GROWTH AND MIXING

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Abstract. Given a bi-Lipschitz measure-preserving homeomorphism of a compact metric measure space of finite dimension, consider the sequence formed by the Lipschitz norms of its iterations. We obtain lower bounds on the growth rate of this sequence assuming that our homeomorphism mixes a Lipschitz function. In particular, we get a universal lower bound which depends on the dimension of the space but not on the rate of mixing. Furthermore, we get a lower bound on the growth rate in the case of rapid mixing. The latter turns out to be sharp: the corresponding example is given by a symbolic dynamical system associated to the Rudin-Shapiro sequence.

1. Introduction and main results

Let (M, ρ, μ) be a compact metric space endowed with a probability Borel measure μ with $\text{supp}(\mu) = M$. Denote by G the group of all bi-Lipschitz homeomorphisms of (M, ρ) which preserve the measure μ. For $\phi \in G$ write $\Gamma(\phi) = \Gamma_\rho(\phi)$ for the maximum of the Lipschitz constants of ϕ and ϕ^{-1}. Note that $\Gamma(\phi)$ is a sub-multiplicative: $\Gamma(\phi\psi) \leq \Gamma(\phi) \cdot \Gamma(\psi)$. Thus $\log \Gamma$ is a pseudo-norm on G, which enables us to consider the group G as a geometric object. In the present note we discuss a link between dynamics of $\phi \in G$ (the rate of mixing) and geometry of the cyclic subgroup of G generated by ϕ (the growth rate of $\Gamma(\phi^n)$ as $n \to \infty$.) On the geometric side, we focus on the quantity

$$\widehat{\Gamma}_n(\phi) := \max_{i=1,\dots,n} \Gamma(\phi^i) .$$

Notations. We write $(f, g)_{L_2}$ for the L_2-scalar product on $L_2(M, \mu)$. We denote by E the space of all Lipschitz functions on M with zero mean with respect to μ. We write $||f||_{L_2}$ for the L_2-norm of a function f, $\text{Lip}(f)$ for the Lipschitz constant of f and $||f||_\infty$ for the uniform norm of f.

Definition 1.1. We say that a diffeomorphism $\phi \in G$ *mixes* a function $f \in E$ if $(f \circ \phi^n, f)_{L_2} \to 0$ as $n \to \infty$.

It is known that there exist volume-preserving diffeomorphisms ϕ of certain smooth closed manifolds M with arbitrarily slow growth of $\widehat{\Gamma}_n(\phi), n \to \infty$ (see e.g. Borichev [3] for $M = \mathbb{T}^2$ and Fuchs [9] for extension of Borichev's results to manifolds admitting an effective $\mathbb{T}^2$-action). As we shall see below, the situation changes if we assume that ϕ mixes a Lipschitz function: in this case the growth rate of $\widehat{\Gamma}_n(\phi)$

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admits a universal lower bound. Furthermore the bound becomes better provided the rate of mixing is decaying sufficiently fast.

To state our first result we need the following invariant of the metric space (M, ρ). Denote by $E_{R,C}$, where $R, C \geq 0$, the subset of functions $f \in E$ with $\mathrm{Lip}(f) \leq R$ and $\|f\|_\infty \leq C$. By the Arzela-Ascoli theorem $E_{R,C}$ is compact with respect to the uniform norm. Denote by $D(R, \epsilon, C)$ the minimal number of $\epsilon/2$-balls (in the uniform norm) needed to cover $E_{R,C}$. Note that for fixed ϵ and C the function $D(R, \epsilon, C)$ is non-decreasing with R. For $t \geq D(0, 1.4, C) = [C/0.7] + 1$ put

$$\tau(t, C) := \sup\{R \geq 0 \ : \ D(R, 1.4, C) \leq t\} .$$

Theorem 1.2. *Assume that a bi-Lipschitz homeomorphism $\phi \in G$ mixes a function $f \in E$ with $||f||_{L_2} = 1$. Then there exists $\alpha > 0$ so that*

$$\widehat{\Gamma}_n(\phi) \geq \frac{\tau(\alpha n, \|f\|_\infty)}{\mathrm{Lip}(f)}$$

for all sufficiently large n.

The proof is given in Section 2.

For a compact subset A of a metric space (X, ρ_1) and $\epsilon > 0$ denote by $\mathcal{N}_\epsilon(A)$ the minimal number of open balls with radius $\epsilon/2$ such that their union covers A. Then the upper box dimension of (A, ρ_1) is defined as

$$\overline{\dim}_B(A) = \overline{\lim_{\epsilon \to 0}} \frac{\log \mathcal{N}_\epsilon(A)}{\log 1/\epsilon}. \tag{1}$$

Let (Y, ρ_2) be a compact metric space and let $\mathcal{D}^A_R(Y) \subset Y^A$ stand for the set of Lipschitz functions $f : A \to Y$ with $\mathrm{Lip}(f) \leq R$, where Y^A is equipped with the uniform distance

$$\mathrm{dist}(f, g) = \sup_{x \in A} \rho_2(f(x), g(x)) .$$

It is easy to show (the proof is analogous to that of Theorem XXV in [11])

$$\mathcal{N}_\epsilon(\mathcal{D}^A_R(Y)) \leq \mathcal{N}_{\epsilon/4}(Y)^{\mathcal{N}_{\epsilon/(4R)}(A)}. \tag{2}$$

For the reader's convenience, we present the proof in the Appendix.

Assume now that the metric space (M, ρ) satisfies the following condition:

Condition 1.3. There exist positive numbers d and κ so that for every $\delta > 0$ one can find a δ-net in (M, ρ) consisting of at most $\kappa \cdot \delta^{-d}$ points.

This condition is immediately verified if (M, ρ) is a smooth manifold of dimension d or if $d > \overline{\dim}_B(M)$. Moreover, it is satisfied for some fractal sets $M \subset \mathbb{R}^n$ where d is the fractal dimension M, e.g. if M is a self–similar set (see Theorem 9.3 [5]).

In what follows $[\alpha]$ denotes the integer part of $\alpha \in \mathbb{R}$. Assume that Condition 1.3 holds. Since $E_{R,C} = \mathcal{D}^M_R([-C, C])$, by (2), we have

$$D(R, \epsilon, C) \leq \left(\left[\frac{4C}{\epsilon} \right] + 1 \right)^{\mathcal{N}_{\epsilon/(4R)}(M)} \leq \left(\left[\frac{4C}{\epsilon} \right] + 1 \right)^{\kappa(\epsilon/(4R))^{-d}} .$$

Therefore $\tau(t, C) \geq \mathrm{const} \cdot \log^{1/d} t$. Thus Theorem 1.2 above yields the following:

Corollary 1.4. *If $\phi \in G$ mixes a Lipschitz function then there exists $\lambda > 0$ so that*

$$\widehat{\Gamma}_n(\phi) \geq \lambda \cdot \log^{\frac{1}{d}} n$$

for all sufficiently large n.

This contrasts sharply with the situation when the growth of the sequence $\Gamma(\phi^n)$ is taken under consideration. In fact, for every slowly increasing function $u : [0; +\infty) \to [0; +\infty)$ there exists a volume-preserving real-analytic diffeomorphism of the 3–torus which mixes a real-analytic function and such that $\Gamma(\phi^n) \leq \text{const} \cdot u(n)$ for infinitely many n. Such diffeomorphisms are presented in Section 6.

As a by-product of our proof of Theorem 1.2 we get the following result. Let ϕ be a bi-Lipschitz homeomorphism of a compact metric space M satisfying Condition 1.3.

Theorem 1.5. *If*

$$\liminf_{n \to \infty} \frac{\widehat{\Gamma}_n(\phi)}{\log^{1/d} n} = 0 \quad (3)$$

then the cyclic subgroup $\{\phi^n\}$ has the identity map as its limit point with respect to C^0-topology.

This theorem has the following application to bi-Lipschitz ergodic theory (the next discussion is stimulated by correspondence with A. Katok). Let T be an automorphism of a probability space (X, σ). A *bi-Lipschitz realization* of (X, T, σ) is a metric isomorphism between (X, T, σ) and (M, ϕ, μ), where ϕ is a bi-Lipschitz homeomorphism of a compact metric space M equipped with a Borel probability measure μ. An objective of bi-Lipschitz ergodic theory is to find restrictions on bi-Lipschitz realizations of various classes of dynamical systems (X, T, σ). The class of interest for us is given by *non-rigid automorphisms* which is defined as follows: Denote by U_T the induced Koopman operator $f \mapsto f \circ T$ of $L_2(X, \sigma)$. We say that T is *non-rigid* [10] if the closure of the cyclic subgroup generated by U_T with respect to strong operator topology does **not** contain the identity operator. Theorem 1.5 shows that *any bi-Lipschitz homeomorphism ϕ satisfying condition* (3) *cannot serve as a bi-Lipschitz realization of a non-rigid dynamical system.*

Let us return to the study of the interplay between growth and mixing: Next we explore the influence of the rate of mixing on the growth of $\widehat{\Gamma}_n(\phi)$. We shall need the following definitions.

Definition 1.6. Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of positive numbers converging to zero as $n \to \infty$. We say that a diffeomorphism $\phi \in G$ *mixes* a function $f \in E$ at the rate $\{a_n\}$ if

$$|(f \circ \phi^n, f)_{L_2}| \leq a_n \ \forall n \in \mathbb{N}.$$

Given a positive sequence $a_n \to 0$, we call a positive integer sequence $\{v(n)\}$ *adjoint* to $\{a_n\}$ if the following conditions hold:

$$\sum_{i : 0 < iv(n) \leq n} a_{iv(n)} \leq \frac{1}{4} \ , \quad (4)$$

and

$$\frac{n}{v(n)} \to \infty \text{ as } n \to \infty . \quad (5)$$

Lemma 1.7. *Every positive sequence $a_n \to 0$ admits an adjoint sequence.*

The proof is given in Section 5.

In the next theorem we assume that the metric space (M, ρ) satisfies Condition 1.3.

Theorem 1.8. *Assume that a bi-Lipschitz homeomorphism $\phi \in G$ mixes a Lipschitz function $f \in E$ with $||f||_{L_2} = 1$ at the rate $\{a_n\}$. Then for every adjoint sequence $\{v(n)\}$ of $\{a_n\}$ we have*

$$\widehat{\Gamma}_n(\phi) \geq \frac{1}{2\kappa^{\frac{1}{d}}\mathrm{Lip}(f)} \cdot \left[\frac{n}{2v(n)}\right]^{1/d} \ \forall n \in \mathbb{N} . \quad (6)$$

In particular, if $\sum a_i < \infty$ then

$$\widehat{\Gamma}_n(\phi) \geq const \cdot n^{\frac{1}{d}} . \quad (7)$$

Note that the second part of the theorem is an immediate consequence of the first part. Indeed, if $\sum a_i < \infty$ then the adjoint sequence can be taken constant, $v(n) \equiv v_0$ and (6) implies (7).

As we shall show in Section 7 below the estimate (7) is asymptotically sharp: It is attained for the shift associated with the Rudin-Shapiro sequence.

Corollary 1.9. *Suppose that $\phi \in G$ mixes a Lipschitz function at the rate $\{a_n\}$ such that $a_n = O(1/n^\nu)$, where $0 < \nu < 1$. Then*

$$\widehat{\Gamma}_n(\phi) \geq const \cdot n^{\frac{\nu}{d}}.$$

Proof. If $a_n \leq c/n^\nu$ for some $\nu \in (0;1)$ then one readily checks that for $C > 0$ large enough there exists a sequence $\{v(n)\}$ adjoint to $\{a_n\}$ such that $v(n) \leq C \cdot n^{1-\nu}$. Thus

$$\widehat{\Gamma}_n(\phi) \geq \frac{1}{2\kappa^{\frac{1}{d}}\mathrm{Lip}(f)} \cdot \left[\frac{n}{2Cn^{1-\nu}}\right]^{1/d} \geq \mathrm{const} \cdot n^{\frac{\nu}{d}}.$$

■

Organization of the paper: In Section 2 we prove the universal lower growth bound given in Theorem 1.2 for a bi-Lipschitz homeomorphism which mixes a Lipschitz function (the case of homeomorphism which mixes an L_2–function is also considered). Furthermore, we prove Theorem 1.5 asserting that if a bi-Lipschitz homeomorphism grows sufficiently slow, it must have strong recurrence properties and in particular must be rigid in the sense of ergodic theory. The section ends with a discussion on comparison of growth rates in finitely generated groups and in groups of homeomorphisms. In Section 3 we prove Theorem 1.8 which relates the growth rate to the rate of mixing. For the proof, we derive an auxiliary fact on "almost orthonormal" sequences of Lipschitz functions. In Section 4 we generalize the main results of the paper to the case of Hölder observables. In Section 5 we prove existence of adjoint sequences used in the formulation of Theorem 1.8.

Next we pass to constructing examples. In Section 6 we present an example which emphasizes the difference between the growth rates of sequences $\widehat{\Gamma}_n(\phi)$ and $\Gamma(\phi^n)$: We construct a volume-preserving real-analytic diffeomorphism of the 3-torus which mixes a real-analytic function and such that $\Gamma(\phi^{n_i})$ grows arbitrarily slowly along a suitable subsequence $n_i \to \infty$. In Section 7 we show that the bound in Theorem 1.8 is sharp: It is attained in the case of a symbolic dynamical system associated to the Rudin-Shapiro sequence.

Finally, in Appendix we prove Kolmogorov-Tihomirov type estimate (2).

2. Recurrence via Arzela-Ascoli compactness

Proof of Theorem 1.2. Suppose that the assertion of the theorem is false. Then, considering a sequence $\alpha_k = 1/k$, $k \in \mathbb{N}$ we get a sequence $\{n_k\}$ so that $n_k/k \geq [\|f\|_\infty/0.7] + 1$ and

$$R_k := \mathrm{Lip}(f) \cdot \widehat{\Gamma}_{n_k}(\phi) < \tau(n_k/k, \|f\|_\infty) .$$

This yields

$$D(R_k, 1.4, \|f\|_\infty) \leq n_k/k < m+1,$$

where $m = [n_k/k] \geq 1$. Consider $m+1$ functions

$$f, f \circ \phi^k, \ldots, f \circ \phi^{mk} .$$

Since

$$\mathrm{Lip}(g \circ \psi) \leq \mathrm{Lip}(g) \cdot \Gamma(\psi) \;\; \forall g \in E, \psi \in G ,$$

these functions lie in the subset $E_{R_k, \|f\|_\infty} \subset E$. Recall that $E_{R_k, \|f\|_\infty}$ can be covered by $D(R_k, 1.4, \|f\|_\infty) \leq m$ balls (in the uniform norm) of the radius 0.7. By the pigeonhole principle, there is a pair of functions from our collection lying in the same ball. In other words for some natural numbers $p > q$ we have $||f \circ \phi^{pk} - f \circ \phi^{qk}||_\infty \leq 1.4$. Put $j = (p-q)k$. We have

$$||f - f \circ \phi^j||_{L_2} \leq ||f - f \circ \phi^j||_\infty \leq 1.4 .$$

Since

$$||f||_{L_2} = ||f \circ \phi^j||_{L_2} = 1,$$

we have

$$(f, f \circ \phi^j)_{L_2} = \frac{1}{2}(||f||^2_{L_2} + ||f \circ \phi^j||^2_{L_2} - ||f - f \circ \phi^j||^2_{L_2}) \geq \frac{1}{2}(1 + 1 - 1.4^2) = 0.02 .$$

Note that $j \geq k$ and thus increasing k we get the above inequality for arbitrarily large values of j. This contradicts the assumption that ϕ mixes f. ■

Denote by H the group of all bi-Lipschitz homeomorphisms (not necessarily measure preserving) of a compact metric space (M, ρ). An argument similar to the one used in the proof above shows that if the growth rate of $\widehat{\Gamma}_n(\phi)$ is sufficiently slow, the cyclic subgroup $\{\phi^n\}$ generated by ϕ has the identity map as its limit point with respect to C^0-topology (cf. a discussion in D'Ambra-Gromov [2, 7.10.C,D]). Here is a precise statement. Denote by Λ the space of Lipschitz self-maps of M. For $\phi \in \Lambda$ write $\mathrm{Lip}(\psi)$ for the Lipschitz constant of ψ. Equip Λ with the C^0-distance

$$\mathrm{dist}(\phi, \psi) = \sup_{x \in M} \rho(\phi(x), \psi(x)) .$$

Denote by Λ_R the subset consisting of all maps ψ from Λ with $\mathrm{Lip}(\psi) \leq R$. This subset is compact with respect to the metric dist by the Arzela-Ascoli theorem. Denote by $\Delta(R, \epsilon)$ the minimal number of $\epsilon/2$-balls required to cover Λ_R. For $t \geq \Delta(0, \epsilon) = \mathcal{N}_\epsilon(M)$ put

$$\theta(t, \epsilon) = \sup\{R \geq 0 \;:\; \Delta(R, \epsilon) \leq t\} .$$

Theorem 2.1. *Let $\phi : M \to M$ be a bi-Lipschitz homeomorphism. Assume that the identity map is not a limit point with respect to C^0-topology for the cyclic subgroup $\{\phi^n\}$. Then for every sequence $\epsilon_n \to 0$ there exists $\alpha > 0$ so that*

$$\widehat{\Gamma}_n(\phi) \geq \theta(\alpha n, \epsilon_n)$$

for all sufficiently large n.

Proof. Suppose that the assertion of the theorem is false. For every $\alpha = 1/k$, $k \in \mathbb{N}$ we can choose $n_k > \max(\mathcal{N}_{\epsilon_k}(M), k)$ so that

$$\widehat{\Gamma}_{n_k}(\phi) < \theta(n_k/k, \epsilon_k).$$

Put $m_k = [n_k/k]$ and $R_k = \widehat{\Gamma}_{n_k}(\phi)$. Since $R_k < \theta(n_k/k, \epsilon_k)$, we obtain

$$\Delta(R_k, \epsilon_k) \leq n_k/k < m_k + 1.$$

Consider $m_k + 1$ maps $\mathbf{1}, \phi^k, \ldots, \phi^{km_k}$. They lie in Λ_{R_k}. Since $\Delta(R_k, \epsilon_k) \leq m_k$, it follows that at least two of these maps lie in the same $\epsilon_k/2$-ball covering of Λ_{R_k}. Therefore there exist $p > q$ so that

$$\text{dist}(\phi^{pk}, \phi^{qk}) \leq \epsilon_k \ .$$

Put $l_k = (p-q)k$, and note that $\text{dist}(\phi^{pk}, \phi^{qk}) = \text{dist}(\mathbf{1}, \phi^{l_k})$. Thus $\text{dist}(\mathbf{1}, \phi^{l_k}) \leq \epsilon_k$, and since k divides l_k we have $l_k \to \infty$. We conclude that $\phi^{l_k} \to \mathbf{1}$, which contradicts the fact that the identity map is not a limit point (with respect to C^0-topology) for the sequence $\{\phi^n\}$. ∎

Remark 2.2. Assume that the metric space (M, ρ) satisfies Condition 1.3 with exponent $d > 0$. Since $\Lambda_R = \mathcal{D}^R_M(M)$, by (2), we have

$$\Delta(R, \epsilon) \leq \mathcal{N}_{\epsilon/4}(M)^{\mathcal{N}_{\epsilon/(4R)}(M)} \leq (\kappa(\epsilon/4)^{-d})^{\kappa(\epsilon/(4R))^{-d}}.$$

Thus

$$\theta(t, \epsilon) \geq \text{const}\frac{\epsilon \cdot \log^{1/d} t}{\log^{1/d} 1/\epsilon} \ .$$

Corollary 2.3. *Let $\phi : M \to M$ be a bi-Lipschitz homeomorphism, where M satisfies Condition 1.3. Assume that the identity map is not a limit point with respect to C^0-topology for the cyclic group $\{\phi^n\}$. Let $\{\eta(n)\}$ be a sequence of positive numbers such that $\eta(n) \to +\infty$ as $n \to +\infty$ and $\eta(n) = o(\log n)$. Then $\widehat{\Gamma}_n(\phi) \geq \eta(n)^{1/d}$ for all sufficiently large n.*

Proof. An application of Theorem 2.1 for $\epsilon_n = (\eta(n)/\log n)^{\frac{1}{2d}}$ gives the existence of $\alpha > 0$ for which

$$\begin{aligned} \widehat{\Gamma}_n(\phi) &\geq \theta(\alpha n, \epsilon_n) \geq \text{const}\frac{\epsilon_n \cdot \log^{1/d} \alpha n}{\log^{1/d} 1/\epsilon_n} \geq \text{const}\frac{\left(\frac{\eta(n)}{\log n}\right)^{\frac{1}{2d}} \cdot \log^{1/d} n}{\log^{1/d} \frac{\log n}{\eta(n)}} \\ &= \text{const}\frac{\left(\frac{\log n}{\eta(n)}\right)^{\frac{1}{2d}}}{\log^{1/d} \frac{\log n}{\eta(n)}} \cdot \eta(n)^{1/d} \geq \eta(n)^{1/d} \end{aligned}$$

for all sufficiently large n. ∎

Theorem 1.5 is an immediate consequence of Corollary 2.3.

Remark 2.4. Consider *any* group H equipped with a pseudo-norm ℓ: $\ell(h) \geq 0$ all $h \in H$, $\ell(h^{-1}) = \ell(h)$ and $\ell(hg) \leq \ell(h) + \ell(g)$. For an element $h \in G$ put

$$\widehat{\ell}_n(h) = \max_{i=1,\dots,n} \ell(h^n) \ .$$

It is instructive to compare possible growth rates of cyclic subgroups in the following two cases:

(i) H is a finitely generated group, ℓ is the word norm;
(ii) H is the group of all bi-Lipschitz homeomorphisms equipped with the pseudo-norm $\ell = \log \Gamma$.

We claim that in the first case, condition

$$\liminf_{n \to \infty} \frac{\widehat{\ell}_n(\phi)}{\log n} = 0 \tag{8}$$

is equivalent to the fact that ϕ is of finite order. Indeed, assume that ϕ satisfies (8). Denote by $H_R \subset H$ the ball of radius R centred at ϕ in the word norm. Denote by K the number of elements in the generating set of H. Then the cardinality of H_R does not exceed K^{R+1}. Condition (8) guarantees that there exists $n > 0$ arbitrarily large so that $\widehat{\ell}_n(\phi) \leq \log n/(2 \log K)$. Consider $n+1$ elements $\mathbf{1}, \phi, \dots, \phi^n$. All of them lie in the set H_R with $R = \widehat{\ell}_n(\phi)$. This set contains at most $K^{R+1} \leq K\sqrt{n}$ elements. Since $K\sqrt{n} < n+1$ for large n we get that among $\mathbf{1}, \phi, \dots, \phi^n$ there are at least two equal elements, hence ϕ is of finite order. The claim follows.

In contrast to this, in the case (ii), the group of bi-Lipschitz homeomorphisms may have elements of infinite order which satisfy (8), see [3, 9]. These elements are "exotic" from the algebraic viewpoint: they cannot be included into any finitely generated subgroup H' of H so that the inclusion

$$(H', \text{word norm}) \hookrightarrow (H, \log \Gamma)$$

is quasi-isometric. It would be interesting to explore more thoroughly the dynamics of these exotic elements.

Corollary 2.3 shows that if such an exotic element is of a "very slow" growth then it has strong recurrence properties. The argument based on the Arzela compactness, which was used in its proof, imitates the argument showing that condition (8) characterizes elements of finite order in finitely generated groups. Let us compare these results for bi-Lipschitz homeomorphisms of d-dimensional spaces. Consider such a homeomorphism, say, ϕ with $\widehat{\ell}_n(\phi) = o(\log n)$, which means that it is algebraically exotic in the sense of the discussion above. If ϕ satisfies a stronger inequality $\widehat{\ell}_n(\phi) \leq (\frac{1}{d} - \epsilon) \log \log n$, it is strongly recurrent by Corollary 2.3 above. We conclude this discussion with the following open problem: explore dynamical properties of those bi-Lipschitz homeomorphisms of d-dimensional spaces whose growth sequence $\widehat{\ell}_n(\phi)$ falls into the gap between $\frac{1}{d} \log \log n$ and $o(\log n)$.

3. Almost orthonormal systems of Lipschitz functions

In this section we prove Theorem 1.8. We start with the following general result on "almost orthonormal" systems of functions:

Theorem 3.1. *Let $\{f_i\}$ be a sequence of linear independent Lipschitz functions from E with $||f_i||_{L_2} = 1$ with the following property: There exists a sequence of positive real numbers $a_n \to 0$ so that $|(f_i, f_j)_{L_2}| \le a_{i-j}$ for all $j < i$. Let $\{v(n)\}$ be an adjoint sequence of $\{a_n\}$. Then*

$$\max_{i=1,\dots,n} \operatorname{Lip}(f_i) \ge \frac{1}{2\kappa^{\frac{1}{d}}} \cdot \left[\frac{n}{2v(n)}\right]^{1/d} \quad \forall n \in \mathbb{N} \,. \tag{9}$$

Lemma 3.2. *Let $f_i \in L_2(M)$, $i = 1, \dots, N$ be a sequence of functions with $||f_i||_{L_2} = 1$ for all i and $|(f_i, f_j)_{L_2}| \le \alpha_{i-j}$ for $j < i$, where $\sum_{i=1}^{N} \alpha_i \le 1/4$. Then for every real numbers $c_1, \dots, c_N$ we have*

$$||\sum_{i=1}^{N} c_i f_i||_{L_2}^2 \ge \frac{1}{2} \sum_{i=1}^{N} c_i^2 \,.$$

Proof. Put

$$h = \sum_{i=1}^{N} c_i f_i \ \text{ and } \ C = \sqrt{\sum_{i=1}^{N} c_i^2} \,.$$

Then

$$||h||_{L_2}^2 = C^2 + I \,,$$

where $I = 2\sum_{j<i} c_i c_j (f_i, f_j)$. By the Cauchy-Schwarz inequality,

$$|I| \le 2 \sum_{p=1}^{N} \sum_{j=1}^{N-p} |c_j| \cdot |c_{j+p}| \cdot \alpha_p \le 2 \cdot \frac{1}{4} \cdot C^2 = C^2/2 \,.$$

Thus

$$||h||_{L_2}^2 \ge C^2 - C^2/2 = C^2/2$$

as required. ■

Proof of Theorem 3.1. We shall assume that $2v(n) \le n$, otherwise the inequality (9) holds by trivial reasons. Put $q(n) = [n/(2v(n))]$ and $\delta = (\kappa/q(n))^{1/d}$. By the definition of κ and d, there exists a δ-net on M consisting of $p \le q(n)$ points. Denote by $E' \subset E$ the codimension p subspace consisting of all those functions which vanish at the points of the net.

Let V be the linear span of the functions $f_{iv(n)}$, $i = 1, \dots, 2q(n)$. Then the dimension of $W := V \cap E'$ is $\ge 2q(n) - p \ge q(n)$. It is well known [4, p.103] that there exists $h \in W$ with

$$||h||_\infty \ge \sqrt{\dim W} ||h||_{L_2} \,. \tag{10}$$

Write $h = \sum_{i=1}^{2q(n)} c_i f_{iv(n)}$. Note that $|(f_{iv(n)}, f_{jv(n)})_{L_2}| \le a_{(i-j)v(n)}$ for $i < j$. Put $\alpha_i = a_{iv(n)}$. By the definition of $v(n)$, we have

$$\sum_{i=1}^{2q(n)} \alpha_i \le \frac{1}{4} \,,$$

and hence by Lemma 3.2

$$||h||_{L_2}^2 \ge C^2/2 \,, \text{ with } \ C = \sqrt{\sum_{i=1}^{2q(n)} c_i^2} \,.$$

We conclude from (10) that

$$||h||_\infty \geq \frac{1}{\sqrt{2}} \cdot \sqrt{q(n)} \cdot C \ .$$

Recall now that h vanishes at all the points of the δ-net. Thus

$$\text{Lip}(h) \geq ||h||_\infty/\delta \geq \frac{1}{\sqrt{2}} \cdot \sqrt{q(n)} \cdot C \cdot (\kappa/q(n))^{-1/d} \ . \tag{11}$$

Next, let us estimate $\text{Lip}(h)$ from above. Put

$$\Pi_n := \max_{i=1,\ldots,n} \text{Lip}(f_i) \ .$$

We have

$$\text{Lip}(h) = \text{Lip}\big(\sum_{i=1}^{2q(n)} c_i f_{iv(n)}\big) \leq \Pi_n \cdot \sqrt{2q(n)} \cdot C \ .$$

Combining this inequality with lower bound (11) we get

$$\Pi_n \geq \frac{1}{2\kappa^{\frac{1}{d}}} \cdot q(n)^{\frac{1}{d}} \ ,$$

as required. ■

Reduction of Theorem 1.8 to Theorem 3.1: We start with the following auxiliary lemma.

Lemma 3.3. *Assume that $\phi \in G$ mixes a function $f \in E$. Then for every $m > 0$ the functions $f, f \circ \phi, \ldots, f \circ \phi^m$ are linearly independent elements of E.*

Proof. Assume that $\|f\|_{L_2} = 1$ and on the contrary that for some m these functions are linearly dependent. Then for some $p \in \mathbb{N}$

$$f \circ \phi^p \in V := \text{Span}(f, f \circ \phi, \ldots, f \circ \phi^{p-1})$$

which implies that *every* function of the form $f \circ \phi^n, n \in \mathbb{Z}$ belongs to V. The space V is finite-dimensional and every element of the sequence $\{f \circ \phi^n\}, n \in \mathbb{Z}$ has unit L_2-norm. Thus this sequence has a subsequence converging to an element $g \in V$ of unit L_2-norm. Since ϕ mixes f, we have $(g, f \circ \phi^n)_{L_2} = 0$ for every $n \in \mathbb{Z}$. It follows that $g = 0$, contrary to $\|g\|_{L_2} = 1$. This completes the proof. ■

Proof of Theorem 1.8. Put $f_i = f \circ \phi^i, i \in \mathbb{N}$. Since ϕ mixes f at the rate $\{a_i\}$ we have $|(f_i, f_j)_{L_2}| \leq a_{i-j}$ for all $j < i$. The functions $\{f_i\}$ are linearly independent by Lemma 3.3. Thus all the assumptions of Theorem 3.1 hold. Theorem 1.8 readily follows from Theorem 3.1 combined with the inequality

$$\max_{i=1,\ldots,n} \text{Lip}(f_i) \leq \widehat{\Gamma}_n(\phi) \cdot \text{Lip}(f) \ .$$

■

Remark 3.4. Assume that $\{f_i\}$ is an orthonormal system (in the L_2-sense) of Lipschitz functions with zero mean. Put

$$\Pi_n := \max_{i=1,\dots,n} \mathrm{Lip}(f_i) .$$

It follows from Theorem 3.1 that

$$\Pi_n \geq \mathrm{const} \cdot n^{\frac{1}{d}} .$$

For an illustration, consider the Euclidean torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Let $\lambda_1 \leq \lambda_2 \leq \dots$ be the sequence of the eigenvalues (taken with their multiplicities) of the Laplace operator. Each λ_n has the form $4\pi^2|v|^2$, where v runs over $\mathbb{Z}^d \setminus \{0\}$. Choose the sequence of eigenfunctions f_n corresponding to λ_i so that the eigenfunctions corresponding to $4\pi^2|v|^2$ are $\sqrt{2}\sin 2\pi(x,v)$ and $\sqrt{2}\cos 2\pi(x,v)$. It follows that

$$\mathrm{Lip}(f_n) \approx |v| \approx \lambda_n^{1/2} \approx n^{1/d} ,$$

where the last asymptotic (up to a multiplicative constant) is just the Weyl law. It follows that the exponent of the power-law in the right hand side of the inequality (9) is sharp.

4. From Lipschitz to Hölder observables

Assume that the metric space (M,ρ) satisfies Condition 1.3 with exponent $d>0$. Let $\phi : (M,\rho,\mu) \to (M,\rho,\mu)$ be a bi-Lipschitz homeomorphism. Suppose that $f : M \to \mathbb{R}$ is a Hölder continuous function with exponent $\beta \in (0;1]$ which is mixed by ϕ. Let ρ_β stand for the metric on M given by $\rho_\beta(x,y) = \rho(x,y)^\beta$. Under the new metric f becomes a Lipschitz function and ϕ remains bi-Lipschitz with $\Gamma_{\rho_\beta}(\phi) = \Gamma(\phi)^\beta$. Moreover the metric space (M,ρ_β) satisfies Condition 1.3 with exponent d/β. By Corollary 1.4, we have

$$\widehat{\Gamma}_n(\phi)^\beta = \widehat{(\Gamma_{\rho_\beta})}_n(\phi) \geq \mathrm{const} \cdot \log^{\frac{\beta}{d}} n$$

which yields the following:

Corollary 4.1. *If $\phi \in G$ mixes a Hölder continuous function then there exists $\lambda > 0$ so that*

$$\widehat{\Gamma}_n(\phi) \geq \lambda \cdot \log^{\frac{1}{d}} n$$

for all natural n.

In the same manner an application of Theorem 1.8 and Corollary 1.9 gives the following:

Corollary 4.2. *Suppose that $\phi \in G$ mixes a Hölder continuous function at the rate $\{a_n\}$ such that $\sum a_n < \infty$. Then there exists $\lambda > 0$ so that*

$$\widehat{\Gamma}_n(\phi) \geq \lambda \cdot n^{\frac{1}{d}}$$

for all natural n. If $a_n = O(1/n^\nu)$, where $0 < \nu < 1$ then there exists $\lambda > 0$ so that

$$\widehat{\Gamma}_n(\phi) \geq \lambda \cdot n^{\frac{\nu}{d}}$$

for all natural n.

5. Existence of an adjoint sequence

Proof of Lemma 1.7. Making a rescaling if necessary assume that $a_n \le 1$ for all n. Choose $N_k \nearrow \infty, k \in \mathbb{N}$ so that $N_1 = 1$ and $a_i \le 1/k$ for all $i \ge N_k$. Put $b_n := 1/k$ for $n \in [N_k; N_{k+1})$. Thus $\{b_n\}$ a non-increasing positive sequence which majorates $\{a_n\}$ and converges to zero.

Define $v(n)$ as *the minimal* integer k with

$$\frac{b_k}{k} < \frac{1}{4n} .$$

Note that $v(n) \to \infty$ as $n \to \infty$. By definition

$$\frac{b_{v(n)-1}}{v(n)-1} \ge \frac{1}{4n} .$$

Thus we get that

$$\frac{v(n)}{4n} \le b_{v(n)-1} + \frac{1}{4n} ,$$

and hence $v(n)/n \to 0$ which yields assumption (5). Furthermore, using monotonicity of b_n and inequality $b_{v(n)}/v(n) < 1/(4n)$ which follows from the definition of $v(n)$ we estimate

$$\sum_{i:0<iv(n)\le n} b_{iv(n)} \le \frac{n}{v(n)} \cdot b_{v(n)} \le \frac{1}{4}$$

and we get assumption (4). ■

6. Slowly growing diffeomorphisms

As we have shown above, if a bi-Lipschitz homeomorphism ϕ of a d-dimensional compact metric space mixes a Lipschitz function, the growth rate of the sequence $\widehat{\Gamma}_n(\phi)$ is at least $\sim \log^{1/d} n$ (see Corollary 1.4). Furthermore, $\widehat{\Gamma}_n(\phi) \ge \text{const} \cdot n^{\nu/d}$ provided the mixing rate is $\sim n^{-\nu}$ for some $\nu \in (0;1)$ (see Corollary 1.9). In this section we work out an example which shows that the behavior of the sequence $\Gamma(\phi^n)$ is essentially different from the one of $\widehat{\Gamma}_n(\phi)$ even in real-analytic category. In addition, this example gives us an opportunity to test our lower bounds on $\widehat{\Gamma}_n(\phi)$ in terms of the rate of mixing.

Consider the three dimensional torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ equipped with the Euclidean metric and the Lebesgue measure. Fix any concave increasing function $u : [0; +\infty) \to [0; +\infty)$ such that

$$\lim_{x\to+\infty} u(x) = +\infty, \; u(1) \ge 1 \text{ and } u(x) \le x^{3/4}.$$

Theorem 6.1. *There exists a real-analytic measure-preserving diffeomorphism $\phi : \mathbb{T}^3 \to \mathbb{T}^3$ with the following properties:*

(i) *ϕ mixes a nonzero real-analytic function at the rate $\{\log u(n)/u(n)^{1/3}\}$;*

(ii) *There exists a positive constant $c_1 > 0$ such that $\Gamma(\phi^n) \le c_1 u(n)$ for infinitely many $n \in \mathbb{N}$;*

(iii) *There exist positive constants* c_2, c_3 *such that*

$$c_2\frac{\sqrt{n}}{\log u(n)} \le \widehat{\Gamma}_n(\phi) \le c_3 u(\sqrt{n})\sqrt{n},$$

where the left hand side inequality holds for every natural n *and the right hand side holds for infinitely many* n.

In particular, this theorem shows that $\Gamma(\phi^n)$ can grow arbitrarily slowly along a subsequence even when ϕ mixes a real-analytic function.

Remark 6.2. Taking $u(x) = x^{3\nu}$, for $0 < \nu < 1/4$, we get a diffeomorphism ϕ which mixes a real-analytic function at the rate $1/n^{\nu-\epsilon}$ (for arbitrary small $\epsilon > 0$) and such that $\widehat{\Gamma}_n(\phi) \ge \text{const}\cdot n^{1/2-\epsilon}$. Notice that applying Corollary 1.9 we get $\widehat{\Gamma}_n(\phi) \ge \text{const}\cdot n^{\nu/3-\epsilon}$. Thus Corollary 1.9 gives a correct prediction of the appearance of a power law in the lower bound for $\widehat{\Gamma}_n(\phi)$, though with a non-optimal exponent. It is an interesting open problem to find the sharp value of the exponent in Corollary 1.9.

Our construction of a diffeomorphism ϕ in Theorem 6.1 and the estimate of the rate of mixing follows the work of Fayad [6] (see also [7]). The main additional difficulty in our situation is due to the fact that we have to keep track of the growth of the differential.

Preliminaries: We denote by $\mathbb{T}$ the circle group $\mathbb{R}/\mathbb{Z}$ which we will constantly identify with the interval $[0;1)$ with addition mod 1. For a real number t denote by $\|t\|$ its distance to the nearest integer number. For an irrational $\alpha \in \mathbb{T}$ denote by $\{q_n\}$ its sequence of denominators, i.e.

$$q_0 = 1, \;\; q_1 = a_1, \;\; q_{n+1} = a_{n+1}q_n + q_{n-1},$$

where $[0; a_1, a_2, \dots]$ is the continued fraction expansion of α. Then

$$\frac{1}{2q_{n+1}} < \|q_n\alpha\| < \frac{1}{q_{n+1}} \quad \text{for each natural } n. \tag{12}$$

Let $T : \mathbb{T} \to \mathbb{T}$ stand for the corresponding ergodic rotation $Tx = x + \alpha$. Every measurable function $\varphi : \mathbb{T} \to \mathbb{R}$ determines the measurable cocycle over the rotation T given by

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x) + \varphi(Tx) + \ldots + \varphi(T^{n-1}x) & \text{if} \quad n > 0 \\ 0 & \text{if} \quad n = 0 \\ -\big(\varphi(T^n x) + \ldots + \varphi(T^{-1}x)\big) & \text{if} \quad n < 0. \end{cases}$$

If $\varphi : \mathbb{T} \to \mathbb{R}$ is a continuous function then

$$\|\varphi^{(m+n)}\|_\infty \le \|\varphi^{(m)}\|_\infty + \|\varphi^{(n)}\|_\infty \text{ and } \|\varphi^{(-n)}\|_\infty = \|\varphi^{(n)}\|_\infty$$

for all integer m, n.

Recall that

$$4\|x\| \le |e^{2\pi i x} - 1| \le 2\pi\|x\| \quad \text{for each real } x. \tag{13}$$

The construction: Let us consider a pair of irrational numbers (α, α') such that the sequences of denominators $\{q_n\}$, $\{q'_n\}$ of convergents for their continued fraction expansion satisfy

$$2u^{-1}(e^{q'_{n-1}}) \le \frac{q_n}{u(q_n)} \le 3u^{-1}(e^{q'_{n-1}}), \;\; 2u^{-1}(e^{q_n}) \le \frac{q'_n}{u(q'_n)} \le 3u^{-1}(e^{q_n}) \tag{14}$$

for any $n \geq n_0(\alpha, \alpha')$. Here n_0 is a sufficiently large positive integer which will be chosen in the course of the proof. For a given pair we consider real analytic functions φ, ψ on $\mathbb{T}$ given by

$$\varphi(x) = \sum_{n=n_0}^{\infty} \frac{\cos 2\pi q_n x}{2\pi q_n u^{-1}(e^{q_n})}, \quad \psi(y) = \sum_{n=n_0}^{\infty} \frac{\cos 2\pi q'_n y}{2\pi q'_n u^{-1}(e^{q'_n})}. \tag{15}$$

Let us consider the volume–preserving diffeomorphism $\phi : \mathbb{T}^3 \to \mathbb{T}^3$ given by

$$\phi(x, y, z) = (x + \alpha, y + \alpha', z + \varphi(x) + \psi(y)).$$

We claim that ϕ has all the properties listed in Theorem 6.1.

Starting growth estimates: Then for each integer n we have

$$\phi^n(x, y, z) = (x + n\alpha, y + n\alpha', z + \varphi^{(n)}(x) + \psi^{(n)}(y))$$

and hence $\Gamma(\phi^n) \sim \max(\|\varphi'^{(n)}\|_\infty, \|\psi'^{(n)}\|_\infty)$.

Lemma 6.3. *For every $x, y \in \mathbb{T}$ and $k \in \mathbb{N}$ we have*

$$|\varphi'^{(q_k)}(x)| \leq \frac{6q_k}{u^{-1}(e^{q_k})}, \quad |\varphi''^{(q_k)}(x)| \leq \frac{6q_k^2}{u^{-1}(e^{q_k})},$$

$$|\psi'^{(q'_k)}(y)| \leq \frac{48q'_k}{u^{-1}(e^{q'_k})}, \quad |\psi''^{(q'_k)}(y)| \leq \frac{48q'^2_k}{u^{-1}(e^{q'_k})}.$$

Proof. Since

$$\varphi^{(m)}(x) = \sum_{n=n_0}^{\infty} \frac{1}{2\pi q_n u^{-1}(e^{q_n})} \operatorname{Re} e^{2\pi i q_n x} \frac{e^{2\pi i m q_n \alpha} - 1}{e^{2\pi i q_n \alpha} - 1},$$

we obtain

$$\varphi'^{(m)}(x) = \sum_{n=n_0}^{\infty} \frac{1}{u^{-1}(e^{q_n})} \operatorname{Im} e^{2\pi i q_n x} \frac{e^{2\pi i m q_n \alpha} - 1}{e^{2\pi i q_n \alpha} - 1}, \tag{16}$$

hence

$$|\varphi'^{(q_k)}(x)| \leq \sum_{n=n_0}^{\infty} \frac{1}{u^{-1}(e^{q_n})} \frac{|e^{2\pi i q_k q_n \alpha} - 1|}{|e^{2\pi i q_n \alpha} - 1|}.$$

In the next chain of inequalities we use that by increasing n_0 we can assume that $\sum_{n=n_0}^{\infty} q_n/u^{-1}(e^{q_n}) < 1/4$. We have

$$\begin{aligned}
&\sum_{n=n_0}^{k-1} \frac{1}{u^{-1}(e^{q_n})} \frac{|e^{2\pi i q_k q_n \alpha} - 1|}{|e^{2\pi i q_n \alpha} - 1|} \\
&\leq \sum_{n=n_0}^{k-1} \frac{2}{u^{-1}(e^{q_n})} \frac{\|q_k q_n \alpha\|}{\|q_n \alpha\|} \leq \sum_{n=n_0}^{k-1} \frac{2}{u^{-1}(e^{q_n})} \frac{q_n \|q_k \alpha\|}{\|q_n \alpha\|} \\
&\leq \sum_{n=n_0}^{k-1} \frac{4}{u^{-1}(e^{q_n})} \frac{q_n q_{n+1}}{q_{k+1}} \leq \frac{4q_k}{q_{k+1}} \sum_{n=n_0}^{k-1} \frac{q_n}{u^{-1}(e^{q_n})} \leq \frac{q_k}{q_{k+1}} \leq \frac{q_k}{q'_k}.
\end{aligned}$$

In view of (14),

$$\frac{q_k}{q'_k} \leq \frac{1}{2u(q'_k)} \frac{q_k}{u^{-1}(e^{q_k})} \leq \frac{q_k}{u^{-1}(e^{q_k})}.$$

It follows that

$$\sum_{n=n_0}^{k-1}\frac{1}{u^{-1}(e^{q_n})}\frac{|e^{2\pi i q_k q_n\alpha}-1|}{|e^{2\pi i q_n\alpha}-1|}\leq\frac{q_k}{u^{-1}(e^{q_k})}\ .$$

Furthermore,

$$\sum_{n=k}^{\infty}\frac{1}{u^{-1}(e^{q_n})}\frac{|e^{2\pi i q_k q_n\alpha}-1|}{|e^{2\pi i q_n\alpha}-1|}\leq\sum_{n=k}^{\infty}\frac{2q_k}{u^{-1}(e^{q_n})}\leq\frac{4q_k}{u^{-1}(e^{q_k})},$$

and the required upper bound for $|\varphi'^{(q_k)}(x)|$ follows.

Since

$$|\varphi''^{(q_k)}(x)|\leq\sum_{n=n_0}^{\infty}\frac{2\pi q_n}{u^{-1}(e^{q_n})}\frac{|e^{2\pi i q_k q_n\alpha}-1|}{|e^{2\pi i q_n\alpha}-1|},$$

similar arguments to those above show that $|\varphi''^{(q_k)}(x)|\leq 48q_n^2/u^{-1}(e^{q_k})$.

The remaining inequalities are proved similarly. ■

Lemma 6.4. *For every natural m and k we have*

$$\|\varphi'^{(m)}\|_\infty\leq\frac{6m}{u^{-1}(e^{q_k})}+q_k,\ \ \|\varphi''^{(m)}\|_\infty\leq\frac{48mq_k}{u^{-1}(e^{q_k})}+q_k,$$

$$\|\psi'^{(m)}\|_\infty\leq\frac{6m}{u^{-1}(e^{q'_k})}+q'_k,\ \ \|\psi''^{(m)}\|_\infty\leq\frac{48mq'_k}{u^{-1}(e^{q'_k})}+q'_k.$$

Proof. Write m as $m=pq_k+r$, where $p=[m/q_k]$ and $0\leq r<q_k$. Then

$$\|\varphi'^{(m)}\|_\infty\leq p\|\varphi'^{(q_k)}\|_\infty+\|\varphi'^{(r)}\|_\infty\leq\frac{m}{q_k}\frac{6q_k}{u^{-1}(e^{q_k})}+r\|\varphi'\|_\infty\leq\frac{6m}{u^{-1}(e^{q_k})}+q_k.$$

The remaining inequalities are proved similarly. ■

A van der Corput like Lemma:[1] For estimating the rate of mixing, we shall need the following version of the van der Corput Lemma:

Lemma 6.5. *Let $f:\mathbb{T}\to\mathbb{R}$ be a C^1 function. Suppose there exist a family $\{(a_j;b_j)\subset\mathbb{T}:j=1,\ldots,s\}$ of pairwise disjoint intervals and a real positive number a such that $|f'(x)|\geq a>0$ for all $x\in\mathbb{T}\setminus\bigcup_{j=1}^s(a_j;b_j)$. Then*

$$\left|\int_{\mathbb{T}}e^{2\pi i f(x)}dx\right|\leq\frac{1}{2\pi}\frac{\|f''\|_\infty}{a^2}+\frac{s}{\pi a}+\sum_{j=1}^{s}(b_j-a_j).\tag{17}$$

Proof. Without loss of generality we can assume that $a_1<b_1<\ldots<a_s<b_s<a_1$. Put $D=\bigcup_{j=1}^s(a_j;b_j)$ and $a_{s+1}=a_1$. Then

$$\begin{aligned}\left|\int_{\mathbb{T}}e^{2\pi i f(x)}\,dx\right| &\leq \left|\int_{\mathbb{T}\setminus D}e^{2\pi i f(x)}\,dx\right|+\sum_{j=1}^{s}(b_j-a_j)\\ &= \left|\int_{\mathbb{T}\setminus D}\frac{1}{2\pi i f'(x)}\,de^{2\pi i f(x)}\right|+\sum_{j=1}^{s}(b_j-a_j).\end{aligned}$$

[1] It is known also as a stationary phase argument.

Integrating by parts gives

$$
\begin{aligned}
&\left|\int_{\mathbb{T}\setminus D} \frac{1}{2\pi i f'(x)}\, de^{2\pi i f(x)}\right| \\
&= \left|\sum_{j=1}^{s}\left(\frac{e^{2\pi i f(a_{j+1})}}{2\pi f'(a_{j+1})} - \frac{e^{2\pi i f(b_j)}}{2\pi f'(b_j)} - \frac{1}{2\pi}\int_{b_j}^{a_{j+1}} e^{2\pi i f(x)}\, d\left(\frac{1}{f'(x)}\right)\right)\right| \\
&= \left|\sum_{j=1}^{s}\left(\frac{e^{2\pi i f(a_{j+1})}}{2\pi f'(a_{j+1})} - \frac{e^{2\pi i f(b_j)}}{2\pi f'(b_j)} + \frac{1}{2\pi}\int_{b_j}^{a_{j+1}} e^{2\pi i f(x)}\frac{f''(x)}{(f'(x))^2}\, dx\right)\right| \\
&\leq \frac{1}{2\pi}\sum_{j=1}^{s}\left[\left(\frac{1}{|f'(a_j)|} + \frac{1}{|f'(b_j)|}\right) + \sum_{j=1}^{s}|a_{j+1}-b_j|\frac{\|f''\|_\infty}{a^2}\right] \\
&\leq \frac{1}{2\pi}\frac{\|f''\|_\infty}{a^2} + \frac{s}{\pi a}.
\end{aligned}
$$

■

Lemma 6.6. *There exists $C > 0$ such that*

$$
I_m := \left|\int_{\mathbb{T}^2} e^{2\pi i(\varphi^{(m)}(x)+\psi^{(m)}(y))}\, dxdy\right| \leq C\frac{\log u(m)}{u(m)^{1/3}}.
$$

Proof. For each m large enough there exists a natural number $k \geq n_0$ such that

$$
u^{-1}(e^{q_k}) \leq \frac{m}{u(m)} \leq u^{-1}(e^{q'_k}) \text{ or } u^{-1}(e^{q'_k}) \leq \frac{m}{u(m)} \leq u^{-1}(e^{q_{k+1}}).
$$

Suppose that $m/u(m) \in [u^{-1}(e^{q_k}); u^{-1}(e^{q'_k})]$. Then

$$
\frac{m}{u(m)} \leq u^{-1}(e^{q'_k}) \leq \frac{q_{k+1}}{2u(q_{k+1})} \leq \frac{q_{k+1}/2}{u(q_{k+1}/2)}
$$

and hence $m \leq q_{k+1}/2$ because of the concavity of u.

Put

$$
a_j = \frac{1}{2q_k}\left(j - \frac{1}{u(m)^{1/3}}\right) - \frac{(m-1)\alpha}{2},\ \ b_j = \frac{1}{2q_k}\left(j + \frac{1}{u(m)^{1/3}}\right) - \frac{(m-1)\alpha}{2}
$$

for $j = 1, \ldots, 2q_k$. If $x \in \mathbb{T} \setminus \bigcup_{j=1}^{2q_k}(a_j; b_j)$, then

$$
1/u(m)^{1/3} \leq \|2q_k(x + (m-1)\alpha/2)\| \leq |\sin 2\pi q_k(x + (m-1)\alpha/2)|.
$$

By (16),

$$
\begin{aligned}
|\varphi'^{(m)}(x)| \geq\ & \frac{1}{u^{-1}(e^{q_k})}\left|\operatorname{Im} e^{2\pi i q_k x}\frac{e^{2\pi i m q_k \alpha}-1}{e^{2\pi i q_k \alpha}-1}\right| \\
& - \sum_{n=n_0}^{k-1}\frac{1}{u^{-1}(e^{q_n})}\frac{|e^{2\pi i m q_n \alpha}-1|}{|e^{2\pi i q_n \alpha}-1|} - \sum_{n=k+1}^{\infty}\frac{1}{u^{-1}(e^{q_n})}\frac{|e^{2\pi i m q_n \alpha}-1|}{|e^{2\pi i q_n \alpha}-1|}.
\end{aligned}
$$

Note that

$$\begin{aligned}
&\left|\operatorname{Im} e^{2\pi i q_k x}\frac{e^{2\pi i m q_k\alpha}-1}{e^{2\pi i q_k\alpha}-1}\right| \\
= &\left|\frac{1}{2i}\left(e^{2\pi i q_k x}\frac{e^{2\pi i q_k m\alpha}-1}{e^{2\pi i q_k\alpha}-1}-e^{-2\pi i q_k x}\frac{e^{-2\pi i q_k m\alpha}-1}{e^{-2\pi i q_k\alpha}-1}\right)\right| \\
= &\left|\frac{1}{2i}\frac{e^{2\pi i q_k m\alpha}-1}{e^{2\pi i q_k\alpha}-1}\left(e^{2\pi i q_k x}-e^{-2\pi i q_k(x+(m-1)\alpha)}\right)\right| \\
= &\frac{|e^{2\pi i q_k m\alpha}-1|}{|e^{2\pi i q_k\alpha}-1|}|\sin 2\pi q_k(x+(m-1)\alpha/2)|.
\end{aligned}$$

Since $m \le q_{k+1}/2$ and $\|q_k\alpha\| < 1/q_{k+1}$, we have

$$\|mq_k\alpha\| \le m\|q_k\alpha\| \le \frac{1}{2}q_{k+1}\|q_k\alpha\| < \frac{1}{2},$$

hence $\|mq_k\alpha\| = m\|q_k\alpha\|$. It follows that

$$\frac{|e^{2\pi i q_k m\alpha}-1|}{|e^{2\pi i q_k\alpha}-1|} \ge \frac{\|q_k m\alpha\|}{2\|q_k\alpha\|} = \frac{m}{2}.$$

Thus

$$\left|\operatorname{Im} e^{2\pi i q_k x}\frac{e^{2\pi i m q_k\alpha}-1}{e^{2\pi i q_k\alpha}-1}\right| \ge \frac{m}{2u(m)^{1/3}u^{-1}(e^{q_k})}\ .$$

Since $\|q_n\alpha\| > 1/(2q_{n+1})$, we have

$$\begin{aligned}
\sum_{n=n_0}^{k-1}\frac{1}{u^{-1}(e^{q_n})}\frac{|e^{2\pi i m q_n\alpha}-1|}{|e^{2\pi i q_n\alpha}-1|} &\le \sum_{n=n_0}^{k-1}\frac{1}{u^{-1}(e^{q_n})}\frac{1}{2\|q_n\alpha\|} \le \sum_{n=n_0}^{k-1}\frac{1}{u^{-1}(e^{q_n})}q_{n+1} \\
&\le q_k\sum_{n=n_0}^{k-1}\frac{1}{u^{-1}(e^{q_n})} \le q_k.
\end{aligned}$$

Moreover

$$\sum_{n=k+1}^{\infty}\frac{1}{u^{-1}(e^{q_n})}\frac{|e^{2\pi i m q_n\alpha}-1|}{|e^{2\pi i q_n\alpha}-1|} \le m\sum_{n=k+1}^{\infty}\frac{2}{u^{-1}(e^{q_n})} \le \frac{4m}{u^{-1}(e^{q_{k+1}})}.$$

Therefore, if $x \in \mathbb{T}\setminus\bigcup_{j=1}^{2q_k}(a_j;b_j)$, then

$$|\varphi'^{(m)}(x)| \ge \frac{m}{2u(m)^{1/3}u^{-1}(e^{q_k})} - q_k - \frac{4m}{u^{-1}(e^{q_{k+1}})}.$$

Since $u^{-1}(e^{q_k}) \le m/u(m) \le m$, we have

$$q_k \le \log u(m) \le \frac{\log u(m)}{u(m)^{2/3}}\frac{m}{u(m)^{1/3}u^{-1}(e^{q_k})}.$$

Moreover, since

$$u^{-1}(e^{q_k}) \le \frac{m}{u(m)} \le q_{k+1} \text{ and } u(m) \le m^{3/4},$$

we have

$$\begin{aligned}
\frac{m}{u^{-1}(e^{q_{k+1}})} &\le \frac{m}{u(m)^{1/3}u^{-1}(e^{q_k})}\frac{m/(u(m))^{2/3}}{u^{-1}(e^{m/u(m)})} \\
&\le \frac{m}{u(m)^{1/3}u^{-1}(e^{q_k})}\frac{(m/u(m))^2}{u^{-1}(e^{m/u(m)})}.
\end{aligned}$$

Therefore, for m large enough,

$$|\varphi'^{(m)}(x)| \geq \frac{m}{4u(m)^{1/3}u^{-1}(e^{q_k})} \text{ for all } x \in \mathbb{T} \setminus \bigcup_{j=1}^{2q_k}(a_j;b_j). \tag{18}$$

On the other hand, by Lemma 6.4,

$$|\varphi''^{(m)}(x)| \leq \frac{48mq_k}{u^{-1}(e^{q_k})} + q_k \leq \frac{50mq_k}{u^{-1}(e^{q_k})}.$$

An application of Lemma 6.5 for the function $\varphi^{(m)}$ and the family of intervals $(a_i;b_i)$, $i = 1,\dots,2q_k$ gives

$$\begin{aligned}
&\left|\int_{\mathbb{T}} e^{2\pi i\varphi^{(m)}(x)}\,dx\right| \\
&\leq \frac{1}{2\pi}\frac{\frac{50mq_k}{u^{-1}(e^{q_k})}}{\left(\frac{m}{4u(m)^{1/3}u^{-1}(e^{q_k})}\right)^2} + \frac{2q_k}{\frac{\pi m}{4u(m)^{1/3}u^{-1}(e^{q_k})}} + \frac{2}{u(m)^{1/3}} \\
&= \frac{400q_k u^{-1}(e^{q_k})u(m)^{2/3}}{\pi m} + \frac{4q_k u^{-1}(e^{q_k})u(m)^{1/3}}{\pi m} + \frac{2}{u(m)^{1/3}} \\
&\leq \frac{200q_k u^{-1}(e^{q_k})u(m)^{2/3}}{m} + \frac{2}{u(m)^{1/3}}.
\end{aligned}$$

Since $u^{-1}(e^{q_k}) \leq m/u(m)$, we have $q_k \leq \log u(m)$ and

$$\frac{q_k u^{-1}(e^{q_k})u(m)^{2/3}}{m} \leq \frac{\log u(m)}{u(m)^{1/3}}.$$

Consequently

$$\left|\int_{\mathbb{T}} e^{2\pi i\varphi^{(m)}(x)}\,dx\right| \leq 202\frac{\log u(m)}{u(m)^{1/3}}.$$

When $m/u(m) \in [u^{-1}(e^{q'_k}); u^{-1}(e^{q_{k+1}})]$, proceeding in the same way we obtain

$$\left|\int_{\mathbb{T}} e^{2\pi i\psi^{(m)}(y)}\,dy\right| \leq 202\frac{\log u(m)}{u(m)^{1/3}}.$$

Therefore for each natural m we have

$$I_m = \left|\int_{\mathbb{T}} e^{2\pi i\varphi^{(m)}(x)}dx\right|\left|\int_{\mathbb{T}} e^{2\pi i\psi^{(m)}(y)}dy\right| \leq 202\frac{\log u(m)}{u(m)^{1/3}}.$$

■

Proof of Theorem 6.1.

(i): Take $f : \mathbb{T}^3 \to \mathbb{R}$ given by $f(x,y,z) = \sin 2\pi z$. Then in view of Lemma 6.6 we obtain

$$\begin{aligned}
|(f\circ\phi^n, f)| &= \frac{1}{2}\left|\operatorname{Im}\int_{\mathbb{T}^2} e^{2\pi i(\varphi^{(n)}(x)+\psi^{(n)}(y))}\,dxdy\right| \\
&\leq \frac{1}{2}\left|\int_{\mathbb{T}^2} e^{2\pi i(\varphi^{(n)}(x)+\psi^{(n)}(y))}\,dxdy\right| \leq \text{const}\cdot\frac{\log u(n)}{u(n)^{1/3}}
\end{aligned}$$

for all $n \in \mathbb{N}$ large enough.

(ii): Since

$$\phi^n(x,y,z) = (x+n\alpha, y+n\alpha', z+\varphi^{(n)}(x)+\psi^{(n)}(y))$$

it suffices to show that $\max(\|\varphi'^{(n)}\|_\infty, \|\psi'^{(n)}\|_\infty) \le c_1 u(n)$ for infinitely many $n \in \mathbb{N}$. By Lemma 6.3 and Lemma 6.4,

$$\|\varphi'^{(q_k)}\|_\infty \le 1 \text{ and } \|\psi'^{(q_k)}\|_\infty \le \frac{6q_k}{u^{-1}(e^{q'_{k-1}})} + q'_{k-1}.$$

From (14) we have

$$q'_{k-1} \le \log u(q_k) \text{ and } u^{-1}(e^{q'_{k-1}}) \ge \frac{q_k}{3u(q_k)}.$$

It follows that

$$\|\psi'^{(q_k)}\|_\infty \le 18u(q_k) + \log u(q_k) \le 20u(q_k)$$

for all k large enough.

(iii): Set

$$g_m := \max(\|\varphi'^{(m)}\|_\infty, \|\psi'^{(m)}\|_\infty) \text{ and } \widehat{g}_m = \max_{0\le i\le m} g_i.$$

It suffices to show that

$$c_2\frac{\sqrt{m}}{\log u(m)} \le \widehat{g}_m \le c_3 u(\sqrt{m})\sqrt{m}\ , \tag{19}$$

where the left hand side inequality holds for every natural m and the right hand side holds for infinitely many m.

By Lemma 6.4,

$$\widehat{g}_m \le \max\left(\frac{6m}{u^{-1}(e^{q_k})} + q_k, \frac{6m}{u^{-1}(e^{q'_k})} + q'_k\right) \tag{20}$$

for every natural m and k. Choose x and y so that

$$\sin(2\pi q_k(x + (m-1)\alpha/2)) = \sin(2\pi q'_k(y + (m-1)\alpha'/2)) = 1\ .$$

Proceeding along the same lines as in the proof of Lemma 6.6 one readily shows that

$$u^{-1}(e^{q_k}) \le \frac{m}{u(m)} \le u^{-1}(e^{q'_k}) \implies g_m \ge |\varphi'^{(m)}(x)| \ge \frac{m}{4u^{-1}(e^{q_k})}, \tag{21}$$

$$u^{-1}(e^{q'_k}) \le \frac{m}{u(m)} \le u^{-1}(e^{q_{k+1}}) \implies g_m \ge |\psi'^{(m)}(y)| \ge \frac{m}{4u^{-1}(e^{q'_k})}. \tag{22}$$

To prove the lower bound on $\widehat{g}_m$ suppose that $u^{-1}(e^{q_k}) \le m/u(m) \le u^{-1}(e^{q'_k})$ (the case of $u^{-1}(e^{q'_k}) \le m/u(m) \le u^{-1}(e^{q_{k+1}})$ is treated similarly).

Case 1. Suppose that $m \le (u^{-1}(e^{q_k}))^2$. Set $m_0 := [u^{-1}(e^{q_k})]$. Then

$$u^{-1}(e^{q_k})/2 \le m_0 \le u^{-1}(e^{q_k}) \le m$$

and

$$u^{-1}(e^{q'_{k-1}}) \le q_k \le \frac{e^{q_k/3}}{2} \le \left(\frac{u^{-1}(e^{q_k})}{2}\right)^{1/4} \le m_0^{1/4} \le \frac{m_0}{u(m_0)} \le u^{-1}(e^{q_k}).$$

Therefore in view of (22), we obtain

$$\widehat{g}_m \ge g_{m_0} \ge \frac{m_0}{4u^{-1}(e^{q'_{k-1}})} \ge \frac{u^{-1}(e^{q_k})}{8u^{-1}(e^{q'_{k-1}})} \ge \frac{u^{-1}(e^{q_k})}{4q_k} \ge \frac{\sqrt{m}}{4\log u(m)}.$$

Case 2. Suppose that $m \geq (u^{-1}(e^{q_k}))^2$. Then in view of (21), we obtain

$$\widehat{g}_m \geq g_m \geq \frac{m}{4u^{-1}(e^{q_k})} \geq \sqrt{m}/4.$$

The desired lower bound on $\widehat{g}_m$ follows.

To prove the upper bound on $\widehat{g}_m$ in formula (19) we take $m = (q_k')^2$. Then

$$\frac{6m}{u^{-1}(e^{q_k})} + q_k = \frac{6(q_k')^2}{u^{-1}(e^{q_k})} + q_k \leq 18u(q_k')q_k' + q_k \leq 20u(q_k')q_k' \leq 20u(\sqrt{m})\sqrt{m}.$$

Moreover

$$\frac{6m}{u^{-1}(e^{q_k'})} + q_k' = \frac{6(q_k')^2}{u^{-1}(e^{q_k'})} + q_k' \leq 2q_k' = 2\sqrt{m}.$$

Finally, from (20) we have

$$\widehat{g}_m \leq 20u(\sqrt{m})\sqrt{m}.$$

This completes the proof. ■

7. Growth of the Rudin-Shapiro shift

In the present section we prove the following result.

Theorem 7.1. *Fix $d > 0$. There exists a bi-Lipschitz homeomorphism ϕ of a compact measure metric space (X, ρ, μ) with the following properties:*

- (i) *The upper box dimension (see formula (1) above) of (X, ρ) equals d. Furthermore, for every $\delta > 0$ there exists a δ-net in X containing at most $\mathrm{const} \cdot \delta^{-d}$ points (see Condition 1.3 above);*
- (ii) *The homeomorphism ϕ mixes a nonzero Lipschitz function $f : X \to \mathbb{R}$ with zero mean at the speediest possible rate, i.e. $(f \circ \phi^k, f)_{L_2(X,\mu)} = 0$ for all $k \neq 0$;*
- (iii) *There exist $c_1, c_2 > 0$ so that the growth rate of ϕ satisfies*

$$c_1 \cdot n^{1/d} \leq \widehat{\Gamma}_n(\phi) \leq c_2 \cdot n^{1/d}$$

for all $n \in \mathbb{N}$.

Thus we confirm that the lower bound (7) in Theorem 1.8 is sharp. As we shall explain below, the homeomorphism ϕ can be chosen as the shift associated to the Rudin-Shapiro sequence.

In what follows we work in the framework of the theory of symbolic dynamical systems associated to substitutions (see [12, 8]). Let us consider a finite alphabet $\mathcal{A}$. Denote by $\mathcal{A}^* = \bigcup_{n\geq 1} \mathcal{A}^n$ the set of all finite words over the alphabet $\mathcal{A}$. A *substitution* on $\mathcal{A}$ is a mapping $\zeta : \mathcal{A} \to \mathcal{A}^*$. Any substitution ζ induces two maps, also denoted by ζ, one from $\mathcal{A}^*$ to $\mathcal{A}^*$ and another from $\mathcal{A}^{\mathbb{N}}$ to $\mathcal{A}^{\mathbb{N}}$ by putting

$$\zeta(a_0 a_1 \dots a_n) = \zeta(a_0)\zeta(a_1)\dots\zeta(a_n) \text{ for every } a_0 a_1 \dots a_n \in \mathcal{A}^*,$$

$$\zeta(a_0 a_1 \dots a_n \dots) = \zeta(a_0)\zeta(a_1)\dots\zeta(a_n)\dots \text{ for every } a_0 a_1 \dots a_n \dots \in \mathcal{A}^{\mathbb{N}}.$$

If there exists a letter $a \in \mathcal{A}$ so that $\zeta(a)$ consists of at least two letters and starts with a, the word $\zeta^n(a)$ starts with $\zeta^{n-1}(a)$ and is strictly longer than $\zeta^{n-1}(a)$. Thus $\zeta^n(a)$ converges in the obvious sense as $n \to \infty$ to an infinite word $v \in \mathcal{A}^{\mathbb{N}}$ such that $\zeta(v) = v$.

We can associate to the sequence v a topological dynamical system as follows. Let $\mathcal{L}(v)$ denote the language of the sequence v, i.e. the set of all finite words (over the alphabet $\mathcal{A}$) which occur in v. Let $X_v \subset \mathcal{A}^{\mathbb{Z}}$ stand for the set of all sequences $x = \{x_n\}_{n\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ such that $x_n x_{n+1} \ldots x_{n+k-1} \in \mathcal{L}(v)$ for all $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Obviously, X_v is a compact subset of $\mathcal{A}^{\mathbb{Z}}$ with the product topology and X_v is invariant under the two-sided Bernoulli shift $\phi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, $[\phi(\{x_k\}_{k\in\mathbb{Z}})]_n = x_{n+1}$. Therefore we can consider ϕ as a homeomorphism of X_v.

A substitution ζ is called *primitive* if there exists $k \geq 1$ such that $\zeta^k(a)$ contains b for every $a, b \in \mathcal{A}$. If ζ is primitive, the space $X = X_v$ does not depend on the choice of v. Furthermore, the corresponding homeomorphism $\phi : X \to X$ is minimal and uniquely ergodic. Unique ergodicity of ϕ can be deduced from the analogous result in [12, Chapter V] for the one-sided shift: Given two words $z, w \in \mathcal{L}(v)$, denote by $\Omega_z(w)$ the number of appearances of z as a sub-word in w. Unique ergodicity of the one-sided shift yields (see [12, Corollary IV.14]) existence of a positive function $\omega : \mathcal{L}(v) \to (0; 1]$ so that for every z

$$\frac{\Omega_z(w)}{\text{length}(w)} \to \omega(z) \ \text{ uniformly in } \ w \ \text{ as } \ \text{length}(w) \to \infty \,. \tag{23}$$

This in turn yields, exactly as in [12, Corollary IV.14], unique ergodicity of the two-sided shift ϕ.

Let us consider the Rudin–Shapiro sequence $v = \{v_n\}_{n\geq 0}$ over the alphabet $\mathcal{A} = \{-1, +1\}$ which is defined by the relation

$$v_0 = 1, \;\; v_{2n} = v_n, \;\; v_{2n+1} = (-1)^n v_n \text{ for any } n \geq 0.$$

It arises from the fixed point $ABACABDB\ldots$ of the primitive substitution $A \mapsto AB, B \mapsto AC, C \mapsto DB, D \mapsto DC$ after replacing A, B by $+1$ and C, D by -1. As above, we associate to the sequence v the topological space $X \subset \mathcal{A}^{\mathbb{Z}}$ and the two-sided shift $\phi : X \to X$. Notice that ϕ is uniquely ergodic as a factor of the corresponding uniquely ergodic substitution system.

Proof of Theorem 7.1. Let μ be the unique ϕ-invariant Borel probability measure on X. We shall show that after a suitable choice of a metric on X, the shift $\phi : X \to X$ possesses properties (i)-(iii) stated in the theorem.

Choosing the metric: Fix a concave increasing function $u : [0; +\infty) \to [0; +\infty)$ such that $u(0) = 0$ and $u(t) \to +\infty$ as $t \to +\infty$. Then define a metric ρ on X by putting $\rho(x, y) = e^{-u(k(x,y))}$, where $k(x, y) = \min\{|k| : x_k \neq y_k, k \in \mathbb{Z}\}$ for two distinct sequences $x, y \in X$. Of course, ϕ is a bi-Lipschitz homeomorphism with $\widehat{\Gamma}_n(\phi) \leq e^{u(n)}$.

Denote by $\{p_n(v)\}$ the complexity of the sequence v, that is $p_n(v)$ is the number of different words of length n occurring in v. As it was shown in [1], $p_n(v) = 8(n-1)$ for every $n \geq 2$ (in fact, a simpler estimate $n \leq p_n(v) \leq \text{const} \cdot n$ is sufficient for our purposes, see Propositions 1.1.1 and 5.4.6 in [8]).

Suppose that $u(t) = d^{-1} \log t$ for all t large enough. Given $k > 2$, put $p = p_{2k-1}(v)$ and consider all possible words $w^{(1)}, \ldots, w^{(p)}$ from $\mathcal{L}(v)$ of length $2k - 1$. Fix arbitrary elements $x^{(i)} \in X$, $i = 1, \ldots, p$ so that $x^{(i)}_{-k+1} x^{(i)}_{-k+2} \ldots x^{(i)}_{k-2} x^{(i)}_{k-1} = w^{(i)}$. Note that the points $x^{(i)}$ lie at the distance $\geq e^{-u(k-1)} = (k-1)^{-1/d}$ one from the other. Furthermore, every point of X lies at the distance $\leq e^{-u(k)} = k^{-1/d}$ from $x^{(i)}$ for some $i = 1, \ldots, p$. Recalling that $p = 16k - 16$ we conclude that the upper box dimension of X equals d, and moreover for every $\delta > 0$ there exists a δ-net in X containing at most $\text{const} \cdot \delta^{-d}$ points. Thus we get property (i) in Theorem 7.1.

MIXING: Consider a function $f : X \to \mathbb{R}$, $f(x) = x_0$. Clearly, f is Lipschitz with respect to ρ. Let us check that $(f \circ \phi^k, f)_{L_2(X,\mu)} = 0$ for all $k \neq 0$. We prove this property by combining the unique ergodicity of ϕ with the following fact, see [8, Proposition 2.2.5]:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} v_n v_{n+k} = 0 \ \ \forall k \in \mathbb{N} .$$

Indeed, there exists a sequence $y \in X$ such that $y_n = v_n$ for all $n \geq 0$ (see Lemma 7.3 below). Then

$$\begin{aligned} \int_X f(\phi^k x) f(x)\, d\mu(x) &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\phi^{k+n} y) f(\phi^n y) \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} v_n v_{n+k} = 0. \end{aligned}$$

This proves property (ii) in Theorem 7.1.

GROWTH BOUNDS: The lower bound (7) in Theorem 1.8 yields $\widehat{\Gamma}_n(\phi) \geq \text{const} \cdot n^{1/d}$. On the other hand

$$\widehat{\Gamma}_n(\phi) \leq e^{u(n)} = n^{1/d},$$

which yields property (iii) in Theorem 7.1.
This completes the proof. ■

Remark 7.2. Let us modify the metric ρ defined above by taking the function $u(t)$ to be of an arbitrarily slow growth. As a result we get an example of a bi-Lipschitz homeomorphism ϕ of a compact metric measure space (M, ρ, μ) of *infinite* box dimension which mixes a Lipschitz function f at the speediest possible rate, that is $(f, f \circ \phi^n)_{L_2} = 0$ for all $n \in \mathbb{N}$, and such that the growth rate of $\widehat{\Gamma}_n(\phi)$ is arbitrarily slow. This illustrates the significance of Condition 1.3 on the metric ρ for the validity of the statement of Theorem 1.8.

We conclude this section with the following lemma which was used in the proof of Theorem 7.1 above.

Lemma 7.3. *There exists a sequence $y \in X$ so that $y_n = v_n$ for all $n \geq 0$.*

Proof. By (23), for every $n \in \mathbb{N}$ the word $v_0 \dots v_n$ appears infinitely many times as a subword in v. Thus we can find a sequence of words of the form $y^{(n)} = y^{(n)}_{-n} \dots y^{(n)}_{-1} v_0 \dots v_n$, $n \in \mathbb{N}$ in the language $\mathcal{L}(v)$. Next we choose a collection $\{\{n^l_k\}_{k\in\mathbb{N}}\}_{l\in\mathbb{N}}$ of increasing sequences of natural numbers by the following inductive procedure: Since $\{y^{(n)}_{-1}\}_{n\in\mathbb{N}}$ takes only two values, we can find an increasing sequence $\{n^1_k\}_{k\in\mathbb{N}}$ such that $\{y^{(n^1_k)}_{-1}\}_{k\in\mathbb{N}}$ is constant. Assume that the sequence $\{n^l_k\}_{k\in\mathbb{N}}$ is already chosen. Choose $\{n^{l+1}_k\}_{k\in\mathbb{N}}$ as a subsequence of $\{n^l_k\}_{k\in\mathbb{N}}$ for which $\{y^{(n^{l+1}_k)}_{-l-1}\}_{k\in\mathbb{N}}$ is constant. Now we can define the desired sequence $y = \{y_k\}_{k\in\mathbb{Z}} \in X$ by putting

$$y_{-k} = y^{(n^k_k)}_{-k} \text{ for } k > 0 \text{ and } y_k = v_k \text{ for } k \geq 0.$$

■

8. Appendix: Kolmogorov-Tihomirov formula

In this section we prove formula (2). Cover A by $n = \mathcal{N}_{\epsilon/(4R)}(A)$ balls $A_1, \dots, A_n$ of radius $\epsilon/(8R)$ centered at $a_1, \dots, a_n \in A$ respectively and cover Y by $m = \mathcal{N}_{\epsilon/4}(Y)$ balls $Y_1, \dots, Y_m$ of radius $\epsilon/8$ centered at $y_1, \dots, y_m$ respectively. Put $I = \{1, \dots, n\}$, $J = \{1, \dots, m\}$. For a map $\sigma : I \to J$ set

$$X_\sigma = \{f \in \mathcal{D}_R^A(Y) \; : \; f(a_i) \in Y_{\sigma(i)} \; \forall i \in I\} \; .$$

Obviously, $\mathcal{D}_R^A(Y)$ is covered by m^n sets X_σ. *Warning:* some of these sets might be in fact empty.

Assume that $f, g \in X_\sigma \cap \mathcal{D}_R^A(Y)$. Take any point $a \in A$. Choose a_i so that $\rho_1(a, a_i) < \epsilon/(8R)$. Then $\rho_2(f(a), f(a_i)) < \epsilon/8$ and $\rho_2(g(a), g(a_i)) < \epsilon/8$ since the Lipschitz constant of f and g is $\leq R$. Furthermore, $\rho_2(f(a_i), y_{\sigma(i)}) < \epsilon/8$ and $\rho_2(g(a_i), y_{\sigma(i)}) < \epsilon/8$. Thus $\rho_2(f(a), g(a)) < \epsilon/2$. Since this is true for all points a in a compact space A we conclude that $\text{dist}(f, g) < \epsilon/2$. It follows that the set $X_\sigma \cap \mathcal{D}_R^A(Y)$ is either empty, or is fully contained in a ball of radius $\epsilon/2$ (in the sense of metric dist) centered at any of its points.

Looking at all $\sigma \in J^I$, we get a covering of $\mathcal{D}_R^A(Y)$ by at most m^n of metric balls of radius $\epsilon/2$, as required.

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