Self-similarity for ergodic flows

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Let us consider a measure-preserving flow \((T_t)_{t \in \mathbb{R}}\) on a probability standard Borel space \((X, \mathcal{B}, \mu)\). The flow \(\mathcal{T} = (T_t)_{t \in \mathbb{R}}\) is called \textbf{self-similar} if there exists \(s \neq \pm 1\) such that the rescaled flow \(\mathcal{T}_s = (T_t)_{st \in \mathbb{R}}\) is isomorphic to the original flow \(\mathcal{T} = (T_t)_{t \in \mathbb{R}}\), this is there exists a measure-preserving automorphism \(S : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)\) such that

\[
S \circ T_t = T_{st} \circ S \quad \text{for all } t \in \mathbb{R}.
\]

If \(s = -1\) then \(\mathcal{T}\) is usually called \textbf{reversible}.

\[\mathcal{I}(\mathcal{T}) := \{s \neq 0 : \mathcal{T} \text{ and } \mathcal{T}_s \text{ are isomorphic}\}\]

\(\mathcal{I}(\mathcal{T})\) is a multiplicative subgroup of \(\mathbb{R}^*\).

\[\mathcal{I}_{\text{aut}}(\mathcal{T}) := \{(t, t') \in \mathbb{R}^2 : T_t \text{ and } T_{t'} \text{ are isomorphic}\}\]
Joining method

By a **joining** between flow $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ on $(X, \mathcal{B}, \mu)$ and $S = (S_t)_{t \in \mathbb{R}}$ on $(Y, \mathcal{C}, \nu)$ we mean any probability measure $\rho$ on $(X \times Y, \mathcal{B} \otimes \mathcal{C})$ such that

- $\rho$ is $(T_t \times S_t)_{t \in \mathbb{R}}$–invariant;
- the projections of $\rho$ on $X$ and $Y$ are equal to $\mu$ and $\nu$ respectively.

$$ \mu \times \nu \in \mathcal{J}(\mathcal{T}, S) := \text{the set of all joinings}. $$

The flows $\mathcal{T}$, $S$ are called **disjoint** in the Furstenberg sense if $\mathcal{J}(\mathcal{T}, S) = \{\mu \times \nu\}$.

If $R : (X, \mu) \to (Y, \nu)$ is an isomorphism of $\mathcal{T}$ and $S$, i.e. $R \circ T_t = S_t \circ R$ then the graph measure $\mu_R$ (the image of $\mu$ via $X \ni x \mapsto (x, Rx) \in X \times Y$) is a joining.

$\{\mu_{T_t} : t \in \mathbb{R}\}$ an important family of self-joinings of $\mathcal{T}$. 
Every joining $\rho \in \mathcal{J}(\mathcal{T}, \mathcal{S})$ defines an operator $V_\rho : L^2(X, \mu) \to L^2(Y, \nu)$ by

$$L^2(X, \mu) \hookrightarrow L^2(X \times Y, \rho)$$

$$V_\rho \downarrow \downarrow pr$$

$$L^2(Y, \nu)$$

$V_\rho : L^2(X, \mu) \to L^2(Y, \nu)$ is an intertwining Markov operator

- $f \geq 0 \implies V_\rho f \geq 0$;
- $V_\rho 1 = 1$, $V_\rho^* 1 = 1$;
- $V_\rho \circ T_t = S_t \circ V_\rho$.

$T_t : L^2(X, \mu) \to L^2(X, \mu)$ standard unitary Koopman operator

$$T_t(f) = f \circ T_t.$$
\( \rho \mapsto V_\rho \) gives a one-to-one correspondence between joinings and intertwining Markov operators (Vershik, Ryzhikov).

\[
\mu \times \nu \quad \longleftrightarrow \quad \int : L^2(X, \mu) \to L^2(Y, \nu),
\]

\[
(\int f)(y) = \int_X f \, \mu
\]

\( \mu T_t \quad \longleftrightarrow \quad T_t \)

ergodic joining \( \rho \quad \longleftrightarrow \quad \) indecomposable operator \( V_\rho \)

\( \rho \) is an ergodic measure for the flow \( (T_t \times S_t)_{t \in \mathbb{R}} \)
Positive entropy: Let $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ be a measure-preserving flow such that $0 < h_\mu(\mathcal{T}) < +\infty$. Then $h_\mu(T_s) = |s|h_\mu(\mathcal{T})$. Since entropy is an invariant for isomorphism of flows $\mathcal{I}(\mathcal{T}) \subset \{-1, 1\}$.

Zero entropy: Let $(h_t)_{t \in \mathbb{R}}$ be the horocycle flow on a compact surface of constant negative curvature $M$. $(h_t)_{t \in \mathbb{R}}$ acts on the unit tangent bundle $UT(M)$ and preserves a unique probability measure $\mu_0$. If $(g_s)_{s \in \mathbb{R}}$ stands for the geodesic flow then

$$g_s \circ h_t \circ g_s^{-1} = h_{te^{-2s}},$$

hence each $s > 0$ is a scale of self-similarity for $(h_t)_{t \in \mathbb{R}}$.

Infinite entropy: Such flows can have also plenty of self-similarities.
Abstract result

Proposition (abstract)

Let \((U_t)\) be a bounded \(C^0\)-semigroup on a separable Banach space \(B\) \((\|U_t\| \leq C)\). Suppose that

\[
B^0 \subset B^\circ \left(= \{ x^* \in B^* : t \mapsto U^*_t x^* \text{ is strongly continuous} \} \right)
\]

is a closed \((U^*_t)\)-invariant separable subspace such that \(0 \in B^0\) is the only fixed point for \((U^*_t)\) on \(B^0\). Suppose that

\[
U^*_{t_n} \to Q : B^0 \to B^* \,*\text{-weakly.}
\]

Then there exists \(E \subset \mathbb{R}\) of full Lebesgue measure such that if

\[
A \circ U^*_s = U^*_r \circ A
\]

for some \(r \in E, s \in \mathbb{R}, A : B^0 \to B^0\), then

\[
A \circ Q = 0 \text{ on } B^0.
\]
Theorem

Let $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ be a weakly mixing flow such that

\[(*) \quad T_{tn} \to Q = \alpha \int_{\mathbb{R}} T_s \, dP(s) + (1 - \alpha) J \text{ weakly in } L^2(X, \mu),\]

where $0 < \alpha \leq 1$, $P \in \mathcal{P}(\mathbb{R})$ and $J \in J(\mathcal{T})$. Then $\mathcal{T}$ and $\mathcal{T}_s$ are disjoint for a.e. $s \in \mathbb{R}$, moreover, $\mathcal{T}_s$ and $T_t$ are disjoint for a.e. $(s, t) \in \mathbb{R}^2$.

Proof. We apply Abstract Proposition to $B = B^* = B^\odot = L^2(X, \mu)$, $U_t = T_{-t}$ and $U_t^* = T_t$. Let $B^0 := L^2_0(X, \mu)$. By ergodicity, zero is the only fixed point of $(U_t^*)$ on $B^0$. Suppose that $r \in E$. We will show that $\mathcal{T}$ and $\mathcal{T}_r$ are disjoint. Let $A : L^2(X, \mu) \to L^2(X, \mu)$ be a joining between $\mathcal{T}$ and $\mathcal{T}_r$, then

$$A \circ T_s = T_{rs} \circ A \text{ for each } s \in \mathbb{R}.$$
It follows that

\[
0 = A \circ Q = A \circ (\alpha \int_{\mathbb{R}} T_s \, dP(s) + (1 - \alpha) J)
\]

\[
= \alpha \int_{\mathbb{R}} A \circ T_s \, dP(s) + (1 - \alpha) A \circ J \text{ on } L^2_0(X, \mu),
\]

hence

\[
\alpha \int_{\mathbb{R}} A \circ T_s \, dP(s) + (1 - \alpha) A \circ J = \int \text{ on } L^2(X, \mu).
\]

By the weak mixing of \( T \), \( \mu \times \mu \in J(T) \) is ergodic, so \( \int \) is indecomposable. Consequently, \( A \circ T_s = \int \) for \( P \)-a.e. \( s \), and hence \( A = \int \circ T_{-t} = \int \).

Abstract Proposition can be applied to the horocycle flow to prove

\[
(h_t)_* \mu \rightarrow \mu_0 \text{ weakly in } \mathcal{P}(UTM).
\]

This gives some new equidistribution results for horocycle flows.
How to verify the property $(\ast)$?

Special flow $T^f$ built over $T : (X, \mu) \to (X, \mu)$ and a positive square integrable $f : X \to \mathbb{R}^+$. 

Suppose that $T$ is rigid, i.e. $T^{qn} \to Id$. Suppose that $(f_0^{(n)})_{n \geq 1}$ is bounded in $L^2(X, \mu)$, where

$$f_0(x) = f(x) - \int f \, d\mu, \quad f_0^{(n)}(x) = f_0(x) + f_0(Tx) + \ldots + f_0(T^{n-1}x)$$
By Prokhorov’s theorem \((f_0^{(q_n)})_*(\mu) \to P\) weakly in \(\mathcal{P}(\mathbb{R})\).

**Theorem (Fr.-Lem. 04)**

\[
T_{mq_n}^f \to \int_{\mathbb{R}} T_s^f dP(s).
\]

- If \(T\) is an irrational rotation by \(\alpha\) on \(S^1\) and \((q_n)\) is the sequence of denominators of the continued fraction expansion of \(\alpha\) then \(T^{q_n} \to \text{Id}\) and by Denjoy-Koksma inequality \(\|f_0^{(q_n)}\|_{\text{sup}} \leq 2 \text{Var} f\), whenever \(f \in BV\). Hence (*) holds.
- Similar result holds for so called interval exchange transformations (which need not to be rigid).

**Theorem (Fr.-Lem. 06)**

If \(T\) is an ergodic IET and \(f \in BV\) then there exists \(a_n \to +\infty\) such that

\[
T_{a_n}^f \to \alpha \int_{\mathbb{R}} T_s^f dP(s) + (1 - \alpha)J
\]

for some \(0 < \alpha \leq 1\) and \(P \in \mathcal{P}(\mathbb{R})\).
By a **translation surface** we mean any \((M, \omega)\), where \(M\) is a compact Riemann surface and \(\omega\) is a holomorphic 1-form (called also Abelian differential). For every direction \(\theta\) \((\theta \in \mathbb{C} \text{ and } |\theta| = 1)\) the Abelian differential determines the direction vector field \(V_{\theta} : M \to TM\) so that \(\omega(V_{\theta}) = \theta\) (except zeros of \(\omega\)). The flow \(F_{\theta}\) associated to \(V_{\theta}\) is called a **translation flow** in the direction \(\theta\). Each ergodic translation flow has a special representation over an ergodic IET and under a piecewise constant function.

**Corollary**

*If \(F\) is weakly mixing translation flow then \(F_s\) and \(F\) are disjoint for a.e. \(s \in \mathbb{R}\), moreover, diffeomorphisms \(F_s\) and \(F_t\) are also disjoint for a.e. \((s, t) \in \mathbb{R}^2\).*

Almost every translation flow is weakly mixing in the product of the moduli space of Abelian differentials and \(S^1\). [Avila, Forni 2007]
Absence of self-similarity

Theorem (Fr.-Lem. 09)

Let $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ be an ergodic flow such that

- $T_{tn} \to \int_{\mathbb{R}} T_s \, dP(s)$ and $\mathcal{T}$ is not rigid, or
- $T_{tn} \to \alpha \int_{\mathbb{R}} T_s \, dP(s) + (1 - \alpha)J$ ($0 < \alpha \leq 1$) and $\mathcal{T}$ is not partially rigid.

Then for each $s \neq \pm 1$ the flows $\mathcal{T}$ and $\mathcal{T}_s$ are not isomorphic.

$\mathcal{T}$ is partially rigid if $T_{sn} \to J \geq a \text{Id}$ with $0 < a \leq 1$. 
Proof. Suppose that $T$ and $T_s$ are isomorphic for some $0 < |s| < 1$. Then

$$T_{st} = S \circ T_t \circ S^{-1} \text{ hence } T_{s^{m}t} = S^{m} \circ T_t \circ S^{-m}.$$ 

It follows that

$$T_{s^{m}t_{n}} = S^{m} \circ T_{t_{n}} \circ S^{-m} \xrightarrow{n \to \infty} S^{m} \int_{\mathbb{R}} T_u dP(u) \circ S^{-m}$$

$$= \int_{\mathbb{R}} S^{m} \circ T_u \circ S^{-m} dP(u) = \int_{\mathbb{R}} T_{s^{m}u} dP(u)$$

$$\xrightarrow{m \to \infty} \int_{\mathbb{R}} T_0 dP(u) = Id.$$ 

Consequently, $T$ is rigid. \qed
von Neumann flows are special flows built over irrational rotations on the circle and under piecewise $C^1$-functions with non-zero sum of jumps. von Neumann proved that such flows are weakly mixing.

**Theorem (Fr.-Lem. 06)**

von Neumann flows are not partially rigid.

**Corollary**

von Neumann flows have no self-similarities.

**Theorem (Fr.-Lem. 09)**

von Neumann flows built over ergodic IETs have no self-similarities.
This approach works also roof functions with zero sum of jumps (piecewise constant). Such flows are partially rigid. For some Diophantine rotations and for a careful choice of discontinuities of the roof function the special flow is mild mixing, which implies the absence of rigidity [Lemańczyk, Lesigne, Frączek 2007].

**Theorem (Fr. 2009)**

_If the genus of \( M \) is greater than 1 then for every stratum \( \mathcal{H}_g(m_1, \ldots, m_\kappa) \) of the moduli space of Abelian differentials there exists a dense subset \( \mathcal{H} \) such that the vertical flow of each \( \omega \in \mathcal{H} \) has no self-similarities._

**Theorem (Kułąga 2009)**

_For every compact surface \( M \) with genus greater than 1 there exists a smooth flow with no self-similarities (zero entropy)._
**Problem:** Give a classification of multiplicative subgroups of $\mathbb{R}^*$ that can be obtained as $\mathcal{I}(\mathcal{T})$.

Danilenko proved that $\mathcal{I}(\mathcal{T})$ is always a Borel subgroup. Recall $\mathcal{I}(\mathcal{T}) = \mathbb{R}^*$ for some horocycle flows.

For each countable subgroup $G \subset \mathbb{R}^*$ there exists an ergodic flow such that $\mathcal{I}(\mathcal{T}) = G$.

**Theorem (Danilenko-Lemańczyk, private communication)**

There exist ergodic flows for which $\mathcal{I}(\mathcal{T})$ is uncountable and has zero Lebesgue measure.

**Theorem (Danilenko, Ryzhikov independently)**

The absence of self-similarity is generic in the set of measure preserving flows $\text{Flow}(X, \mathcal{B}, \mu)$.

The distance $d_\mathcal{F}$ on $\text{Flow}(X, \mathcal{B}, \mu)$ is given by

$$d_\mathcal{F}((S_t)_{t \in \mathbb{R}}, (T_t)_{t \in \mathbb{R}}) = \sup_{0 \leq t \leq 1} d(S_t, T_t).$$