^{(\alpha + \alpha^3)A} = \text{Id}$. Therefore $\|H\| = \|(\alpha + \alpha^3)A\| \in 2\pi \mathbb{N}_0$. \qed

**Theorem 7.3.** Suppose that $\alpha$ is the golden ratio. Then for every $C^2$-cocycle $\varphi : \mathbb{T} \to SU(2)$, we have $d(\varphi) \in 2\pi \mathbb{N}_0$.

**Proof.** Fix $n \in \mathbb{N}$ such that $2\alpha^{2n}[1/2\alpha^{2n}] \geq 4/5$. Set $I_j = [2(j - 1)\alpha^{2n}, 2j\alpha^{2n}]$ for $j \in E = \{1, \ldots, [1/2\alpha^{2n}]\}$ and $\varepsilon_n = \frac{1}{2\alpha^{2n}} \int_{I_j} |D(D(\varphi(q))(\varphi(q))^{-1})|d\lambda$. By Proposition 6.6, $\varepsilon_n$ tends to zero. For $i = 1, 2$ define
\[ E_i = \left\{ j \in E; \frac{1}{2\alpha^{2n}q_{2n+i}^2} \int_{I_j} |D(D(\varphi(q_{2n+i}))(\varphi(q_{2n+i}))^{-1})|d\lambda \leq 10\varepsilon_{2n+i} \right\}. \]
Then
\[ \varepsilon_{2n+i} = \frac{1}{q_{2n+i}^2} \int |D(D\varphi^{(q_{2n+i})}) (\varphi^{(q_{2n+i})})^{-1})|d\lambda \]
\[ \geq \frac{1}{q_{2n+i}^2} \sum_{j \in E \setminus E_j} \int |D(D\varphi^{(q_{2n+i})}) (\varphi^{(q_{2n+i})})^{-1})|d\lambda \]
\[ \geq 20\alpha^{2n} \varepsilon_{2n+i} ([1/2\alpha^{2n}] - \#E_i) \]

Hence
\[ \#E_i \geq [1/2\alpha^{2n}] \left(1 - \frac{1}{10} \frac{1/2\alpha^{2n}}{[1/2\alpha^{2n}]}\right) \geq \frac{7}{8}[1/2\alpha^{2n}] \]
for \( i = 1, 2 \). Therefore
\[ \#(E_1 \cap E_2) \geq \#E_1 + \#E_2 - \#E \geq \frac{3}{4}[1/2\alpha^{2n}] \].

Define
\[ G_n = \bigcup_{j \in E_1 \cap E_2} [(2j - 2)\alpha^{2n}, (2j - 1)\alpha^{2n}] \].

Observe that \( y \in G_n \) implies
\[ \frac{1}{2\alpha^{2n}q_{2n+i}^2} \int_{\gamma + \alpha^{2n}} \int |D(D\varphi^{(q_{2n+i})}) (\varphi^{(q_{2n+i})})^{-1})|d\lambda \leq 10\varepsilon_{2n+i} \]
for \( i = 1, 2 \) and
\[ \lambda(G_n) \geq \alpha^{2n} \#(E_1 \cap E_2) \geq \frac{3}{8} 2\alpha^{2n}[1/2\alpha^{2n}] \geq \frac{3}{10} . \]

Set \( G' = \bigcap_{n \in \mathbb{N}} \bigcup_{k > n} G_k \). Then \( \lambda(G') \geq 3/10 \). Since \( \frac{1}{n} D\varphi^{(n)}(\varphi^{(n)})^{-1} \to \psi \) almost everywhere, we see that the set
\[ G = \left\{ x \in G', \frac{1}{n} D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1} \to \psi(x) \right\} \]
has positive measure.

For every \( y \in \mathbb{T} \) denote by \( \varphi_y : \mathbb{T} \to SU(2) \) the \( C^2 \)-cocycle \( \varphi_y(x) = \varphi(x + y) \). Suppose that \( y \in G \). Then \( \frac{1}{n} D\varphi_y^{(n)}(0)(\varphi_y^{(n)}(0))^{-1} \to \psi(y) \) and there exists an increasing sequence \( \{n_k\}_{k \in \mathbb{N}} \) of natural numbers such that \( y \in G_{n_k} \) for any natural \( k \). Hence
\[ \alpha^{2n_k} \int_0^{\alpha^{2n_k}} |D(D\varphi_y^{(q_{2n_k+i})}) (\varphi_y^{(q_{2n_k+i})})^{-1})|d\lambda \leq 20(\alpha^{2n_k}q_{2n_k+i}^2)\varepsilon_{2n_k+i} \]
for \( i = 1, 2 \). Since the sequence \( \{\alpha^n q_{n+i}\}_{n \in \mathbb{N}} \) converges for \( i = 1, 2 \) and \( \varepsilon_n \) tends to zero, letting \( k \to \infty \) we have
\[ \lim_{k \to \infty} \alpha^{2n_k} \int_0^{\alpha^{2n_k}} |D(D\varphi_y^{(q_{2n_k+i})}) (\varphi_y^{(q_{2n_k+i})})^{-1})|d\lambda = 0 \]
for \( i = 1, 2 \). By Lemma 7.2, \(|\psi(y)|\) \( \in 2\pi\mathbb{N} \) for every \( y \in G \). Since \( d(\varphi) = |\psi(y)| \) for a.e. \( y \in \mathbb{T} \), we conclude that \( d(\varphi) \in 2\pi\mathbb{N} \).

## 8. The 2-Dimensional Case

This section will deal with properties of smooth cocycles over ergodic rotations on the 2-dimensional torus with values in \( SU(2) \). By \( \mathbb{T}^2 \) we will mean the group \( \mathbb{R}^2/\mathbb{Z}^2 \). We will identify functions on \( \mathbb{T}^2 \) with functions on \( \mathbb{R}^2 \) periodic in each coordinate with period 1. Suppose that \( T(x_1, x_2) = (x_1 + \alpha, x_2 + \beta) \) is an ergodic rotation on \( \mathbb{T}^2 \). Let \( \varphi : \mathbb{T}^2 \to SU(2) \) be a \( C^1 \)-cocycle over the rotation \( T \). Analysis similar to that in Section 2 shows that there exists \( \psi_i \in L^2(\mathbb{T}^2, su(2)) \), \( i = 1, 2 \) such that

\[
\frac{1}{n} \frac{\partial}{\partial x_i} \varphi^{(n)}(\varphi^{(n)})^{-1} \rightarrow \psi_i \quad \text{in} \quad L^2(\mathbb{T}^2, su(2)).
\]

Moreover, \(|\psi_i|\) is a \( \lambda \otimes \lambda \)-a.e. constant function and \( \varphi(\tilde{x})\psi_i(T\tilde{x})\varphi(\tilde{x})^{-1} = \psi_i(\tilde{x}) \) for \( \lambda \otimes \lambda \)-a.e. \( \tilde{x} \in \mathbb{T} \times \mathbb{T} \) for \( i = 1, 2 \).

**Definition 2.** The pair

\[
(\|\psi_1\|, \|\psi_2\|) = \lim_{n \to \infty} \frac{1}{n} \left( \left\| \frac{\partial}{\partial x_1} \varphi^{(n)}(\varphi^{(n)})^{-1} \right\|_{L^1}, \left\| \frac{\partial}{\partial x_2} \varphi^{(n)}(\varphi^{(n)})^{-1} \right\|_{L^1} \right)
\]

will be called the degree of the cocycle \( \varphi : \mathbb{T}^2 \to SU(2) \) and denoted by \( d(\varphi) \).

Similarly, one can prove the following

**Theorem 8.1.** If \( d(\varphi) \neq 0 \), then \( \varphi \) is cohomologous to a diagonal cocycle

\[
\mathbb{T}^2 \ni \tilde{x} \mapsto \begin{bmatrix} \gamma(\tilde{x}) & 0 \\ 0 & \gamma(\tilde{x}) \end{bmatrix} \in SU(2), \quad \text{where} \quad \gamma : \mathbb{T} \to \mathbb{T} \quad \text{is measurable. Moreover,}
\]

the skew product \( T_\gamma : \mathbb{T}^2 \times \mathbb{T} \to \mathbb{T}^2 \times \mathbb{T} \) is ergodic and it is mixing on the ortho-complement of the space of functions depending only on the first two variables.

Analysis similar to that in the proof of Theorem 6.1 gives

**Theorem 8.2.** Let \( \varphi : \mathbb{T}^2 \to SU(2) \) be a \( C^2 \)-cocycle with \( d(\varphi) \neq 0 \). Suppose that the sequence \( \left\{ \frac{1}{n} \frac{\partial}{\partial x_i} \varphi^{(n)}(\varphi^{(n)})^{-1} \right\}_{n \in \mathbb{N}} \) is uniformly convergent and \( \left\{ \frac{\partial}{\partial x_i} \left( \frac{1}{n} \frac{\partial}{\partial x_i} \varphi^{(n)}(\varphi^{(n)})^{-1} \right) \right\}_{n \in \mathbb{N}} \) is bounded in \( L^2(\mathbb{T}^2, su(2)) \) for \( i = 1, 2 \). Then the Lebesgue component in the spectrum of \( T_\varphi \) has countable multiplicity.

By \( BV^\#(\mathbb{T}^2, SU(2)) \) we mean the set of all measurable functions \( f : \mathbb{T} \to SU(2) \) such that

- the functions \( f(x, \cdot), f(\cdot, x) : \mathbb{T} \to SU(2) \) are of bounded variation for any \( x \in \mathbb{T} \);

- the functions \( \frac{\partial}{\partial x_i} f(f)^{-1}, \frac{\partial}{\partial x_j} f(f)^{-1} : \mathbb{T}^2 \to su(2) \) are Riemann integrable for \( i = 1, 2 \).

Then we immediately get the following

**Lemma 8.3.** Let \( \varphi : \mathbb{T}^2 \to SU(2) \) be a \( C^2 \)-cocycle. Suppose that \( \varphi \) is cohomologous to a diagonal cocycle with a transfer function in \( BV^\#(\mathbb{T}^2, SU(2)) \).
Then the sequence \( \{ \frac{1}{n} \frac{\partial}{\partial x_1} \varphi^{(n)}(\varphi^{(n)})^{-1} \} \) is uniformly convergent and 
\( \{ \frac{1}{n} \frac{\partial}{\partial x_i} \varphi^{(n)}(\varphi^{(n)})^{-1} \} \) is uniformly bounded for \( i = 1, 2 \).

It is easy to check that if \( \varphi \) is cohomologous to a diagonal cocycle via a \( C^1 \) transfer function, then \( d(\varphi) \in 2\pi(\mathbb{N}_0 \times \mathbb{N}_0) \). However, in the next section we show that for every ergodic rotation \( T(x_1, x_2) = (x_1 + \alpha, x_2 + \beta) \) there exists a smooth cocycle whose degree is equal to \( 2\pi(\beta, |\alpha|) \).

### 9. Cocycles Over Flows

Let \( \omega \) be an irrational number. By \( S : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{T}^2 \) we mean the ergodic flow defined by
\[
S_t(x_1, x_2) = (x_1 + t\omega, x_2 + t).
\]

Let \( \varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2) \) be a smooth cocycle over \( S \), i.e.
\[
\varphi_{t+s}(\vec{x}) = \varphi_t(\vec{x}) \varphi_s(S_t\vec{x})
\]
for all \( t, s \in \mathbb{R}, \vec{x} \in \mathbb{T}^2 \) or equivalently, \( \varphi \) is the fundamental matrix solution for a linear differential system
\[
\frac{d}{dt} y(t) = y(t) A(S_t\vec{x}),
\]
where \( A : \mathbb{T}^2 \to su(2) \), i.e. \( \varphi \) satisfies
\[
\begin{align*}
\frac{d}{dt} \varphi_t(\vec{x}) &= \varphi_t(\vec{x}) A(S_t\vec{x}) \\
\varphi_0(\vec{x}) &= \text{Id}.
\end{align*}
\]
Then
\[
\frac{\partial}{\partial x_i} \varphi_{t+s}(\vec{x}) \varphi_{t+s}(\vec{x})^{-1} = \frac{\partial}{\partial x_i} \varphi_t(\vec{x}) \varphi_t(\vec{x})^{-1} + \text{Ad}_{\varphi_t(\vec{x})} \frac{\partial}{\partial x_i} \varphi_s(S_t\vec{x}) \varphi_s(S_t\vec{x})^{-1}.
\]
Hence
\[
\left\| \frac{\partial}{\partial x_i} \varphi_{t+s}(\varphi_{t+s})^{-1} \right\|_{L^1} \leq \left\| \frac{\partial}{\partial x_i} \varphi_t(\varphi_t)^{-1} \right\|_{L^1} + \left\| \frac{\partial}{\partial x_i} \varphi_s(\varphi_s)^{-1} \right\|_{L^1}.
\]
It follows that the limit
\[
\lim_{t \to \infty} \frac{1}{|t|} \left\| \frac{\partial}{\partial x_i} \varphi_t(\varphi_t)^{-1} \right\|_{L^1}
\]
eexists for \( i = 1, 2 \).

**Definition 3.** The pair
\[
\lim_{t \to \infty} \frac{1}{|t|} \left( \left\| \frac{\partial}{\partial x_1} \varphi_t(\varphi_t)^{-1} \right\|_{L^1}, \left\| \frac{\partial}{\partial x_2} \varphi_t(\varphi_t)^{-1} \right\|_{L^1} \right)
\]
will be called the degree of the cocycle \( \varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2) \) and denoted by \( d(\varphi) \).

For a given cocycle \( \varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2) \) over the flow \( S \), by \( \hat{\varphi} : \mathbb{T} \to SU(2) \) we will mean the cocycle over the rotation \( T = x + \omega \) defined by \( \hat{\varphi}(x) = \varphi_1(x, 0) \). Then \( \hat{\varphi}^{(n)}(x) = \varphi_n(x, 0) \).

**Lemma 9.1.** \( d(\varphi) = (1, |\omega|) d(\hat{\varphi}) \).
Proof. First, observe that
\[
\varphi_{x_2}(x_1 - x_2\omega, 0)\varphi_n(x_1, x_2) = \varphi_{n+x_2}(x_1 - x_2\omega, 0) = \varphi_n(x_1 - x_2\omega, 0)\varphi_{x_2}(x_1 - x_2\omega + n\omega, 0).
\]
Hence
\[
\varphi_n(x_1, x_2) = \varphi_{x_2}(x_1 - x_2\omega, 0)^{-1}\varphi^{(n)}(x_1 - x_2\omega)\varphi_{x_2}(x_1 - x_2\omega + n\omega, 0)
\]
for all \(x_1, x_2 \in \mathbb{R}\) and \(n \in \mathbb{N}\). Fix \((x_1, x_2) \in [0, 1] \times [0, 1]\). Then
\[
\frac{\partial}{\partial x_1} \varphi_n(x_1, x_2)\varphi_n(x_1, x_2)^{-1}
\]
\[
= -\varphi_{x_2}(x_1 - x_2\omega, 0)^{-1}\frac{\partial}{\partial x_1} \varphi_{x_2}(x_1 - x_2\omega, 0)
\]
\[
+ \text{Ad}_{\varphi_{x_2}(x_1 - x_2\omega, 0)}^{-1}(D\varphi^{(n)}(x_1 - x_2\omega)\varphi^{(n)}(x_1 - x_2\omega)^{-1})
\]
\[
+ \text{Ad}_{\varphi_{x_2}(x_1 - x_2\omega, 0)}^{-1}\varphi^{(n)}(x_1 - x_2\omega)
\]
\[
\left(\frac{\partial}{\partial x_1} \varphi_{x_2}(x_1 - x_2\omega + n\omega, 0)\varphi_{x_2}(x_1 - x_2\omega + n\omega, 0)^{-1}\right).
\]
It follows that
\[
\left\|\frac{\partial}{\partial x_1} \varphi_n(\varphi_n)^{-1}\right\|_{L^1} - \left\|D\varphi^{(n)}(\varphi^{(n)})^{-1}\right\|_{L^1}
\]
\[
= \left\|\frac{\partial}{\partial x_1} \varphi_n(\varphi_n)^{-1}\right\|_{L^1} - \int_0^1 \left\|D\varphi^{(n)}(x_1 - x_2\omega)\varphi^{(n)}(x_1 - x_2\omega)^{-1}\right\|dx_1 dx_2
\]
\[
\leq 2 \int_0^1 \left\|\frac{\partial}{\partial x_1} \varphi_{x_2}(x_1 - x_2\omega, 0)\varphi_{x_2}(x_1 - x_2\omega, 0)^{-1}\right\|dx_1 dx_2.
\]
Therefore
\[
\lim_{n \to \infty} \frac{1}{n} \left\|\frac{\partial}{\partial x_1} \varphi_n(\varphi_n)^{-1}\right\|_{L^1} = \lim_{n \to \infty} \frac{1}{n} \left\|D\varphi^{(n)}(\varphi^{(n)})^{-1}\right\|_{L^1} = d(\hat{\varphi}).
\]
Similarly,
\[
\frac{\partial}{\partial x_2} \varphi_n(x_1, x_2)\varphi_n(x_1, x_2)^{-1}
\]
\[
= -\varphi_{x_2}(x_1 - x_2\omega, 0)^{-1}\frac{\partial}{\partial t} \varphi_{x_2}(x_1 - x_2\omega, 0)
\]
\[
+ \omega \varphi_{x_2}(x_1 - x_2\omega, 0)^{-1}\frac{\partial}{\partial x_1} \varphi_{x_2}(x_1 - x_2\omega, 0)
\]
\[
- \omega \text{Ad}_{\varphi_{x_2}(x_1 - x_2\omega, 0)}^{-1}(D\varphi^{(n)}(x_1 - x_2\omega)\varphi^{(n)}(x_1 - x_2\omega)^{-1})
\]
It is easy to see that 
\[ \text{homotopy} \]
Therefore 
\[ \text{and the proof is complete.} \]

It follows that 
\[
\left| \frac{\partial}{\partial x_2} \varphi_n(\varphi_n)^{-1} \right|_{L^1} - |\omega| \left| \frac{\partial}{\partial x_1} \varphi_n(x_1 - x_2\omega, 0) \varphi_n(x_1 - x_2\omega, 0)^{-1} \right| dx_1 dx_2 
\]
\[
+ 2|\omega| \int_0^1 \int_0^1 \left| \frac{\partial}{\partial x_1} \varphi_n(x_1 - x_2\omega, 0) \varphi_n(x_1 - x_2\omega, 0)^{-1} \right| dx_1 dx_2.
\]

Therefore 
\[
\lim_{n \to \infty} \frac{1}{n} \left| \frac{\partial}{\partial x_2} \varphi_n(\varphi_n)^{-1} \right|_{L^1} = |\omega| \lim_{n \to \infty} \frac{1}{n} \left| \frac{\partial}{\partial x_1} \varphi_n(\varphi_n)^{-1} \right|_{L^1} = |\omega| d(\hat{\varphi}),
\]
and the proof is complete. \( \square \)

**Lemma 9.2.** For any C^2-cocycle \( \psi : \mathbb{T} \to SU(2) \) over the rotation \( T \) there exists a C^2-cocycle \( \varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2) \) over the flow \( S \) such that \( \hat{\varphi} = \psi \).

**Proof.** Since the fundamental group of \( SU(2) \) is trivial, we can choose a C^2-homotopy \( \psi : [0, 1] \times \mathbb{T} \to SU(2) \) such that 
\[
\psi(t, x) = \begin{cases} 
\text{Id} & \text{for } t \in [0, 1/4] \\
\psi(x) & \text{for } t \in [3/4, 1]. 
\end{cases}
\]

By \( \varphi : \mathbb{R} \times \mathbb{T} \to SU(2) \) we mean the C^2-function determined by 
\[
\psi(n + t, x) = \psi(n)(x) \psi(t, x + n\omega)
\]
for any \( t \in [0, 1] \) and \( n \in \mathbb{Z} \). Then it is easy to check that 
\[
\psi(n + t, x) = \psi(n)(x) \psi(t, x + n\omega) 
\]
for any \( t \in \mathbb{R} \) and \( n \in \mathbb{Z} \). Let \( \varphi : \mathbb{R} \times \mathbb{R}^2 \to SU(2) \) be defined by 
\[
\varphi_t(x_1, x_2) = \psi(x_2, x_1 - x_2\omega)^{-1} \psi(t + x_2, x_1 - x_2\omega).
\]
It is easy to see that \( \varphi_t(x_1 + 1, x_2) = \varphi_t(x_1, x_2) \) and \( \varphi_t(x_1, x_2 + 1) = \varphi_t(x_1, x_2) \), by (14). Then \( \varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2) \) is a C^2-function and 
\[
\varphi_{t+s}(x) = \psi(x_2, x_1 - x_2\omega)^{-1} \psi(t + s + x_2, x_1 - x_2\omega)
\]
\[
= \psi(x_2, x_1 - x_2\omega)^{-1} \psi(t + x_2, x_1 - x_2\omega) \psi(x_2 + t, (x_1 + t\omega) - (x_2 + t)\omega)
\]
\[
\times \psi(s + (x_2 + t), (x_1 + t\omega) - (x_2 + t)\omega)
\]
\[
= \varphi(x) \varphi_s(S_t x).
\]
Moreover,
\[ \phi(x) = \varphi_1(x, 0) = \psi(0, x)^{-1} \psi(1, x) = \psi(x), \]
which completes the proof. \(\square\)

Suppose that \(\alpha, \beta, 1\) are independent over \(\mathbb{Q}\). Set \(\omega = \alpha/\beta\).

**Theorem 9.3.** For every ergodic rotation \(T(x_1, x_2) = (x_1 + \alpha, x_2 + \beta)\) and for every natural \(k\) there exists a \(C^2\)-cocycle over \(T\) whose degree is equal to \(2\pi k(|\beta|, |\alpha|)\).

**Proof.** Let \(S\) denote the ergodic flow given by (13). Suppose that \(\varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2)\) is a \(C^2\)-cocycle over \(S\) such that \(d(\varphi) = 2\pi k\). Consider the cocycle \(\varphi_\beta : \mathbb{T}^2 \to SU(2)\) over the rotation \(T = S\beta\). Then \(\varphi_\beta = \varphi_\beta\), and
\[
\lim_{n \to \infty} \frac{1}{n} \left\| \frac{\partial}{\partial x_i} \varphi_\beta^n (\varphi_\beta^{-1}) \right\| = |\beta| \lim_{n \to \infty} \frac{1}{|\beta|} \left\| \frac{\partial}{\partial x_i} \varphi_\beta^n (\varphi_\beta^{-1}) \right\|.
\]
It follows that
\[ d(\varphi_\beta) = |\beta| d(\varphi) = |\beta|(1, |\omega|) d(\varphi) = (|\beta|, |\alpha|) d(\tilde{\varphi}), \]
which proves the theorem. \(\square\)

Suppose that \(\beta \in (0, 1)\). Let \(\varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2)\) be a \(C^2\)-cocycle over \(S\) such that \(\tilde{\varphi}\) is a diagonal \(C^2\)-cocycle with nonzero degree. Set \(T = S\beta\) and \(\psi = \varphi_\beta\). Let \(p : \mathbb{T}^2 \to SU(2)\) be a \(BV^\#\)-function such that
\[ p(x_1, x_2) = \varphi_{x_2}(x_1 - x_2 \omega, 0)^{-1} \]
for \((x_1, x_2) \in \mathbb{R} \times [0, 1)\). Then
\[ p(T(x_1, x_2)) = \begin{cases} \varphi_{x_2+\beta}(x_1 - x_2 \omega, 0)^{-1} & \text{for } x_2 \in [0, 1 - \beta) \\ \varphi_{x_2+\beta-1}(x_1 - (x_2 - 1) \omega, 0)^{-1} & \text{for } x_2 \in [1 - \beta, 1). \end{cases} \]
Moreover,
\[ \varphi_{x_2+\beta}(x_1 - x_2 \omega, 0) = \varphi_{x_2}(x_1 - x_2 \omega, 0) \varphi_\beta(x_1, x_2) \]
and
\[ \varphi_{x_2+\beta-1}(x_1 - (x_2 - 1) \omega, 0) = \varphi_{-1}(x_1 - (x_2 - 1) \omega, 0) \varphi_{x_2+\beta}(x_1 - x_2 \omega, 0) = \varphi_{1}(x_1 - x_2 \omega, 0)^{-1} \varphi_{x_2+\beta}(x_1 - x_2 \omega, 0). \]
It follows that \(\psi(\bar{x}) = p(\bar{x}) \delta(\bar{x}) p(T \bar{x})^{-1}\), where \(\delta : \mathbb{T}^2 \to SU(2)\) is the diagonal \(BV^\#\)-cocycle given by
\[ \delta(x_1, x_2) = \begin{cases} \text{Id} & \text{for } x_2 \in [0, 1 - \beta) \\ \phi(x_1 - x_2 \omega) & \text{for } x_2 \in [1 - \beta, 1). \end{cases} \]

**Lemma 9.4.** Let \(\phi : \mathbb{T}^2 \to \mathbb{T}\) be a cocycle over the rotation \(T(x_1, x_2) = (x_1 + \alpha, x_2 + \beta)\). Suppose that \(\phi|\mathbb{T} \times [0, \gamma), \phi|\mathbb{T} \times [\gamma, 1)\) are \(C^1\)-functions, where \(\gamma\) is irrational. If \(d(\phi(\cdot, 0)) \neq d(\phi(\cdot, \gamma))\), then \(\phi\) is not a coboundary.
Proof. Set $I_1 = [0, \gamma), I_2 = [\gamma, 1), a_1 = d(\phi(\cdot, 0))$ and $a_2 = d(\phi(\cdot, \gamma))$. Then there exists a function $\phi : \mathbb{T}^2 \to \mathbb{R}$ such that $\phi|\mathbb{T} \times I_j$ is of class $C^1$ for $j = 1, 2$ and $\phi(x_1, x_2) = \exp 2\pi i (\tilde{\phi}(x_1, x_2) + a_j x_1)$ for any $(x_1, x_2) \in \mathbb{T} \times I_j$.

Next note that

$$\phi^{(n)}(x_1, x_2) = \exp 2\pi i (\tilde{\phi}(x_1, x_2) + (a_1 S^n_1(x_2) + a_2 S^n_2(x_2)) x_1 + c_n(x_2)),$$

where $S^n_1(x) = \sum_{k=0}^{n-1} 1_k(x + k\beta)$ and $c_n(x) = \sum_{k=0}^{n-1} k\alpha(a_1 1_{I_1} + a_2 1_{I_2})(x + k\beta)$.

Since the rotation by $\beta$ is uniquely ergodic,

$$\frac{1}{n} (a_1 S^n_1 + a_2 S^n_2) \to a_1 \gamma + a_2 (1 - \gamma)$$

uniformly. Since $a_1 \neq a_2$ and $\gamma$ is irrational, there exists $S > 0$ and $n_0 \in \mathbb{N}$ such that $|a_1 S^n_1(x) + a_2 S^n_2(x)| \geq nS$ for all $x \in \mathbb{T}$ and $n \geq n_0$. Applying integration by parts, we get

$$\left| \int_{\mathbb{T}^2} \phi^{(n)}(x_1, x_2) dx_1 dx_2 \right| \leq \int_0^1 \left| \int_0^1 e^{2\pi i (\tilde{\phi}(x_1, x_2) + (a_1 S^n_1(x_2) + a_2 S^n_2(x_2)) x_1)} dx_1 \right| dx_2$$

$$= \int_0^1 \left| \frac{1}{2\pi |a_1 S^n_1(x_2) + a_2 S^n_2(x_2)|} \right| \left| \int_0^1 e^{2\pi i \tilde{\phi}(x_1, x_2)} de^{2\pi i (a_1 S^n_1(x_2) + a_2 S^n_2(x_2)) x_1} \right| dx_2$$

$$\leq \int_0^1 \left| \frac{1}{2\pi |a_1 S^n_1(x_2) + a_2 S^n_2(x_2)|} \right| \left| \int_0^1 e^{2\pi i (a_1 S^n_1(x_2) + a_2 S^n_2(x_2)) x_1} e^{2\pi i \tilde{\phi}(x_1, x_2)} dx_2 \right| dx_2$$

$$\leq \int_0^1 \frac{1}{nS} \left| \int_0^1 e^{2\pi i \tilde{\phi}(x_1, x_2)} e^{(a_1 S^n_1(x_2) + a_2 S^n_2(x_2)) x_1} \frac{\partial}{\partial x_1} \tilde{\phi}(x_1, x_2) dx_2 \right| dx_1$$

$$\leq \int_{\mathbb{T}^2} \frac{1}{nS} \left| \frac{\partial}{\partial x_1} \tilde{\phi}^{(n)}(x_1, x_2) \right| dx_1 dx_2.$$

Since $\frac{\partial}{\partial x_1} \tilde{\phi} \in L^1(\mathbb{T}^2, \mathbb{C})$,

$$\frac{1}{n} \frac{\partial}{\partial x_1} \tilde{\phi}^{(n)} \to \int_{\mathbb{T}^2} \frac{\partial}{\partial x_1} \tilde{\phi}(x_1, x_2) dx_1 dx_2 = 0$$

in $L^1(\mathbb{T}^2, \mathbb{C})$, by the Birkhoff ergodic theorem, and the proof is complete. \qed

This leads to the following conclusion.

**Corollary 9.5.** For every ergodic rotation $T$ on $\mathbb{T}^2$ there exists a $C^2$-cocycle $\psi$ with nonzero degree such that the Lebesgue component in the spectrum of $T_\psi$ has countable multiplicity and $\psi$ is not cohomologous to any diagonal $C^1$-cocycle.
Proof. Let \( \hat{\varphi} : \mathbb{T} \to \mathbb{T} \) be a \( C^2 \)-function with nonzero topological degree. Let \( \varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2) \) be a \( C^2 \)-cocycle over \( S \) such that \( \hat{\varphi} = \begin{bmatrix} \varphi & 0 \\ 0 & (\varphi)^{-1} \end{bmatrix} \). Define \( \psi = \varphi \beta \). Then \( d(\psi) = 2\pi(|\beta|, |\alpha|)|d(\varphi)| \neq 0 \). Moreover, \( \psi \) and the diagonal cocycle \( \delta : \mathbb{T}^2 \to SU(2) \) given by

\[
\delta(x_1, x_2) = \begin{cases} 
\text{Id} & \text{for } x_2 \in [0, 1 - \beta) \\
\hat{\varphi}(x_1 - x_2\omega) & \text{for } x_2 \in [1 - \beta, 1)
\end{cases}
\]

are cohomologous with a transfer function in \( BV^\#(\mathbb{T}^2, SU(2)) \). Applying Theorem 8.2 and Lemma 8.3, we get the first part of our claim.

Next suppose that \( \psi \) is cohomologous to a diagonal \( C^1 \)-cocycle. Then it is easy to see that the cocycle \( \eta : \mathbb{T}^2 \to \mathbb{T} \) given by

\[
\eta(x_1, x_2) = \begin{cases} 
\text{Id} & \text{for } x_2 \in [0, 1 - \beta) \\
\hat{\varphi}(x_1 - x_2\omega) & \text{for } x_2 \in [1 - \beta, 1)
\end{cases}
\]

is cohomologous to a \( C^1 \)-cocycle \( g : \mathbb{T}^2 \to \mathbb{T} \). Applying Lemma 9.4 for \( \phi = \eta g^{-1} \) and \( \gamma = 1 - \beta \) we find that \( \eta g^{-1} \) is not a coboundary, which completes the proof.

References


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