ON HAUSDORFF DIMENSION OF THE SET OF CLOSED ORBITS FOR A CYLINDRICAL TRANSFORMATION

K. FRĄCZEK AND M. LEMAŃCZYK

Abstract. We deal with Besicovitch's problem of existence of discrete orbits for transitive cylindrical transformations $T_\varphi : (x,t) \mapsto (x+\alpha, t+\varphi(x))$ where $Tx = x+\alpha$ is an irrational rotation on the circle $\mathbb{T}$ and $\varphi : \mathbb{T} \to \mathbb{R}$ is continuous, i.e. we try to estimate how big can be the set $D(\alpha,\varphi) := \{x \in \mathbb{T} : |\varphi^{(n)}(x)| \to +\infty \text{ as } |n| \to +\infty\}$. We show that for almost every α there exists φ such that the Hausdorff dimension of $D(\alpha,\varphi)$ is at least $1/2$. We also provide a Diophantine condition on α that guarantees the existence of φ such that the dimension of $D(\alpha,\varphi)$ is positive. Finally, for some multidimensional rotations T on $\mathbb{T}^d$, $d \geq 3$, we construct smooth φ so that the Hausdorff dimension of $D(\alpha,\varphi)$ is positive.

1. Introduction

Let $T : X \to X$ be a minimal homeomorphism of a compact metric space (X,d) and let $\varphi : X \to \mathbb{R}$ be a continuous function. Denote by $T_\varphi : X \times \mathbb{R} \to X \times \mathbb{R}$ the corresponding cylindrical transformation

$$T_\varphi(x,t) = (Tx, t+\varphi(x)).$$

Then $T^n_\varphi(x,t) = (T^n x, t+\varphi^{(n)}(x))$ for every integer n, where

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x)+\varphi(Tx)+\ldots+\varphi(T^{n-1}x) & \text{if} \quad n>0 \\ 0 & \text{if} \quad n=0 \\ -(\varphi(T^{-1}x)+\ldots+\varphi(T^{n+1}x)+\varphi(T^n x)) & \text{if} \quad n<0. \end{cases}$$

Cylindrical transformations appear naturally when studying some autonomous ordinary differential equations on $\mathbb{R}^3$ or on other non-compact manifolds (cf. [17] and Section 8). Moreover, in the case of circle rotations T, cylindrical transformations yield a broad and interesting class of homeomorphisms of the plane. If $Tx = x+\alpha$ then every continuous function $\varphi : \mathbb{T} \to \mathbb{R}$ defines the homeomorphism $f_{\alpha,\varphi} : \mathbb{R}^2 \to \mathbb{R}^2$, $f_{\alpha,\varphi}(re^{2\pi i\omega}) = re^{\varphi(\omega)+2\pi i(\omega+\alpha)}$ for $r \geq 0$ and $\omega \in \mathbb{T}$. The homeomorphism $f_{\alpha,\varphi}$ has a fixed point at zero and restricted to $\mathbb{R}^2 \setminus \{(0,0)\}$ is topologically isomorphic to T_φ via the map

$$\mathbb{T} \times \mathbb{R} \ni (\omega, r) \mapsto e^r e^{2\pi i\omega} \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

A surprising property of general cylindrical transformations is that they are never minimal, that is, there are points whose orbits are not dense [5], [16], (see also [13] for a more general non-minimality result in case of homeomorphisms of the cylinder $\mathbb{T} \times \mathbb{R}$) and the problem of classifying minimal subsets for such transformations is still open[1]. Clearly, a minimal subset arises if we are given a discrete orbit.

2000 *Mathematics Subject Classification.* 37B05, 37C45, 37C29.

Research partially supported by Polish MNiSzW grant N N201 384834; partially supported by Marie Curie "Transfer of Knowledge" EU program – project MTKD-CT-2005-030042 (TODEQ).

[1] The situation changes if instead of minimality we consider so called positive minimality i.e. for a continuous map T (not necessarily invertible) of a locally compact space X we require all semi-orbits $\{T^n x : n \geq 0\}$, $x \in X$, to be dense in X. As it has already been noticed in [4] (see Chapter I, Exercise 11 or the subsequent article [6]) if there is a recurrent point in X and

Recall that a subset S of a topological space is discrete if every point $x \in S$ has a neighborhood U such that $S \cap U = \{x\}$. Moreover, the orbit of $(x,t) \in X \times \mathbb{R}$ for the cylindrical transformation T_φ is discrete if and only if

$$|\varphi^{(n)}(x)| \to +\infty \text{ as } |n| \to +\infty.$$

If T is uniquely ergodic with μ the only T–invariant measure and if $\int_X \varphi\, d\mu \neq 0$ then by unique ergodicity, $\varphi^{(n)}/n \to \int \varphi\, d\mu$ uniformly as $|n| \to +\infty$. Therefore $|\varphi^{(n)}(x)| \to +\infty$ as $|n| \to +\infty$ for each $x \in X$, i.e. φ is transient, and hence every orbit of T_φ is discrete. It follows that the partition of $X \times \mathbb{R}$ into orbits of T_φ yields the decomposition into minimal components. Yet, in one more situation $X \times \mathbb{R}$ is the union of minimal components – it is the case when $\int_X \varphi\, d\mu = 0$ and $\varphi(x) = j(x) - j(Tx)$ for a continuous function $j : X \to \mathbb{R}$, i.e. when φ is a coboundary; indeed, the minimal components are of the form $\{(x, j(x)+a) : x \in X\}$, $a \in \mathbb{R}$. Clearly, in this case there are no discrete orbits, in fact, j exists if and only if each orbit of T_φ is bounded [8].

When we restrict our considerations to T which is a minimal rotation on a compact metric group (the case which is well known to be uniquely ergodic) then we have the following.

Proposition 1 (see [3], [8] or [14]). *If φ is not transient nor φ is a coboundary then T_φ has a dense orbit, i.e. T_φ is topologically transitive.*

Note that, by Proposition 1, it follows that if T is a minimal rotation, φ has zero mean and T_φ has a discrete orbit then T_φ is automatically topologically transitive.

From now on we will only deal with the transitive case and we assume that T is a minimal rotation on X. In this case the set of transitive points is G_δ and dense, however it is always a proper subset of $X \times \mathbb{R}$ since T_φ is not minimal. This set is usually also large from the measure-theoretic point of view; indeed, if we assume ergodicity of T_φ (with respect to the product of Haar measure on X and Lebesgue measure on $\mathbb{R}$) then each open subset of $X \times \mathbb{R}$ has positive measure and since $X \times \mathbb{R}$ is second countable, the complement of the set of transitive points has measure zero. Even in case of T a minimal rotation, the problem of classifying possible minimal subsets for the corresponding cylindrical transformations remains open. It is even open in case of irrational rotations on the circle, although in the latter case we would like to emphasize that if φ is too smooth then there are no minimal subsets at all. More precisely, if $\varphi : \mathbb{T} \to \mathbb{R}$ is of bounded variation then T_φ has no minimal subset (see [15] and [16]). However, Besicovitch [5] already in 1951 showed that, if we require φ to be only continuous, then for T_φ a minimal subset can exist, namely, despite its topological transitivity T_φ can have a discrete orbit. The problem of coexistence of dense orbits (topological transitivity) and discrete orbits for cylindrical transformations when T is a minimal rotation is called the Besicovitch problem, and we will call the cylindrical transformation T_φ Besicovitch if indeed dense and discrete orbits for T_φ coexist. In the present paper we will deal with the Besicovitch problem for rotations on finite dimensional tori $\mathbb{T}^d$.

In Section 2 we show that the phenomenon discovered by Besicovitch [5] – for a particular irrational α there exists a continuous $\varphi : \mathbb{T} \to \mathbb{R}$ such that T_φ is transitive and admits discrete orbits – in fact happens for each irrational α. In other words, we show that for every irrational $\alpha \in \mathbb{T}$ there exists a continuous $\varphi : \mathbb{T} \to \mathbb{R}$ such

T is positively minimal then X has to be compact. Take now an arbitrary continuous map T of a locally compact space X and suppose that $M \subset X$ is positively minimal. Then M is locally compact and it follows that either M is a discrete orbit or M is compact. Therefore there are no positively minimal subsets for transitive T_φ as above – if $M \subset X \times \mathbb{R}$ is positively minimal then it cannot be a discrete orbit and if M is compact then by [8], φ is a coboundary and therefore T_φ is not transitive.

that T_φ is Besicovitch; note that this implies the existence of Besicovitch cylindrical transformations over each minimal rotation on $\mathbb{T}^d$, $d \geq 2$. Indeed, if R is a rotation on $\mathbb{T}^{d-1}$ such that $T \times R$ is minimal then the cylindrical transformation $(T \times R)_{\bar{\varphi}}$, with $\bar{\varphi}(x_1, \ldots, x_d) = \varphi(x_1)$, is Besicovitch if T_φ is Besicovitch (this follows from Proposition 1 since $\widetilde{\varphi}$ is not transient and since no orbit of $(T \times R)_{\widetilde{\varphi}}$ is bounded, $\widetilde{\varphi}$ is not a coboundary).

In Section 3, for α satisfying some Diophantine conditions, our construction of φ is improved and we obtain γ–Hölder continuous functions φ such that T_φ is Besicovitch (it turns out however that in all our constructions $\gamma < 1/2$). A slight modification of the construction yields $\varphi : \mathbb{T} \to \mathbb{R}$ whose Fourier coefficients are $\mathrm{O}(\log |n|/|n|)$; see Section 4. We have already mentioned that in case φ is of bounded variation, T_φ is not Besicovitch. This is one more (direct) consequence of the classical Denjoy-Koksma inequality (see e.g. [11] or [9]); indeed, the inequality

$$|\varphi^{(q_n)}(x)| \leq \mathrm{Var}\ \varphi \quad \text{for each } x \in \mathbb{T}, \tag{1}$$

where (q_n) is the sequence of denominators of α means in particular that the orbit of each point (x,t) is not discrete (in fact T_φ has no minimal subsets at all; see [15] and [16]). In [2], an inequality similar to (1) has been proved in L^2 for functions whose Fourier coefficients are $\mathrm{O}(1/|n|)$, we have been however unable to decide whether there exists a continuous $\varphi : \mathbb{T} \to \mathbb{R}$ whose Fourier coefficients are $\mathrm{O}(1/|n|)$ and which is Besicovitch for some rotation $Tx = x + \alpha$. Recently, in [12], for every minimal odometer T the existence of Besicovitch cylindrical transformation has been proved to exist.

We then pass to deal with the Besicovitch problem for minimal rotations on higher dimensional tori $\mathbb{T}^d$, $d \geq 2$. As it was shown by Yoccoz in [19], the Denjoy-Koksma inequality does not hold anymore in higher dimensions and one can expect that among smooth cylindrical transformations there are Besicovitch cylindrical maps. Such are indeed shown to exist in Section 7. More precisely, we prove that for every $r \geq 1$ there exist $d \geq 3$, a minimal rotation $T : \mathbb{T}^d \to \mathbb{T}^d$ and $\varphi : \mathbb{T}^d \to \mathbb{R}$ of class C^r such that T_φ is Besicovitch; the construction is based on Yoccoz's method from [19].

Once we know that Besicovitch cylindrical transformations exist for each minimal rotation on $\mathbb{T}^d$, another natural problem arises to discuss the size of the set of points whose orbits are discrete. More precisely, we will deal with the set

$$D(\alpha, \varphi) = \{x \in \mathbb{T}^d : \lim_{n \to \pm\infty} |\varphi^{(n)}(x)| \to +\infty\}.$$

By our standing assumption of transitivity, $\int \varphi(x)\, dx = 0$ and thus T_φ is recurrent as an infinite measure-preserving system (see [1], [18]), so for a.e. $x \in \mathbb{T}^d$ there exists $k_n = k_n(x) \to +\infty$ such that $\varphi^{(k_n)}(x) \to 0$, hence $D(\alpha, \varphi)$ has zero Lebesgue measure. Moreover, the set of transitive points for T_φ is G_δ dense, so the set of points whose orbits are discrete is a first category set. Furthermore, this set is equal to $D(\alpha, \varphi) \times \mathbb{R}$, so $D(\alpha, \varphi)$ is a first category subset of $\mathbb{T}^d$ (which is dense if it is nonempty). Consequently, $D(\alpha, \varphi)$ is small from both the topological and the measure theoretical point of view. We are interested in the Hausdorff dimension of $D(\alpha, \varphi)$.

If $d = 1$ then for almost every $\alpha \in \mathbb{T}$, using a modification of the construction from Section 2, we built a continuous function $\varphi : \mathbb{T} \to \mathbb{R}$ with zero mean such that $\dim_H D(\alpha, \varphi) > 0$. Moreover, we give a lower bound on the Hausdorff dimension related to some Diophantine condition of α; see Section 6. We also study the coexistence problem of discrete orbits of different types, more precisely, the size of sets

$$D^{s_- s_+}(\alpha, \varphi) = \{x \in \mathbb{T} : \lim_{m \to -\infty} \varphi^{(m)}(x) \to s_-\infty \text{ and } \lim_{m \to +\infty} \varphi^{(m)}(x) \to s_+\infty\}$$

for $s_-, s_+ \in \{-,+\}$ is investigated. We show that the coexistence of all types of discrete orbits appears for some (transitive) cylindrical transformations; we mention that the same phenomenon was also observed for cylindrical transformations over odometers in [12]. For almost every $\alpha \in \mathbb{T}$ we construct a class of examples for which $\dim_H D^{s_-s_+}(\alpha,\varphi) \geq 1/2$ for every pair (s_-, s_+). This gives evidence that for transitive homeomorphisms of the plane the coexistence of orbits with completely different behavior is possible. Indeed, returning to the homeomorphisms of the plane mentioned at the beginning of this section, let us consider $f_{\alpha,\varphi} : \mathbb{R}^2 \to \mathbb{R}^2$. Then for each $\omega \in \mathbb{T}$ points from the ray $\mathrm{Ray}(\omega) = \{re^{i\omega} : r > 0\}$ generate orbits of the same type and

- if $\omega \in D^{--}(\alpha,\varphi)$ then each $x \in \mathrm{Ray}(\omega)$ generates a homoclinic orbit attracted by zero;
- if $\omega \in D^{++}(\alpha,\varphi)$ then each $x \in \mathrm{Ray}(\omega)$ generates a discrete orbit;
- if $\omega \in D^{-+}(\alpha,\varphi)(D^{+-}(\alpha,\varphi))$ then for each $x \in \mathrm{Ray}(\omega)$ one semi-orbit is attracted by zero and another semiorbit escapes to the infinity.

Therefore there exists a transitive homeomorphism $f_{\alpha,\varphi} : \mathbb{R}^2 \to \mathbb{R}^2$ such that each of the above orbit type appears and the Hausdorff dimension of the set of the corresponding points is no smaller than 3/2.

In the higher dimensional case, for every $r \geq 1$ we construct $\alpha \in \mathbb{T}^d$ and a C^r–function $\varphi : \mathbb{T}^d \to \mathbb{R}$ with zero mean such that $\dim_H D^{++}(\alpha,\varphi) > 0$; see Section 7.

As an application, in Section 8 we demonstrate a family of continuous (or even Hölder) perturbations of some integrable systems which completely destroys its integrable dynamics. More precisely, the perturbed systems have plenty of orbits which are dense, homoclinic and heteroclinic to limit cycles.

The authors are grateful to the referee for pointing out the references [6] and [15].

2. Construction

By $\mathbb{T}$ we will mean the group $\mathbb{R}/\mathbb{Z}$ which most of time will be treated as $[0,1)$ with addition mod 1. By $\{t\}$ we denote the fractional part of t and $\|t\|$ is the distance of t from the set of integers. Denote by $\lfloor t \rfloor$ and $\lceil t \rceil$ the floor and the ceiling of t respectively.

We will show that for each irrational $\alpha \in \mathbb{T}$ we can construct a continuous $\varphi : \mathbb{T} \to \mathbb{R}$ so that the corresponding cylindrical flow T_φ is Besicovitch, i.e. it is topologically transitive but it has some minimal orbits. By Proposition 1, we only need to construct a continuous function φ with integral zero and such that

$$\varphi^{(n)}(0) \to +\infty \ \text{ when } \ |n| \to +\infty;$$

indeed such a φ is neither a coboundary nor transient, so T_φ must be topologically transitive.

Fix an irrational $\alpha \in [0,1)$. Let (p_n/q_n) be the sequence of convergents of α, i.e.

$$\begin{aligned} q_{-1} = 0, \quad q_0 = 1, \quad q_n &= a_n q_{n-1} + q_{n-2} \text{ for } n \geq 1 \\ p_{-1} = 1, \quad p_0 = 0, \quad p_n &= a_n p_{n-1} + p_{n-2} \text{ for } n \geq 1, \end{aligned}$$

where $[0; a_1, a_2, \ldots]$ is the continued fraction of α. We have (see e.g. [10])

$$\frac{1}{2q_nq_{n+1}} < (-1)^n \left(\alpha - \frac{p_n}{q_n}\right) = \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_nq_{n+1}}, \tag{2}$$

hence

$$\frac{1}{2q_{n+1}} < \|q_n\alpha\| < \frac{1}{q_{n+1}}. \tag{3}$$

Let (M_n) be a sequence of natural numbers such that

$$M_n \to +\infty \tag{4}$$

and

$$\sum_{n=1}^{\infty} \frac{M_n}{q_{n-1}} < +\infty. \tag{5}$$

Set

$$L_n = M_n \cdot \frac{q_n q_{n+1}}{q_{n-1}}. \tag{6}$$

In view of (6) and (5),

$$\sum_{n=1}^{\infty} \frac{L_n}{q_n q_{n+1}} < +\infty. \tag{7}$$

We now define $f_n : [0,1) \to \mathbb{R}^+$ which is Lipschitz continuous (with the Lipschitz constant equal to L_n), $1/q_n$–periodic, $f_n(0) = 0$ and $|f_n'(x)| = L_n$ for all $x \in [0,1)$ except for the integer multiples of $\frac{1}{2q_n}$. Notice that $f_n(y) = L_n y$ for $y \in [0, 1/(2q_n))$ and $f_n(y) = L_n(1/q_n - y)$ for $y \in [1/(2q_n), 1/q_n]$. Using $1/q_n$–periodicity of f_n and (2) for each $x \in [0,1)$ we have

$$|f_n(x+\alpha) - f_n(x)| = \left| f_n(x+\alpha) - f_n(x + \frac{p_n}{q_n}) \right| \leq L_n \left| \alpha - \frac{p_n}{q_n} \right| < \frac{L_n}{q_n q_{n+1}},$$

so

$$\|f_n(\,\cdot\, + \alpha) - f_n(\,\cdot\,)\|_{C(\mathbb{T})} < \frac{L_n}{q_n q_{n+1}} \tag{8}$$

and it follows from (7) that the series

$$\varphi(x) = \sum_{n=1}^{\infty} (f_n(x+\alpha) - f_n(x))$$

converges uniformly, so φ is continuous and clearly $\int_0^1 \varphi(x)\,dx = 0$. For each integer k we have

$$\varphi^{(k)}(x) = \sum_{n=1}^{\infty} (f_n(x + k\alpha) - f_n(x)),$$

in particular

$$\varphi^{(k)}(0) = \sum_{n=1}^{\infty} f_n(k\alpha). \tag{9}$$

We will show that $\varphi^{(k)}(0) \to +\infty$ when $|k| \to +\infty$. Fix a nonzero integer k. There is a unique $n = n(k) \geq 0$ such that $q_n \leq |k| < q_{n+1}$. By (2) applied to $n+1$ we have

$$\frac{|k|}{2q_{n+1}q_{n+2}} < \left| k\alpha - k\frac{p_{n+1}}{q_{n+1}} \right| < \frac{|k|}{q_{n+1}q_{n+2}} < \frac{1}{q_{n+2}},$$

so

$$\left| k\alpha - \frac{kp_{n+1}}{q_{n+1}} \right| > \frac{q_n}{2q_{n+1}q_{n+2}}.$$

Moreover.

$$\begin{aligned} \left| k\alpha - \frac{kp_{n+1}}{q_{n+1}} \right| &< \frac{1}{q_{n+2}} = \frac{1}{q_{n+1}} - \left(\frac{1}{q_{n+1}} - \frac{1}{q_{n+2}} \right) = \frac{1}{q_{n+1}} - \frac{q_{n+2} - q_{n+1}}{q_{n+1}q_{n+2}} \\ &\leq \frac{1}{q_{n+1}} - \frac{q_n}{q_{n+1}q_{n+2}} < \frac{1}{q_{n+1}} - \frac{q_n}{2q_{n+1}q_{n+2}}. \end{aligned}$$

Since f_{n+1} is $1/q_{n+1}$–periodic and $f_{n+1}(-x) = f_{n+1}(x)$,

$$f_{n+1}(k\alpha) = f_{n+1}\left(k\alpha - k\frac{p_{n+1}}{q_{n+1}}\right) = f_{n+1}\left(\left|k\alpha - k\frac{p_{n+1}}{q_{n+1}}\right|\right).$$

As

$$\frac{q_n}{2q_{n+1}q_{n+2}} < \left|k\alpha - \frac{kp_{n+1}}{q_{n+1}}\right| < \frac{1}{q_{n+1}} - \frac{q_n}{2q_{n+1}q_{n+2}},$$

by the definition of f_{n+1} and (6),

$$f_{n+1}(k\alpha) \geq f_{n+1}\left(\frac{q_n}{2q_{n+1}q_{n+2}}\right) = L_{n+1} \cdot \frac{q_n}{2q_{n+1}q_{n+2}} = M_{n+1}/2.$$

Since all functions f_l are nonnegative, it follows that

$$\varphi^{(k)}(0) = \sum_{l=1}^{\infty} f_l(k\alpha) \geq f_{n+1}(k\alpha) \geq M_{n+1}/2$$

which tends to $+\infty$ in view of (4) and of the fact that $n = n(k) \to +\infty$ when $|k| \to +\infty$.

3. Hölder continuity condition

We need the following simple lemma.

Lemma 2. *Let (X, d) be a compact metric space. Let $(w_n)_{n=1}^{\infty}$ be an increasing sequence of positive real numbers with $w_n \to +\infty$ such that for every $0 < \beta < 1$ there exists $D_\beta > 0$ for which*

$$\sum_{k=1}^{n} w_k^{\beta} \leq D_\beta w_n^{\beta} \quad \text{and} \quad \sum_{k=n}^{\infty} \frac{1}{w_k^{\beta}} \leq \frac{D_\beta}{w_n^{\beta}} \text{ for all } n \in \mathbb{N}. \tag{10}$$

Assume that $\varphi(x) = \sum_{n=1}^{\infty} \varphi_n(x)$, where $\varphi_n : X \to \mathbb{R}$ is Lipschitz continuous with a Lipschitz constant $L(\varphi_n) = L_n$ such that for some $0 < \gamma < 1$ we have

$$L_n \leq w_n^{1-\gamma} \quad \text{and} \quad \|\varphi_n\|_{C(X)} \leq \frac{1}{w_n^{\gamma}} \quad \text{for } n \geq 1.$$

Then $\varphi : X \to \mathbb{R}$ is γ–Hölder continuous.

Proof. Suppose that

$$\frac{1}{w_{n+1}} < d(x,y) \leq \frac{1}{w_n}.$$

Then

$$\begin{aligned}
|\varphi(x) - \varphi(y)| &\leq \sum_{k=1}^{n} |\varphi_k(x) - \varphi_k(y)| + \sum_{k=n+1}^{\infty} |\varphi_k(x) - \varphi_k(y)| \\
&\leq \sum_{k=1}^{n} L_k d(x,y) + 2\sum_{k=n+1}^{\infty} \|\varphi_k\|_{C(X)} \\
&\leq d(x,y)\sum_{k=1}^{n} w_k^{1-\gamma} + 2\sum_{k=n+1}^{\infty} \frac{1}{w_k^{\gamma}} \\
&\leq D_{1-\gamma} d(x,y) w_n^{1-\gamma} + 2D_\gamma \frac{1}{w_{n+1}^{\gamma}} \leq C d(x,y)^{\gamma}
\end{aligned}$$

because $d(x,y)^{1-\gamma} \leq \frac{1}{w_n^{1-\gamma}}$ and $\frac{1}{w_{n+1}^{\gamma}} \leq d(x,y)^{\gamma}$. □

Remark 1. Recall that if $(v_n)_{n=1}^{\infty}$ is a lacunary sequence, i.e. there exists $A > 1$ such that $v_{n+1} > Av_n$ for all $n \geq 1$, then for each $1 \leq k < n$ we have $v_n > A^{n-k}v_k$, so

$$\sum_{k=1}^{n} v_k < \sum_{k=1}^{n} \frac{v_n}{A^{n-k}} < v_n \frac{A}{A-1}$$

and (by changing the role of n and k)

$$\sum_{k=n}^{\infty} \frac{1}{v_k} < \sum_{k=n}^{\infty} \frac{1}{A^{k-n} v_n} < \frac{1}{v_n} \frac{A}{A-1}.$$

Note that if $(v_n)_{n=1}^{\infty}$ is lacunary, then for each $0 < \beta < 1$ also $(v_n^{\beta})_{n=1}^{\infty}$ is lacunary (with A replaced by A^{β}) and therefore the assumption (10) is satisfied in this case.

Let $(q_n)_{n=0}^{\infty}$ be the sequence of denominators of an irrational number $\alpha \in \mathbb{T}$. Note that

$$q_{n+2} = a_{n+2}q_{n+1} + q_n \geq q_{n+1} + q_n > 2q_n.$$

Therefore, the sequence $(w_n)_{n=0}^{\infty}$, $w_n := q_n q_{n+1}$ is lacunary with $A = 2$. It follows that for each $0 < \beta < 1$ setting $D_{\beta} = 2^{\beta}/(2^{\beta} - 1)$ we have

$$\sum_{k=1}^{n} w_k^{\beta} \leq D_{\beta} w_n^{\beta} \quad \text{and} \quad \sum_{k=n}^{\infty} \frac{1}{w_k^{\beta}} \leq \frac{D_{\beta}}{w_n^{\beta}} \text{ for all } n \in \mathbb{N}. \tag{11}$$

Notation. For every $a \geq 1$ denote by $DC(a)$ the set of irrational numbers $\alpha \in \mathbb{T}$ satisfying the Diophantine condition

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{1}{C|q|^{1+a}} \text{ for all } p, q \in \mathbb{Z},\ q \neq 0$$

for some constant $C > 0$. Recall that (see [10]) $\alpha \in DC(a)$ if and only if there exists $C > 0$ such that $q_{n+1} \leq C \cdot q_n^a$ for all $n \geq 1$. Moreover, if $a > 1$ then $DC(a)$ has full Lebesgue measure.

Theorem 3. *Assume that $\alpha \in DC(a)$ for some $a \geq 1$. Then for every $0 < \gamma < \frac{1}{(1+a)a}$ there exists a γ–Hölder continuous function $\varphi : \mathbb{T} \to \mathbb{R}$ with zero mean such that the cylindrical transformation $T_{\varphi} : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$, $T_{\varphi}(x, s) = (x + \alpha, s + \varphi(x))$ is Besicovitch.*

Proof. Let C be a positive constant such that

$$q_{n+1} \leq C \cdot q_n^a \text{ for all } n \geq 1. \tag{12}$$

For any $0 < \gamma < \frac{1}{(1+a)a}$ we set

$$M_n = \frac{q_{n-1}}{2(q_n q_{n+1})^{\gamma}}.$$

Then we construct φ in the same manner as in Section 2. By (12),

$$\frac{q_{n-1}}{(q_n q_{n+1})^{\gamma}} \geq \frac{q_{n-1}}{C^{\gamma} q_{n-1}^{\gamma a} C^{\gamma(1+a)} q_{n-1}^{\gamma a^2}} = \frac{q_{n-1}^{1-\gamma(1+a)a}}{C^{\gamma(2+a)}},$$

and therefore (4) holds. But $M_n/q_{n-1} = 1/(2w_n^{\gamma})$ and $(w_n^{\gamma})_{n=0}^{\infty}$ is lacunary, so also (5) is satisfied. Let $\varphi_n(x) = f_n(x + \alpha) - f_n(x)$. Then

$$L(\varphi_n) = 2L_n = 2M_n \cdot \frac{q_n q_{n+1}}{q_{n-1}} = (q_n q_{n+1})^{1-\gamma} = w_n^{1-\gamma}$$

and, by (8) and (6),

$$\|\varphi_n\|_{C(\mathbb{T})} \leq \frac{L_n}{q_n q_{n+1}} = \frac{M_n}{q_{n-1}} \leq \frac{1}{w_n^{\gamma}}.$$

By Lemma 2 together with (11), the function φ is γ–Hölder continuous. □

4. Function of class $O(\log|n|/|n|)$

Assume that $\alpha\in[0,1)$ is irrational and suppose that $(M_m)_{m\geq 1}$ satisfies (4) and (5). Set $\delta_m=\frac{q_{m-1}}{q_mq_{m+1}}$. Then $M_m=L_m\delta_m$. Let us consider a piecewise linear function $f_m:[0,1)\to\mathbb{R}^+$ which is $1/q_m$–periodic and

$$f_m(x)=\begin{cases} L_mx & \text{if} \quad 0\leq x\leq\delta_m\\ M_m & \text{if} \quad \delta_m\leq x\leq\frac{1}{q_m}-\delta_m\\ L_m(\frac{1}{q_m}-x) & \text{if} \quad \frac{1}{q_m}-\delta_m\leq x\leq\frac{1}{q_m}.\end{cases}$$

Next, consider $\varphi(x)=\sum_{m=1}^{\infty}(f_m(x+\alpha)-f_m(x))$. An analysis similar to that in Section 2 shows that φ is correctly defined and it is continuous. Moreover, $\varphi^{(k)}(0)\to+\infty$ as $|k|\to+\infty$. Let $g_m:[0,1)\to\mathbb{R}^+$ be given by

$$g_m(x)=\begin{cases} x & \text{if} \quad 0\leq x\leq q_m\delta_m\\ q_m\delta_m & \text{if} \quad q_m\delta_m\leq x\leq 1-q_m\delta_m\\ 1-x & \text{if} \quad 1-q_m\delta_m\leq x\leq 1.\end{cases}$$

Therefore $f_m(x)=\frac{L_m}{q_m}g_m(q_mx)$ for $x\in[0,1/q_m)$ and for $n\neq 0$ we have

$$\begin{aligned}\widehat{g}_m(n) &= \int_0^1 g_m(x)e^{-2\pi inx}\,dx=\frac{1}{2\pi in}\int_0^1 g_m'(x)e^{-2\pi inx}\,dx\\ &= \frac{1}{2\pi in}\left(\int_0^{q_m\delta_m}e^{-2\pi inx}\,dx-\int_{1-q_m\delta_m}^1 e^{-2\pi inx}\,dx\right)\\ &= \frac{1}{4\pi^2n^2}\left(e^{-2\pi inq_m\delta_m}+e^{2\pi inq_m\delta_m}-2\right)=-\frac{1}{\pi^2n^2}\sin^2\pi nq_m\delta_m.\end{aligned}$$

Since f_k is $1/q_k$–periodic,

$$\widehat{f}_k(n)=\sum_{j=0}^{q_k-1}e^{-2\pi in\frac{j}{q_k}}\int_0^{1/q_k}f_k(x)e^{-2\pi inx}\,dx,$$

whence $\widehat{f}_k(n)=0$ if q_k does not divide n and moreover

$$\begin{aligned}\widehat{f}_k(q_ks) &= q_k\int_0^{1/q_k}f_k(x)e^{-2\pi iq_ksx}\,dx=q_k\int_0^{1/q_k}\frac{L_k}{q_k}g_k(q_kx)e^{-2\pi iq_kx}\,dx\\ &= L_k\int_0^1 g_k(y)e^{-2\pi isy}\left(\frac{1}{q_k}\,dy\right)=\frac{L_k}{q_k}\widehat{g}_k(s)\end{aligned}$$

for each $s\in\mathbb{Z}$. It follows that

$$\begin{aligned}\widehat{\varphi}(n) &= (e^{2\pi in\alpha}-1)\sum_{k=1}^{\infty}\widehat{f}_k(n)=(e^{2\pi in\alpha}-1)\sum_{k\geq 1:q_k|n}\frac{L_k}{q_k}\widehat{g}_k(\frac{n}{q_k})\\ &= (e^{2\pi in\alpha}-1)\sum_{k\geq 1:q_k|n}\frac{L_k}{q_k}\cdot\frac{-\sin^2\pi n\delta_k}{\pi^2\frac{n^2}{q_k^2}}\\ &= \frac{1-e^{2\pi in\alpha}}{\pi^2n^2}\sum_{k\geq 1:q_k|n}L_kq_k\sin^2\pi n\delta_k.\end{aligned}$$

Thus

$$|\widehat{\varphi}(n)|=|\widehat{\varphi}(-n)|=\frac{2|\sin\pi n\alpha|}{\pi^2n^2}\sum_{k\geq 1:q_k|n}L_kq_k\sin^2\pi n\delta_k. \tag{13}$$

Remark 2. Recall that

$$|\sin\pi x|\leq\pi\|x\|\leq\pi|x| \quad \text{for all} \quad x\in\mathbb{R} \tag{14}$$

and

$$\sin \pi x \geq 2x \quad \text{for all} \quad x \in [0, 1/2]. \tag{15}$$

Lemma 4. *Assume that $\alpha \in [0,1)$ is an irrational number such that*

$$\left(\frac{\log q_k}{k q_{k-1}^2}\right)_{k\geq 1} \text{ is increasing and } \frac{\log q_k}{k q_{k-1}^2} \to +\infty \text{ as } k \to +\infty.$$

Then there exists $(M_k)_{k\in\mathbb{N}}$ with $M_k \to +\infty$ such that for φ above we have $\widehat{\varphi}(n) = \mathrm{O}(\log|n|/|n|)$.

Proof. Fix $0 < \varepsilon < 1$ and let

$$M_k = \min\left(\frac{\log q_k}{k q_{k-1}^2}, q_{k-1}^{\varepsilon}\right).$$

Then $(M_k)_{k\geq 1}$ is increasing and $M_k \to +\infty$. Next note that the sequence $(\delta_k)_{k=1}^{\infty}$ is decreasing. Indeed, since $q_k^2 < q_{k+1}^2 < \log q_{k+2} < q_{k+2}$, it follows that

$$\delta_k = \frac{q_{k-1}}{q_k q_{k+1}} > \frac{q_{k-1} q_k}{q_{k+1} q_{k+2}} \geq \frac{q_k}{q_{k+1} q_{k+2}} = \delta_{k+1}.$$

Since $L_k = M_k/\delta_k$, we see that $(L_k)_{k=1}^{\infty}$ is increasing. Fix $n > 0$ and let $m \geq 0$ be the largest number such that q_m divides n. Note that if $m = 0$, i.e. q_k does not divide n for all $k \geq 1$, then $\widehat{\varphi}(n) = 0$, so assume that $m \geq 1$.

First suppose that $n \geq q_{m+1}$. Since $m q_m^2 M_m \leq (m+1) q_m^2 M_{m+1} \leq \log q_{m+1}$, by (13),

$$\begin{aligned} |\widehat{\varphi}(n)| &\leq \frac{2}{\pi^2 n^2} \sum_{k\geq 1: q_k | n} L_k q_k \leq \frac{2m L_m q_m}{\pi^2 n^2} \\ &\leq \frac{2m L_m q_m}{q_{m+1}} \frac{1}{n} = \frac{2m M_m q_m^2}{q_{m-1}} \frac{1}{n} \leq \frac{\log q_{m+1}}{n} \leq \frac{\log n}{n}. \end{aligned}$$

Next, suppose that $n \leq q_{m+1}$. Since $n = s q_m$, by (13), (14) and (3), we have

$$\begin{aligned} |\widehat{\varphi}(n)| &\leq \frac{2\|s q_m \alpha\|}{\pi^2 n^2} \sum_{k\geq 1: q_k | n} L_k q_k \sin^2 \pi n \delta_k \\ &\leq \frac{2s}{\pi^2 q_{m+1} n^2} \left(L_m q_m (\pi n \delta_m)^2 + \sum_{k=1}^{m-1} L_k q_k\right). \end{aligned}$$

Moreover,

$$\frac{s L_m q_m (\pi n \delta_m)^2}{\pi^2 q_{m+1} n^2} = \frac{s q_m}{q_{m+1}} L_m \delta_m^2 = \frac{n}{q_{m+1}} M_m \frac{q_{m-1}}{q_m q_{m+1}} \leq \frac{M_m q_{m-1}}{q_m q_{m+1}} \leq \frac{1}{q_{m+1}} \leq \frac{1}{n}$$

and

$$\begin{aligned} \frac{s}{\pi^2 q_{m+1} n^2} \sum_{k=1}^{m-1} L_k q_k &\leq \frac{m L_{m-1} q_{m-1} s}{q_{m+1} n^2} = \frac{m M_{m-1} q_{m-1}^2 q_m s}{q_{m-2} q_{m+1} n^2} \\ &= \frac{m M_{m-1} q_{m-1}^2}{q_{m-2} q_{m+1} n} \leq \frac{q_m}{q_{m-2} q_{m+1} n} \leq \frac{1}{n}. \end{aligned}$$

Consequently, $\widehat{\varphi}(n) = \mathrm{O}(\log|n|/|n|)$. □

In this way we have proved the following.

Theorem 5. *There exist an irrational rotation $Tx = x + \alpha$ and a continuous function $\varphi : \mathbb{T} \to \mathbb{R}$ such that the cylindrical transformation T_φ is Besicovitch and $\widehat{\varphi}(n) = \mathrm{O}(\log|n|/|n|)$.*

Remark 3. Note that this is not true that $\widehat{\varphi}(n) = \mathrm{O}(1/|n|)$. Indeed, take $m \geq 1$ and choose $s \in \mathbb{N}$ such that $q_{m+1}/(4q_{m-1}) < s < q_{m+1}/(2q_{m-1})$. Set $n = sq_m$. Since

$$\|n\alpha\| = \|sq_m\alpha\| \leq s\|q_m\alpha\| < 1/2 \text{ and } n\delta_m = s\frac{q_{m-1}}{q_{m+1}} < 1/2,$$

by (15) and (3), it follows that

$$|\sin \pi n\alpha| \geq 2\|sq_m\alpha\| = 2s\|q_m\alpha\| > \frac{1}{4q_{m-1}},$$

$$|\sin \pi n\delta_m| \geq 2n\delta_m = 2s\frac{q_{m-1}}{q_{m+1}} > 1/2.$$

Therefore, from (13),

$$\begin{aligned} |\widehat{\varphi}(n)| &\geq \frac{2|\sin \pi n\alpha|}{\pi^2 n^2} L_m q_m \sin^2 \pi n\delta_m \geq \frac{1}{4\pi^2 n^2}\frac{L_m q_m}{q_{m-1}} = \frac{1}{4\pi^2 n^2}\frac{M_m q_{m+1} q_m^2}{q_{m-1}^2} \\ &= \frac{1}{4\pi^2 n}\frac{M_m q_{m+1} q_m}{s q_{m-1}^2} > \frac{1}{2\pi^2 n}\frac{M_m q_m}{q_{m-1}} > \frac{M_m}{2\pi^2 n}. \end{aligned}$$

Consequently,

$$|n\widehat{\varphi}(n)| \geq \frac{M_m}{2\pi^2} \to +\infty.$$

5. Variants of the construction

Fix an irrational $\alpha \in [0,1)$ and let (q_{k_n}) be a subsequence of the sequence of denominators of α such that all k_n are even (or all are odd). Let (M_n) be a sequence of natural numbers such that

$$M_n \to +\infty \tag{16}$$

and

$$\sum_{n=1}^{\infty} \frac{M_n}{q_{k_{n-1}}} < +\infty. \tag{17}$$

Set

$$L_n = M_n \cdot \frac{q_{k_n} q_{k_n+1}}{q_{k_{n-1}}} \text{ and } \delta_n = \frac{q_{k_{n-1}}}{q_{k_n} q_{k_n+1}}. \tag{18}$$

In view of (17) and (18),

$$\sum_{n=1}^{\infty} \frac{L_n}{q_{k_n} q_{k_n+1}} < +\infty. \tag{19}$$

We now define $f_n : [0,1) \to \mathbb{R}^+$ a piecewise linear Lipschitz continuous function which $1/q_{k_n}$–periodic and

$$f_n(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x \leq \frac{\delta_n}{8} \\ L_n\left(x - \frac{\delta_n}{8}\right) & \text{if} \quad \frac{\delta_n}{8} \leq x \leq \frac{\delta_n}{2} + \frac{\delta_n}{8} \\ \frac{M_n}{2} & \text{if} \quad \frac{\delta_n}{2} + \frac{\delta_n}{8} \leq x \leq \frac{1}{4q_{k_n}} - \frac{\delta_n}{2} \\ L_n\left(x - \frac{1}{4q_{k_n}}\right) + M_n & \text{if} \quad \frac{1}{4q_{k_n}} - \frac{\delta_n}{2} \leq x \leq \frac{1}{4q_{k_n}} + \frac{\delta_n}{2} \\ \frac{3M_n}{2} & \text{if} \quad \frac{1}{4q_{k_n}} + \frac{\delta_n}{2} \leq x \leq \frac{1}{2q_{k_n}} - \frac{\delta_n}{2} - \frac{\delta_n}{8} \\ L_n\left(x - \frac{1}{2q_{k_n}} + \frac{\delta_n}{8}\right) + 2M_n & \text{if} \quad \frac{1}{2q_{k_n}} - \frac{\delta_n}{2} - \frac{\delta_n}{8} \leq x \leq \frac{1}{2q_{k_n}} - \frac{\delta_n}{8} \\ 2M_n & \text{if} \quad \frac{1}{2q_{k_n}} - \frac{\delta_n}{8} \leq x \leq \frac{1}{2q_{k_n}}, \end{cases}$$

and such that $f_n\left(1/q_{k_n} - x\right) = f_n(x)$ for all $x \in [1/2q_{k_n}, 1/q_{k_n}]$.

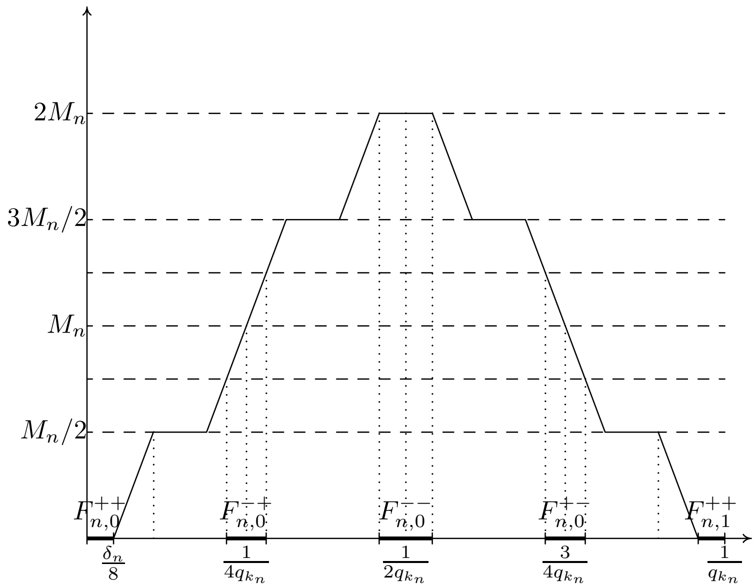

FIGURE 1. The graph of f_n

The Lipschitz constant of f_n is L_n, f_n takes only nonnegative values, $f_n(0) = 0$ and $\|f_n\|_{C(\mathbb{T})} \le 2M_n$. Moreover $f_n(-x) = f_n(x)$ for all $x \in [0, 1)$. Using $1/q_{k_n}$–periodicity of f_n and (2) for each $x \in [0, 1)$ we have

$$\begin{aligned} |f_n(x+\alpha) - f_n(x)| &= \left| f_n(x+\alpha) - f_n\left(x + \frac{p_{k_n}}{q_{k_n}}\right)\right| \\ &\le L_n \left|\alpha - \frac{p_{k_n}}{q_{k_n}}\right| < L_n \cdot \frac{1}{q_{k_n} q_{k_n+1}}, \end{aligned}$$

so $\|f_n(\,\cdot\, + \alpha) - f_n(\,\cdot\,)\|_{C(\mathbb{T})} < \frac{L_n}{q_{k_n} q_{k_n+1}}$ and it follows from (19) that the series

$$\varphi(x) = \sum_{n=1}^{\infty} (f_n(x+\alpha) - f_n(x))$$

converges uniformly, so φ is continuous and clearly $\int_0^1 \varphi(x)\,dx = 0$. For each integer k we have

$$\varphi^{(k)}(x) = \sum_{n=1}^{\infty} (f_n(x+k\alpha) - f_n(x)). \tag{20}$$

Remark 4. Suppose additionally that

$$q_{k_{n+1}} > 16 q_{k_n} q_{k_n+1}/q_{k_{n-1}} \text{ and } q_{k_n-1} \ge 4 q_{k_{n-1}} \text{ for all } n \in \mathbb{N}. \tag{21}$$

It follows that

$$\frac{1}{q_{k_n}} - \frac{1}{q_{k_n+1}} = \frac{q_{k_n+1} - q_{k_n}}{q_{k_n} q_{k_n+1}} \ge \frac{q_{k_n-1}}{q_{k_n} q_{k_n+1}} \ge \frac{4 q_{k_{n-1}}}{4 q_{k_n} q_{k_n+1}} = 4\delta_n. \tag{22}$$

Set

$$F^{++}_{n,j} = \left[-\frac{\delta_n}{8}, \frac{\delta_n}{8}\right] + \frac{j}{q_{k_n}}$$

and let $F^{++} = \bigcap_{n=1}^{\infty} \bigcup_{j=0}^{q_{k_n}-1} F^{++}_{n,j}$. In view of (21),

$$|F^{++}_{n,j}| = \frac{\delta_n}{4} = \frac{q_{k_{n-1}}}{4 q_{k_n} q_{k_n+1}} > \frac{4}{q_{k_{n+1}}},$$

and hence there exist at least two intervals of the form $F^{++}_{n+1,l}$ which are included in $F^{++}_{n,j}$. Consequently the set F^{++} is uncountable. We will show that if $x\in F^{++}$ then $\varphi^{(n)}(x)\to+\infty$ as $|n|\to+\infty$.

Assume that $x\in F^{++}$. Then there exists a sequence $(j_l)_{l=1}^{\infty}$ of natural numbers such that $x\in F^{++}_{l,j_l}$ for all $l\in\mathbb{N}$. Let $x=\frac{j_l}{q_{k_l}}+x_l$, where $|x_l|\le\delta_l/8$. Then

$$f_l\left(x+m\alpha\right)-f_l\left(x\right)=f_l\left(x_l+m\alpha\right)-f_l\left(x_l\right).$$

Fix integer $m\neq0$ and assume that $q_{k_{n-1}}\le|m|<q_{k_n}$. Since $|x_l|\le\delta_l/8$ and $f_l\ge0$ for every $l\in\mathbb{N}$, by the definition of f_l, $f_l(x_l)=0$, so

$$f_l\left(x+m\alpha\right)-f_l\left(x\right)=f_l\left(x_l+m\alpha\right)-f_l\left(x_l\right)=f_l\left(x_l+m\alpha\right)\ge0. \tag{23}$$

Since $|m|<q_{k_n}$, in view of (2) and (22), we have

$$\begin{aligned}\left|x_n+m\alpha-\frac{mp_{k_n}}{q_{k_n}}\right| &\le |x_n|+|m|\left|\alpha-\frac{p_{k_n}}{q_{k_n}}\right|\le|x_n|+\frac{|m|}{q_{k_n}q_{k_n+1}}\\ &\le \frac{\delta_n}{8}+\frac{1}{q_{k_n+1}}<\frac{1}{q_{k_n}}-2\delta_n.\end{aligned}$$

Since $|m|\ge q_{k_{n-1}}$, in view of (2), we have

$$\left|x_n+m\alpha-\frac{mp_{k_n}}{q_{k_n}}\right|\ge|m|\left|\alpha-\frac{p_{k_n}}{q_{k_n}}\right|-|x_n|\ge\frac{q_{k_{n-1}}}{2q_{k_n}q_{k_n+1}}-\frac{\delta_n}{8}=\frac{\delta_n}{2}-\frac{\delta_n}{8}=\frac{\delta_n}{8}+\frac{\delta_n}{4}.$$

By the definition of f_n it follows that

$$f_n\left(\left|x_n+m\alpha-\frac{mp_{k_n}}{q_{k_n}}\right|\right)\ge f_n\left(\frac{\delta_n}{8}+\frac{\delta_n}{4}\right)=L_n\frac{\delta_n}{4}=M_n/4.$$

Therefore, using additionally (23), we obtain

$$\begin{aligned}f_n\left(x+m\alpha\right)-f_n\left(x\right) &= f_n\left(x_n+m\alpha\right)=f_n\left(x_n+m\alpha-\frac{mp_{k_n}}{q_{k_n}}\right)\\ &= f_n\left(\left|x_n+m\alpha-\frac{mp_{k_n}}{q_{k_n}}\right|\right)\ge M_n/4.\end{aligned}$$

Consequently, using (20) and (23) again,

$$\varphi^{(m)}(x)=\sum_{l=1}^{\infty}(f_l\left(x+m\alpha\right)-f_l\left(x\right))\ge f_n\left(x+m\alpha\right)-f_n\left(x\right)\ge\frac{M_n}{4}.$$

Remark 5. Suppose additionally that

$$M_{n+1}\ge33M_n\text{ for all }n\in\mathbb{N}. \tag{24}$$

We will prove that the set of all $x\in[0,1)$ for which $\varphi^{(m)}(x)\to+\infty$ as $m\to+\infty$ and simultaneously $\varphi^{(m)}(x)\to-\infty$ as $m\to-\infty$ is uncountable.

Set

$$F^{-+}_{n,j}=\left[-\frac{\delta_n}{4},\frac{\delta_n}{4}\right]+\frac{1}{4q_{k_n}}+\frac{j}{q_{k_n}}$$

and let $F^{-+}=\bigcap_{n=1}^{\infty}\bigcup_{j=0}^{q_{k_n}-1}F^{-+}_{n,j}$. In view of (21),

$$|F^{-+}_{n,j}|=\frac{\delta_n}{2}=\frac{q_{k_{n-1}}}{2q_{k_n}q_{k_n+1}}>\frac{4}{q_{k_{n+1}}},$$

and hence there exist at least two intervals of the form $F^{-+}_{n+1,l}$ which are included in $F^{-+}_{n,j}$. Consequently the set F^{-+} is uncountable. We will prove that if $x\in F^{-+}$ then $\varphi^{(m)}(x)\to+\infty$ as $m\to+\infty$ and $\varphi^{(m)}(x)\to-\infty$ as $m\to-\infty$.

Assume that $x \in F^{-+}$. Then there exists a sequence $(j_l)_{l=1}^{\infty}$ for natural numbers such that $x \in F^{-+}_{l,j_l}$ for all $l \in \mathbb{N}$. Let $x = \frac{j_l}{q_{k_l}} + \frac{1}{4q_{k_l}} + x_l$, where $|x_l| \le \delta_l/4$. Then

$$f_l\left(x+m\alpha\right)-f_l\left(x\right)=f_l\left(\frac{1}{4q_{k_l}}+x_l+m\alpha\right)-f_l\left(\frac{1}{4q_{k_l}}+x_l\right). \tag{25}$$

For every $l \ge 1$ we have

$$\frac{1}{4q_{k_l}}-\frac{\delta_l}{4}<\frac{1}{4q_{k_l}}+x_l<\frac{1}{4q_{k_l}}+\frac{\delta_l}{4},$$

and hence

$$f_l\left(\frac{1}{4q_{k_l}}+x_l\right)=L_lx_l+M_l. \tag{26}$$

Fix $m > 0$ and assume that $q_{k_{n-1}}/4 \le m < q_{k_n}/4$. Since k_l is even, for every $l \ge 1$ we have

$$\frac{1}{4q_{k_l}}-\frac{\delta_l}{4}<\frac{1}{4q_{k_l}}+x_l\le\frac{1}{4q_{k_l}}+x_l+m\alpha-\frac{mp_{k_l}}{q_{k_l}}.$$

If additionally $l \ge n$ then, by (2),

$$\frac{1}{4q_{k_l}}+x_l+m\alpha-\frac{mp_{k_l}}{q_{k_l}}<\frac{1}{4q_{k_l}}+\frac{\delta_l}{4}+\frac{q_{k_n}}{4q_{k_l}q_{k_l+1}}<\frac{1}{4q_{k_l}}+\frac{\delta_l}{4}+\frac{1}{4q_{k_l+1}}.$$

In view of (22), $\frac{1}{q_{k_l+1}} \le \frac{1}{q_{k_l}} - 4\delta_l$. It follows that

$$\frac{1}{4q_{k_l}}+x_l+m\alpha-\frac{mp_{k_l}}{q_{k_l}}<\frac{1}{2q_{k_l}}-\frac{3\delta_l}{4},$$

hence

$$\frac{1}{4q_{k_l}}+x_l+m\alpha-\frac{mp_{k_l}}{q_{k_l}}\in\left(1/(4q_{k_l})-\delta_l/4,1/(2q_{k_l})-3\delta_l/4\right).$$

Case 1. Suppose that $l \ge n$ and

$$\frac{1}{4q_{k_l}}+x_l+m\alpha-\frac{mp_{k_l}}{q_{k_l}}\in\left(1/(4q_{k_l})-\delta_l/4,1/(4q_{k_l})+\delta_l/2\right].$$

By the definition of f_l,

$$\begin{aligned} f_l\left(\frac{1}{4q_{k_l}}+x_l+m\alpha\right) &= f_l\left(\frac{1}{4q_{k_l}}+x_l+m\alpha-\frac{mp_{k_l}}{q_{k_l}}\right) \\ &= L_l\left(x_l+m\alpha-\frac{mp_{k_l}}{q_{k_l}}\right)+M_l. \end{aligned}$$

Hence, by (25), (26) and (2),

$$\begin{aligned} &f_l\left(x+m\alpha\right)-f_l\left(x\right)=f_l\left(\frac{1}{4q_{k_l}}+x_l+m\alpha\right)-f_l\left(\frac{1}{4q_{k_l}}+x_l\right) \\ &= L_l\left(m\alpha-\frac{mp_{k_l}}{q_{k_l}}\right)\ge L_l\frac{q_{k_{n-1}}}{8q_{k_l}q_{k_l+1}}=M_l\frac{q_{k_{n-1}}}{8q_{k_{l-1}}}. \end{aligned}$$

Case 2. Suppose that $l \ge n$ and

$$\frac{1}{4q_{k_l}}+x_l+m\alpha-\frac{mp_{k_l}}{q_{k_l}}\in\left(1/(4q_{k_l})+\delta_l/2,1/(2q_{k_l})-3\delta_l/4\right).$$

By the definition of f_l,

$$f_l\left(\frac{1}{4q_{k_l}}+x_l+m\alpha\right)=f_l\left(\frac{1}{4q_{k_l}}+x_l+m\alpha-\frac{mp_{k_l}}{q_{k_l}}\right)=\frac{3}{2}M_l.$$

Moreover, by (26),

$$f_l\left(\frac{1}{4q_{k_l}}+x_l\right)=L_lx_l+M_l\leq L_l\frac{\delta_l}{4}+M_l=\frac{5}{4}M_l.$$

Therefore, using (25),

$$f_l\left(x+m\alpha\right)-f_l\left(x\right)=f_l\left(\frac{1}{4q_{k_l}}+x_l+m\alpha\right)-f_l\left(\frac{1}{4q_{k_l}}+x_l\right)\geq\frac{1}{4}M_l.$$

In summary, we have $f_n\left(x+m\alpha\right)-f_n\left(x\right)\geq M_n/8$ and $f_l\left(x+m\alpha\right)-f_l\left(x\right)>0$ for $l>n$. Moreover, $|f_l\left(x+m\alpha\right)-f_l\left(x\right)|\leq 2M_l$ for all $l\geq 1$, in particular for $l<n$. By (20) and (24), it follows that

$$\varphi^{(m)}(x)=\sum_{l=1}^{\infty}(f_l\left(x+m\alpha\right)-f_l\left(x\right))\geq\frac{M_n}{8}-\sum_{l=1}^{n-1}2M_l\geq\frac{M_n}{8}-2\frac{M_n}{32}\geq\frac{M_n}{16}$$

whenever $q_{k_{n-1}}/4\leq m<q_{k_n}/4$. Consequently, $\varphi^{(m)}(x)\to+\infty$ as $m\to+\infty$.

Similar argument will show also that $\varphi^{(-m)}(x)\leq -M_n/16$ whenever $x\in F^{-+}$ and $q_{k_{n-1}}/4\leq -m<q_{k_n}/4$. Since k_l is even, $|x_l|\leq\delta_l/8$ and m is negative, for every $l\geq 1$ we have

$$\frac{1}{4q_{k_l}}+x_l+m\left(\alpha-\frac{p_{k_l}}{q_{k_l}}\right)\leq\frac{1}{4q_{k_l}}+x_l<\frac{1}{4q_{k_l}}+\frac{\delta_l}{4}.$$

Suppose additionally that $l\geq n$. Since $m>-q_{k_n}/4$, by (2) and (22),

$$\begin{aligned}\frac{1}{4q_{k_l}}+x_l+m\left(\alpha-\frac{p_{k_l}}{q_{k_l}}\right)&>\frac{1}{4q_{k_l}}-\frac{\delta_l}{4}+\frac{m}{q_{k_l}q_{k_l+1}}>\frac{1}{4q_{k_l}}-\frac{\delta_l}{4}-\frac{q_{k_n}}{4q_{k_l}q_{k_l+1}}\\&>\frac{1}{4q_{k_l}}-\frac{1}{4q_{k_l+1}}-\frac{\delta_l}{4}>\frac{3\delta_l}{4}.\end{aligned}$$

Hence

$$\frac{1}{4q_{k_l}}+x_l+m\alpha-\frac{mp_{k_l}}{q_{k_l}}\in\left(3\delta_l/4,1/(4q_{k_l})+\delta_l/4\right).$$

Case 1. Suppose that $l\geq n$ and

$$\frac{1}{4q_{k_l}}+x_l+m\alpha-\frac{mp_{k_l}}{q_{k_l}}\in\left(1/(4q_{k_l})-\delta_l/2,1/(4q_{k_l})+\delta_l/4\right].$$

By the definition of f_l,

$$\begin{aligned}f_l\left(\frac{1}{4q_{k_l}}+x_l+m\alpha\right)&=f_l\left(\frac{1}{4q_{k_l}}+x_l+m\alpha-\frac{mp_{k_l}}{q_{k_l}}\right)\\&=L_l\left(x_l+m\alpha-\frac{mp_{k_l}}{q_{k_l}}\right)+M_l.\end{aligned}$$

Hence, by (25), (26) and (2),

$$\begin{aligned}&f_l\left(x+m\alpha\right)-f_l\left(x\right)=f_l\left(\frac{1}{4q_{k_l}}+x_l+m\alpha\right)-f_l\left(\frac{1}{4q_{k_l}}+x_l\right)\\&=L_lm\left(\alpha-\frac{p_{k_l}}{q_{k_l}}\right)\leq -L_l\frac{q_{k_{n-1}}}{8q_{k_l}q_{k_l+1}}=-M_l\frac{q_{k_{n-1}}}{8q_{k_{l-1}}}.\end{aligned}$$

Case 2. Suppose that $l\geq n$ and

$$\frac{1}{4q_{k_l}}+x_l+m\alpha-\frac{mp_{k_l}}{q_{k_l}}\in\left(3\delta_l/4,1/(4q_{k_l})-\delta_l/2\right).$$

By the definition of f_l,

$$f_l\left(\frac{1}{4q_{k_l}}+x_l+m\alpha\right)=f_l\left(\frac{1}{4q_{k_l}}+x_l+m\alpha-\frac{mp_{k_l}}{q_{k_l}}\right)=M_l/2.$$

Moreover, by (26),

$$f_l\left(\frac{1}{4q_{k_l}}+x_l\right)=L_lx_l+M_l\geq -L_l\frac{\delta_l}{4}+M_l=\frac{3}{4}M_l.$$

Therefore, using (25),

$$f_l\left(x+m\alpha\right)-f_l\left(x\right)=f_l\left(\frac{1}{4q_{k_l}}+x_l+m\alpha\right)-f_l\left(\frac{1}{4q_{k_l}}+x_l\right)\leq-\frac{1}{4}M_l.$$

In summary, we have $f_n\left(x+m\alpha\right)-f_n\left(x\right)\leq -M_n/8$ and $f_l\left(x+m\alpha\right)-f_l\left(x\right)<0$ for $l>n$. Moreover, $f_l\left(x+m\alpha\right)-f_l\left(x\right)\leq 2M_l$ for all $l\geq 1$. By (20) and (24), it follows that

$$\varphi^{(m)}(x)=\sum_{l=1}^{\infty}(f_l\left(x+m\alpha\right)-f_l\left(x\right))\leq-\frac{M_n}{8}+\sum_{l=1}^{n-1}2M_l\leq-\frac{M_n}{8}+2\frac{M_n}{32}\leq-\frac{M_n}{16}$$

whenever $q_{k_{n-1}}/4\leq -m<q_{k_n}/4$. Consequently, $\varphi^{(m)}(x)\to-\infty$ as $m\to-\infty$.

Remark 6. In a similar way we can prove that the sets

$$F^{--}=\bigcap_{n=1}^{\infty}\bigcup_{j=0}^{q_{k_n}-1}\left(\left[-\frac{\delta_n}{8},\frac{\delta_n}{8}\right]+\frac{1}{2q_{k_n}}+\frac{j}{q_{k_n}}\right)$$

and

$$F^{+-}=\bigcap_{n=1}^{\infty}\bigcup_{j=0}^{q_{k_n}-1}\left(\left[-\frac{\delta_n}{4},\frac{\delta_n}{4}\right]+\frac{3}{4q_{k_n}}+\frac{j}{q_{k_n}}\right)$$

are uncountable and $x\in F^{--}$ implies $\varphi^{(m)}(x)\to-\infty$ as $|m|\to+\infty$ and $x\in F^{+-}$ implies $\varphi^{(m)}(x)\to+\infty$ as $m\to-\infty$ and $\varphi^{(m)}(x)\to-\infty$ as $m\to+\infty$.

Proposition 6. *For every irrational $\alpha\in\mathbb{T}$ there exists a continuous function $\varphi:\mathbb{T}\to\mathbb{R}$ with zero mean such that the set $D^{s_-s_+}(\alpha,\varphi)$ is uncountable for every $s_-,s_+\in\{-,+\}$.* □

In the next section, under some additional assumptions on α, we will show that the sets $D^{s_-s_+}(\alpha,\varphi)$ may have positive Hausdorff dimension.

6. Hausdorff dimension of $D^{s_-s_+}(\alpha,\varphi)$

Let $(E_k)_{k\geq 0}$ be a sequence of subsets of $[0,1]$ such that each E_k is a union of a finite number of disjoint closed intervals (called k–th level basic intervals). Suppose that each interval of E_{k-1} includes at least $m_k\geq 2$ intervals of E_k, and the maximum length of k–th level intervals tends to zero as $k\to+\infty$. Let us consider the set

$$F=\bigcap_{k=0}^{\infty}E_k.$$

We will use the following criterion to estimate the Hausdorff dimension from below.

Proposition 7 (see Example 4.6 in [7] and its proof)**.** *Suppose that the k–th level intervals are separated by gaps of length at least ε_k so that $\varepsilon_k\geq\varepsilon_{k+1}>0$ for every $k\in\mathbb{N}$. Then*

$$\dim_H F\geq\liminf_{k\to\infty}\frac{\log(m_1\dots m_{k-1})}{-\log(m_k\varepsilon_k)}.$$

Remark 7. Let us consider two intervals $A,B\subset\mathbb{R}$ of length a and b respectively and let $h>0$. Suppose that $a<h<b$. Then there exist at least $\lfloor\frac{b-a}{h}\rfloor$ intervals of the form $A+kh$, $k\in\mathbb{Z}$ included in B. Moreover, if $b\geq 4h$ then $\lfloor\frac{b-a}{h}\rfloor\geq\frac{b}{4h}$.

Fix $s_+, s_- \in \{+,-\}$ and let

$$E_n^{s_-s_+} := \bigcup_{j=0}^{q_{k_n}-1} F_{n,j}^{s_-s_+} = \bigcup_{j=0}^{q_{k_n}-1} (F_{n,0}^{s_-s_+} + \frac{j}{q_{k_n}}).$$

Then $F^{s_-s_+} = \bigcap_{n=0}^{\infty} E_n^{s_-s_+}$. As we have already noticed

$$|F_{n,j}^{s_-s_+}| \geq \delta_n/4 \geq \frac{4}{q_{k_{n+1}}} \text{ and} \frac{1}{q_{k_{n+1}}} \geq \delta_{n+1} \geq |F_{n+1,l}^{s_-s_+}|,$$

so $|F_{n,j}^{s_-s_+}| \geq 4h$ and $h \geq |F_{n+1,l}^{s_-s_+}|$, where $h = 1/q_{k_{n+1}}$, and then by Remark 7,

$$m_{n+1} \geq \left\lfloor \frac{|F_{n,j}^{s_-s_+}| - |F_{n+1,l}^{s_-s_+}|}{h} \right\rfloor \geq \frac{|F_{n,j}^{s_-s_+}|}{4h} \geq \frac{\delta_n q_{k_{n+1}}}{16} = \frac{q_{k_{n-1}} q_{k_{n+1}}}{16 q_{k_n} q_{k_n+1}}.$$

Moreover,

$$\varepsilon_n = \frac{1}{q_{k_n}} - |F_{n,0}^{s_-s_+}| \geq \frac{1}{q_{k_n}} - \frac{\delta_n}{2} \geq \frac{1}{2q_{k_n}}.$$

It follows that

$$m_1 \ldots m_{n-1} \geq m_{n-1} \geq \frac{q_{k_{n-3}} q_{k_{n-1}}}{16 q_{k_{n-2}} q_{k_{n-2}+1}} \geq \frac{q_{k_{n-1}}}{16 q_{k_{n-2}} q_{k_{n-2}+1}}.$$

Moreover,

$$m_n \varepsilon_n \geq \frac{q_{k_{n-2}} q_{k_n}}{16 q_{k_{n-1}} q_{k_{n-1}+1}} \frac{1}{2q_{k_n}} = \frac{q_{k_{n-2}}}{32 q_{k_{n-1}} q_{k_{n-1}+1}} > \frac{1}{32 q_{k_{n-1}} q_{k_{n-1}+1}}.$$

Thus

$$\frac{\log(m_1 \ldots m_{n-1})}{-\log(m_n \varepsilon_n)} \geq \frac{\log q_{k_{n-1}} - \log 16 - \log q_{k_{n-2}} - \log q_{k_{n-2}+1}}{\log q_{k_{n-1}} + \log q_{k_{n-1}+1} + \log 32}. \tag{27}$$

Theorem 8. *Suppose that there exist $\gamma \geq 1$ and $C > 0$ such that*

$$q_{n+1} \leq C q_n^{\gamma} \text{ for infinitely many } n \in \mathbb{N}. \tag{28}$$

Then there exists a continuous function $\varphi : \mathbb{T} \to \mathbb{R}$ with zero mean such that $\dim_H D^{s_-s_+}(\alpha, \varphi) \geq 1/(1+\gamma)$ for all $s_+, s_- \in \{+,-\}$.

Proof. By assumption, we can find a subsequence (q_{k_n}) of even denominators of α (or odd) such that

$$q_{k_n} \geq (q_{k_{n-1}} q_{k_{n-1}+1})^n,\ q_{k_n-1} \geq 4 q_{k_{n-1}} \text{ and } q_{k_n+1} \leq C q_{k_n}^{\gamma} \text{ for all } n \in \mathbb{N}. \tag{29}$$

Take $M_n = 33^n$ and let us consider the function φ constructed in Section 5. Then (16), (17), (21) and (24) hold. It follows that $F^{s_-s_+} \subset D^{s_-s_+}(\alpha, \varphi)$. Moreover, by Proposition 7, (27) and (29),

$$\begin{aligned}
\dim_H F^{s_-s_+} &\geq \liminf_{n\to\infty} \frac{\log(m_1 \ldots m_{n-1})}{-\log(m_n \varepsilon_n)} \\
&\geq \liminf_{n\to\infty} \frac{\log q_{k_{n-1}} - \log 16 - \log q_{k_{n-2}} - \log q_{k_{n-2}+1}}{\log q_{k_{n-1}} + \log q_{k_{n-1}+1} + \log 32} \\
&\geq \lim_{n\to\infty} \frac{(1-1/n)\log q_{k_{n-1}} - \log 16}{(1+\gamma)\log q_{k_{n-1}} + \log C + \log 32} = \frac{1}{1+\gamma},
\end{aligned}$$

which completes the proof. □

Remark 8. In particular, for every $\alpha \in DC(\gamma)$ the condition (28) holds for some $C > 0$, so the statement of Theorem 8 remains valid.

Remark 9. It is of course true that $\bigcup_{n\in\mathbb{Z}}(F^{s_-s_+}+n\alpha)\subset D^{s_-s_+}(\alpha,\varphi)$, however

$$\dim_H \bigcup_{n\in\mathbb{Z}}(F^{s_-s_+}+n\alpha)=\sup_{n\in\mathbb{Z}}\dim_H(F^{s_-s_+}+n\alpha)=\dim_H F^{s_-s_+}.$$

Notice also that for each $n\geq 1$ the set $F^{s_-s_+}$ is covered by q_{k_n} intervals each of which has length $\delta_n/4$. By Proposition 4.1 in [7], using (29), it follows that

$$\dim_H F^{s_-s_+}\leq \liminf_{n\to\infty}\frac{\log q_{k_n}}{-\log\left(q_{k_{n-1}}/(4q_{k_n}q_{k_n+1})\right)}\leq\frac{1}{2}.$$

It follows that if we use only sets $F^{s_-s_+}$ we cannot estimate the Hausdorff dimension of $D^{s_-s_+}(\alpha,\varphi)$ from below by a number greater than $1/2$. Note finally that if α has bounded partial quotients then $\dim_H F^{s_-s_+}=\frac{1}{2}$.

Theorem 9. *For almost every $\alpha\in\mathbb{T}$ there exists a continuous function $\varphi:\mathbb{T}\to\mathbb{R}$ with zero mean such that $\dim_H D^{s_-s_+}(\alpha,\varphi)\geq 1/2$ for all $s_+,s_-\in\{+,-\}$.*

Proof. Recall that (see [10]) for a.e. $\alpha\in\mathbb{T}$ there exist $C>0$ and an increasing sequence (k_n) of natural numbers such that

$$q_{k_n+1}\leq C\cdot q_{k_n}\log q_{k_n}\ \text{ for all } n\in\mathbb{N}.$$

By the proof of Theorem 8, there exists a continuous function $\varphi:\mathbb{T}\to\mathbb{R}$ with zero mean such that

$$\dim_H D^{s_-s_+}(\alpha,\varphi)\geq\liminf_{n\to\infty}\frac{\log q_{k_n}}{\log q_{k_n}+\log q_{k_n+1}}$$

for all $s_+,s_-\in\{+,-\}$. However,

$$\frac{\log q_{k_n}}{\log q_{k_n}+\log q_{k_n+1}}\geq\frac{\log q_{k_n}}{2\log q_{k_n}+\log\log q_{k_n}+\log C}\to 1/2$$

as $n\to+\infty$. Consequently, $\dim_H D^{s_-s_+}(\alpha,\varphi)\geq 1/2$. $\square$

7. Smooth cylindrical transformations over rotations on higher dimensional tori

In this section we will deal with cylindrical transformations over rotations on higher dimensional tori. More precisely, we will construct some examples of cylindrical transformations T_φ of class C^r, $r\geq 1$ admitting dense and discrete orbits and such that the set $D(\alpha,\varphi)$ has positive Hausdorff dimension; the construction is based on Yoccoz's method in [19].

Fix $d\geq 3$. Let $a>1$ be a real number such that $\bar{a}:=2a^{d-1}-a^d-1>0$. (The derivative of the function $P(a)=2a^{d-1}-a^d-1$ is $d-2>0$ at $a=1$ and $P(1)=0$, so P takes positive values on a nonempty interval $(1,b)$.) Let α be an irrational number such that there exists $C_0>4$ for which

$$4q_n^{a^d}\leq q_{n+1}\leq C_0q_n^{a^d}\ \text{ for all } n\in\mathbb{N}. \tag{30}$$

Fix $0<\varepsilon<\bar{a}$ and set $\varphi(x)=\sum_{n=1}^{\infty}(f_n(x+\alpha)-f_n(x))$, where

$$f_n(x)=\frac{q_{n+1}}{q_n^{1+\bar{a}-\varepsilon}}(1-\cos 2\pi q_nx).$$

Note that $f_n(x)\geq 0$ for $x\in\mathbb{T}$ and $n\geq 1$. Let

$$F_n=\bigcup_{j=0}^{q_n-1}\left(\frac{j}{q_n}+\left[-\frac{1}{\sqrt{q_nq_{n+1}}},\frac{1}{\sqrt{q_nq_{n+1}}}\right]\right)\ \text{ and } F=\bigcap_{n=1}^{\infty}F_n.$$

Lemma 10. *The function $\varphi : \mathbb{T} \to \mathbb{R}$ is of class $C^{\lceil \bar{a}-\varepsilon \rceil}$ and there exist $\theta = \theta(C_0, d, a, \varepsilon) > 0$ and $K = K(a, \varepsilon) > 0$ such that $\varphi^{(m)}(x) \geq -K$ for all $x \in F$ and $m \in \mathbb{Z}$. If additionally $q_n^{a^{d-1}} \leq |m| \leq \frac{1}{4} q_{n+1}$ and $q_n^{\bar{a}} \geq 64C_0$ then $\varphi^{(m)}(x) \geq \theta |m|^{\varepsilon/a^d} - K$.*

Proof. Since

$$f_n(x+\alpha) - f_n(x) = \frac{2q_{n+1}}{q_n^{1+\bar{a}-\varepsilon}} \sin \pi q_n \alpha \cdot \sin 2\pi q_n (x + \alpha/2),$$

by (14) and (2),

$$\begin{aligned} \left| \frac{d^k}{dx^k}(f_n(x+\alpha) - f_n(x)) \right| &= \frac{q_{n+1}}{q_n^{1+\bar{a}-\varepsilon}} 2|\sin \pi q_n \alpha| \left| \frac{d^k}{dx^k} \sin 2\pi q_n (x+\alpha/2) \right| \\ &\leq \frac{q_{n+1}}{q_n^{1+\bar{a}-\varepsilon}} 2\pi \|q_n \alpha\| (2\pi q_n)^k \leq \frac{(2\pi)^{k+1}}{q_n^{1+\bar{a}-\varepsilon-k}}. \end{aligned}$$

Moreover $\sum_{n=1}^{\infty} 1/q_n^{1+\bar{a}-\varepsilon-k} < +\infty$ for $0 \leq k \leq \lceil \bar{a} - \varepsilon \rceil$, so $\varphi \in C^{\lceil \bar{a}-\varepsilon \rceil}(\mathbb{T}, \mathbb{R})$.

Suppose that $x \in F$. Then $x \in F_n$ for every $n \geq 1$ and hence $\|q_n x\| \leq \sqrt{q_n}/\sqrt{q_{n+1}}$. In view of (15),

$$\sin^2 \pi q_n x \leq \pi^2 \|q_n x\|^2 \leq \pi^2 q_n / q_{n+1}.$$

Therefore for every $m \in \mathbb{Z}$

$$\begin{aligned} f_n(x+m\alpha) - f_n(x) &\geq -f_n(x) = -\frac{q_{n+1}}{q_n^{1+\bar{a}-\varepsilon}}(1 - \cos 2\pi q_n x) \\ &= -2\frac{q_{n+1}}{q_n^{1+\bar{a}-\varepsilon}} \sin^2 \pi q_n x \geq -2\pi^2 \frac{q_{n+1}}{q_n^{1+\bar{a}-\varepsilon}} \frac{q_n}{q_{n+1}} = -\frac{2\pi^2}{q_n^{\bar{a}-\varepsilon}}. \end{aligned}$$

Let

$$K = K(a, \varepsilon) := 2\pi^2 \sum_{n=1}^{\infty} \frac{1}{4^{n(\bar{a}-\varepsilon)}} = \frac{2\pi^2}{4^{\bar{a}-\varepsilon} - 1}.$$

In view of (30), $q_n \geq 4^n$, so

$$\varphi^{(m)}(x) = \sum_{n=1}^{\infty} (f_n(x+m\alpha) - f_n(x)) \geq -2\pi^2 \sum_{n=1}^{\infty} \frac{1}{q_n^{\bar{a}-\varepsilon}} \geq -K. \tag{31}$$

Suppose additionally that $q_n^{a^{d-1}} \leq |m| \leq \frac{1}{4} q_{n+1}$. By (2),

$$\|mq_n\alpha\| \leq |m| \|q_n \alpha\| < 1/4, \text{ and hence } \|mq_n\alpha\| = |m| \|q_n\alpha\|.$$

Therefore

$$\begin{aligned} f_n(x+m\alpha) - f_n(x) &= \frac{q_{n+1}}{q_n^{1+\bar{a}-\varepsilon}}(\cos 2\pi q_n x - \cos 2\pi q_n(x+m\alpha)) \\ &= \frac{q_{n+1}}{q_n^{1+\bar{a}-\varepsilon}}\left(\cos 2\pi(\pm\|q_n x\|) - \cos 2\pi(\pm\|q_n x\| + \|mq_n\alpha\|)\right) \\ &= \frac{q_{n+1}}{q_n^{1+\bar{a}-\varepsilon}} 2 \sin \pi(\pm 2\|q_n x\| + \|mq_n\alpha\|) \sin \pi \|mq_n \alpha\| \\ &= \frac{q_{n+1}}{q_n^{1+\bar{a}-\varepsilon}} 2 \sin \pi(\pm 2\|q_n x\| + |m| \|q_n\alpha\|) \sin \pi |m| \|q_n \alpha\|. \end{aligned}$$

By (30),

$$\sqrt{q_n q_{n+1}} \leq \sqrt{C_0} q_n^{\frac{1+a^d}{2}} = \sqrt{C_0} q_n^{a^{d-1} - \bar{a}/2} \leq \sqrt{\frac{C_0}{q_n^{\bar{a}}}} |m| \leq \frac{|m|}{8},$$

whenever $q_n^{\bar a} \geq 64C_0$. Moreover, by (2), $\frac{|m|}{2q_{n+1}} \leq |m|\|q_n\alpha\| < 1/4$ and hence

$$\pm 2\|q_nx\| + |m|\|q_n\alpha\| \geq |m|\|q_n\alpha\| - 2\|q_nx\| \geq \frac{|m|}{2q_{n+1}} - 2\frac{\sqrt{q_nq_{n+1}}}{q_{n+1}} \geq \frac{|m|}{4q_{n+1}}.$$

Similarly

$$\pm 2\|q_nx\| + |m|\|q_n\alpha\| \leq |m|\|q_n\alpha\| + 2\|q_nx\| \leq \frac{|m|}{q_{n+1}} + 2\frac{\sqrt{q_nq_{n+1}}}{q_{n+1}} \leq \frac{2|m|}{q_{n+1}} \leq \frac{1}{2}.$$

By (15), it follows that

$$\sin\pi|m|\|q_n\alpha\| \geq 2|m|\|q_n\alpha\| \geq \frac{|m|}{q_{n+1}}$$

and

$$\sin\pi(\pm 2\|q_nx\| + |m|\|q_n\alpha\|) \geq 2(\pm 2\|q_nx\| + |m|\|q_n\alpha\|) \geq \frac{|m|}{2q_{n+1}}.$$

Thus

$$\begin{aligned} f_n(x+m\alpha) - f_n(x) &= \frac{2q_{n+1}}{q_n^{1+\bar a-\varepsilon}}\sin\pi(\pm 2\|q_nx\| + |m|\|q_n\alpha\|)\sin\pi|m|\|q_n\alpha\| \\ &\geq \frac{m^2}{q_n^{1+\bar a-\varepsilon}q_{n+1}} \geq \frac{q_n^{2a^{d-1}}}{C_0q_n^{1+\bar a-\varepsilon}q_n^{a^d}} = \frac{q_n^\varepsilon}{C_0} \geq \frac{q_{n+1}^{\varepsilon/a^d}}{C_0^{1+\frac{\varepsilon}{a^d}}} \geq \frac{|m|^{\varepsilon/a^d}}{C_0^{1+\frac{\varepsilon}{a^d}}}. \end{aligned}$$

In view of the proof of (31), we conclude that

$$\varphi^{(m)}(x) = f_n(x+m\alpha) - f_n(x) + \sum_{l\neq n}^{\infty}(f_l(x+m\alpha) - f_l(x)) \geq \frac{|m|^{\varepsilon/a^d}}{C_0^{1+\frac{\varepsilon}{a^d}}} - K.$$

□

Set $C = 2\lceil 4^{a+2}\rceil$. Let $\alpha_1,\ldots,\alpha_d$ be irrational numbers. Let $(q_n^{(j)})_{n=1}^\infty$ stand for the sequence of denominators of α_j for $j=1,\ldots,d$. Assume that the denominators of $\alpha_1,\ldots,\alpha_d$ satisfy the following inequalities

$$4(q_n^{(j)})^a \leq q_n^{(j+1)} \leq C(q_n^{(j)})^a \text{ for all } 1\leq j\leq d \text{ and } n\geq 1, \tag{32}$$

in which we use the notation $q_n^{(d+1)} = q_{n+1}^{(1)}$ and $q_n^{(0)} = q_{n-1}^{(d)}$. It is easy to see that

$$4(q_n^{(j)})^{a^{d-1}} \leq 4^{1+\ldots+a^{d-2}}(q_n^{(j)})^{a^{d-1}} \leq q_{n+1}^{(j-1)} \tag{33}$$

and

$$4(q_n^{(j)})^{a^d} \leq q_{n+1}^{(j)} \leq C^{1+a+\ldots+a^{d-1}}(q_n^{(j)})^{a^d} = C_a(q_n^{(j)})^{a^d} \tag{34}$$

for all $1\leq j\leq d$ and $n\in\mathbb{N}$.

Remark 10. Denote by $\mathfrak{C}(a)$ the set of all $(\alpha_1,\ldots,\alpha_d)\in\mathbb{T}^d$ satisfying (32). In Appendix B we will show that for uncountably many $(\alpha_1,\ldots,\alpha_d)\in\mathfrak{C}(a)$ the rotation on $\mathbb{T}^d$ by the vector $(\alpha_1,\ldots,\alpha_d)$ is minimal.

Take $(\alpha_1,\ldots,\alpha_d)\in\mathfrak{C}(a)$ such that the rotation $T:\mathbb{T}^d\to\mathbb{T}^d$ given by

$$T(x_1,\ldots,x_d) = (x_1+\alpha_1,\ldots,x_d+\alpha_d)$$

is minimal. Let $0<\varepsilon<\bar a$ and set $\varphi_j(x) = \sum_{n=1}^\infty(f_{n,j}(x+\alpha_j) - f_{n,j}(x))$, where

$$f_{n,j}(x) = \frac{q_{n+1}^{(j)}}{(q_n^{(j)})^{1+\bar a-\varepsilon}}(1-\cos 2\pi q_n^{(j)}x) \text{ for } j=1,\ldots,d.$$

Let us consider the function $\varphi:\mathbb{T}^d\to\mathbb{R}$,

$$\varphi(x_1,\ldots,x_d) = \varphi_1(x_1)+\ldots+\varphi_d(x_d).$$

For $1 \leq j \leq d$ set

$$F^{(j)} = \bigcap_{n=1}^{\infty} \bigcup_{l=0}^{q_n^{(j)}-1} \left(\frac{l}{q_n^{(j)}} + \left[-\frac{1}{\sqrt{q_n^{(j)} q_{n+1}^{(j)}}}, \frac{1}{\sqrt{q_n^{(j)} q_{n+1}^{(j)}}} \right] \right).$$

Theorem 11. *The function $\varphi : \mathbb{T}^d \to \mathbb{R}$ is of class $C^{\lceil \bar{a}-\varepsilon \rceil}$ and*

$$\varphi^{(m)}(x_1, \ldots, x_d) \to +\infty \text{ as } |m| \to +\infty$$

for each $(x_1, \ldots, x_d) \in F^{(1)} \times \ldots \times F^{(d)}$.

Proof. For each integer m with $|m| \geq q_1^{(d)}/4$ there exist $n \in \mathbb{N}$ and $1 \leq j \leq d$ such that $\frac{1}{4} q_{n+1}^{(j-1)} \leq |m| \leq \frac{1}{4} q_{n+1}^{(j)}$. Note that if $|m| \to +\infty$ then $n \to +\infty$, so $(q_n^{(j)})^{\overline{a}} \geq 64 C_a$ whenever $|m|$ is large enough. From (33),

$$(q_n^{(j)})^{a^{d-1}} \leq \frac{1}{4} q_{n+1}^{(j-1)} \leq |m| \leq \frac{1}{4} q_{n+1}^{(j)}.$$

By (34) and Lemma 10, for every $(x_1, \ldots, x_d) \in F^{(1)} \times \ldots \times F^{(d)}$ we have

$$\begin{aligned} \varphi^{(m)}(x_1, \ldots, x_d) &= \sum_{k=1}^{d} \varphi_k^{(m)}(x_k) \geq \varphi_j^{(m)}(x_j) - (d-1)K(a, \varepsilon) \\ &\geq \theta(C_a, d, a, \varepsilon) |m|^{\varepsilon/a^d} - dK(a, \varepsilon). \end{aligned}$$

Consequently,

$$\varphi^{(m)}(x_1, \ldots, x_d) \to +\infty \text{ as } |m| \to +\infty.$$

□

Proposition 12. *For every $1 \leq j \leq d$ we have $\dim_H F^{(j)} \geq \frac{1}{1+a^d}$, in particular $\dim_H(F^{(1)} \times \ldots \times F^{(d)}) \geq \frac{d}{1+a^d}$.*

Proof. Fix $1 \leq j \leq d$ and we will write q_n instead of $q_n^{(j)}$. Note that to calculate the Hausdorff dimension of $F^{(j)}$ we can use the scheme presented at the beginning of Section 6 and Remark 7 in which

$$a = \frac{2}{\sqrt{q_n q_{n+1}}}, \quad b = \frac{2}{\sqrt{q_{n-1} q_n}} \quad \text{and} \quad h = \frac{1}{q_n}.$$

Here

$$\varepsilon_n = \frac{1}{q_n} - \frac{2}{\sqrt{q_n q_{n+1}}} \geq \frac{1}{2q_n} \quad \text{and} \quad m_n \geq \frac{\frac{2}{\sqrt{q_{n-1} q_n}}}{4 \frac{1}{q_n}} = \frac{\sqrt{q_n}}{2\sqrt{q_{n-1}}}.$$

Thus

$$m_1 \ldots m_{n-1} \geq \frac{\sqrt{q_{n-1}}}{2^{n-1}}$$

and using (34)

$$m_n \varepsilon_n \geq \frac{1}{4\sqrt{q_{n-1} q_n}} \geq \frac{1}{4\sqrt{C_a} q_{n-1}^{\frac{1+a^d}{2}}}.$$

Moreover, $q_{n+1} \geq q_n^{a^d}$, and hence (reminding that $q_1 \geq 4$)

$$\log q_n \geq \log q_1^{a^{d(n-1)}} \geq a^{d(n-1)} \log 4.$$

By Proposition 7, it follows that

$$\dim_H F^{(j)} \geq \liminf_{n \to \infty} \frac{\frac{1}{2} \log q_{n-1} - (n-1) \log 2}{\frac{1+a^d}{2} \log q_{n-1} + \log 4 + \log \sqrt{C_a}} = \frac{1}{1+a^d}.$$

Since

$$\dim_H(F^{(1)} \times \ldots \times F^{(d)}) \geq \dim_H F^{(1)} + \ldots + \dim_H F^{(d)},$$

the proof is complete. □

Remark 11. For every $d \geq 3$ in order to obtain the maximum of $\bar{a}$ we have to choose of $a = 2(d-1)/d$. Then

$$\bar{a} = \frac{2^d}{d}\left(\frac{d-1}{d}\right)^{d-1} - 1 > \frac{2^d}{ed} - 1.$$

In particular, for $d = 3$ we have $\bar{a} = 8/3 \cdot (2/3)^2 = 32/27 < 2$, so we can only choose $\varepsilon > 0$ so that $\lceil \bar{a} - \varepsilon \rceil = 1$. However, if $d \to +\infty$ then the degree of smoothness of φ grows exponentially.

8. Differential equations and flows

Consider now a system of differential equations on $\mathbb{T}^d \times \mathbb{T} \times \mathbb{R}$

$$\begin{array}{lcl} \frac{d\overline{x}}{dt} & = & \overline{\alpha} \\ \frac{dz}{dt} & = & 1 \\ \frac{dy}{dt} & = & f(\overline{x}, z), \end{array} \tag{35}$$

where $\overline{\alpha} = (\alpha_1, \ldots, \alpha_d)$ induces a minimal rotation T on $\mathbb{T}^d$ and $f : \mathbb{T}^d \times \mathbb{T} \to \mathbb{R}$ is of class C^r for some $r \geq 0$. Denote by $(\Phi_t)_{t\in\mathbb{R}}$ the corresponding flow on $\mathbb{T}^{d+1} \times \mathbb{R}$

$$\Phi_t(\overline{x}, z, y) = \left(x + t\overline{\alpha}, z + t, \int_0^t f(\overline{x} + s\overline{\alpha}, z + s)\, ds \right).$$

Then $\mathbb{T}^d \times \{0\} \times \mathbb{R} = \mathbb{T}^d \times \mathbb{R}$ is a global section for $(\Phi_t)_{t\in\mathbb{R}}$ and the Poincaré map is given by the formula

$$(\overline{x}, y) \mapsto (\overline{x} + \overline{\alpha}, y + \varphi(\overline{x})),$$

where

$$\varphi(\overline{x}) = \int_0^1 f(\overline{x} + s\overline{\alpha}, s)\, ds, \tag{36}$$

and hence $\varphi : \mathbb{T}^d \to \mathbb{R}$ is also of C^r class. Reciprocally, if $\varphi : \mathbb{T}^d \to \mathbb{R}$ is of C^r class then we can find $f : \mathbb{T}^d \times \mathbb{T} \to \mathbb{R}$ which is of C^r class so that (36) holds (for example $f(\overline{x}, z) = \varphi(\overline{x} - z\overline{\alpha})b(z)$ will do provided $b : [0, 1] \to \mathbb{R}$ is smooth, $\int_0^1 b(t)\, dt = 1$ and $b|_{[0,\eta]} = 0 = b|_{[1-\eta,1]}$ for some $0 < \eta < 1/2$).

Now the flow $(\Phi_t)_{t\in\mathbb{R}}$ is topologically the same as the suspension flow over T_φ. In particular, closed orbits of $(\Phi_t)_{t\in\mathbb{R}}$ which we may identify with closed orbits of the suspension flow are in a natural correspondence with closed orbits of T_φ. More generally, each minimal subset for the suspension flow is of the form $M \times [0, 1)$ where $M \subset \mathbb{T}^d \times \mathbb{R}$ is a minimal subset for T_φ. We will now make use of our knowledge about properties of T_φ to derive properties of $(\Phi_t)_{t\in\mathbb{R}}$.

First of all, we note that $(\Phi_t)_{t\in\mathbb{R}}$ is never minimal. Then, note that

$$\int_{\mathbb{T}^d} \varphi(\overline{x})\, d\overline{x} = \int_{\mathbb{T}^d\times\mathbb{T}} f(\overline{x}, z)\, d\overline{x}dz, \tag{37}$$

so we should constantly assume that the latter integral vanishes; otherwise $\mathbb{T}^d \times \mathbb{T} \times \mathbb{R}$ is foliated into closed orbits of $(\Phi_t)_{t\in\mathbb{R}}$ (each orbit being homeomorphic to $\mathbb{R}$). When the integral vanishes and $\varphi(\overline{x}) = j(\overline{x}) - j(T\overline{x})$ for a continuous $j : \mathbb{T}^d \to \mathbb{R}$ then again $\mathbb{T}^d \times \mathbb{T} \times \mathbb{R}$ is foliated into minimal components of $(\Phi_t)_{t\in\mathbb{R}}$, however now each minimal component is compact. This situation is equivalent to saying that there exists an orbit of $(\Phi_t)_{t\in\mathbb{R}}$ which is relatively compact (and then all orbits are relatively compact). Finally, if the integral (37) vanishes and no orbit of $(\Phi_t)_{t\in\mathbb{R}}$ is relatively compact then $(\Phi_t)_{t\in\mathbb{R}}$ is topologically transitive, that is, there is a dense orbit (and since it is not minimal there are orbits which are not dense). A natural question arises to decide whether in the transitive and smooth ($r \geq 1$) case closed orbits can exist. The answer is negative if $d = 1$ (in fact, for $d = 1$ the flow corresponding to (35) has no minimal subset). We can now interpret the results of Section 7 as the positive answer to the question in case $d \geq 3$ (although with some restriction on the degree of smoothness of the vector field in (35)).

8.1. **Differential equations on $\mathbb{R}^3$.** For any irrational number $\alpha > 0$ let us consider the system of differential equations on $\mathbb{R}^3$

$$\begin{aligned} x' &= -2\pi y + 2\pi\alpha xz \\ y' &= 2\pi x + 2\pi\alpha yz \\ z' &= \pi\alpha(1 - x^2 - y^2 + z^2). \end{aligned} \tag{38}$$

It is easily checked that $\ln((\sqrt{x^2+y^2}+1)^2 + z^2) - \ln((\sqrt{x^2+y^2}-1)^2 + z^2)$ is a first integral of (38). For $a > 0$ the set of $(x, y, z) \in \mathbb{R}^3$ satisfying

$$(\sqrt{x^2+y^2}+1)^2 + z^2 = a((\sqrt{x^2+y^2}-1)^2 + z^2), \tag{39}$$

is a torus; indeed, by setting $r = \sqrt{x^2+y^2}$, (39) is equivalent to

$$\left(r - \frac{a+1}{a-1}\right)^2 + z^2 = \frac{4a}{(a-1)^2}.$$

It follows that the corresponding family of tori establishes an invariant foliation of $\mathbb{R}^3 \setminus (\mathbb{S}_0 \cup \mathbb{R}_0)$, where

$$\mathbb{S}_0 = \{(x, y, z) : x^2 + y^2 = 1, z = 0\} \text{ and } \mathbb{R}_0 = \{(x, y, z) : x = y = 0\}.$$

Moreover, the flow corresponding to (38) acts on each invariant torus as the linear flow in the direction $(\alpha, 1)$ – we will see a relevant computation in a perturbed situation in a while.

The aim of this section is to give (for every irrational α) a continuous perturbation of (38) which completely destroys its integrable dynamics. We will show that under some special continuous perturbation of (38) the resulting systems have plenty of orbits which are dense, homoclinic and heteroclinic to limit cycles. The class of considered perturbations is in the spirit of Chapter XIX in [17].

Let $\psi : \mathbb{T}^2 \to \mathbb{R}$ be a continuous function. Set

$$\omega(x, y, z) = \frac{\arg(x^2 + y^2 + z^2 - 1 - 2zi)}{2\pi} \text{ and } \theta(x, y) = \frac{\arg(x + yi)}{2\pi}$$

for every $\mathbb{R}^3 \setminus (\mathbb{S}_0 \cup \mathbb{R}_0)$ and denote by $F : \mathbb{R}^3 \setminus (\mathbb{S}_0 \cup \mathbb{R}_0) \to \mathbb{R}$ the continuous function

$$F(x, y, z) = \ln\sqrt{\frac{(\sqrt{x^2+y^2}+1)^2 + z^2}{(\sqrt{x^2+y^2}-1)^2 + z^2}} \cdot \psi\left(\omega(x, y, z), \theta(x, y)\right).$$

We will deal with the perturbed differential equation

$$\begin{aligned} x' &= -2\pi y + 2\pi\alpha xz + \frac{(1 - x^2 - y^2 + z^2)x}{2\sqrt{x^2+y^2}} F(x, y, z) \\ y' &= 2\pi x + 2\pi\alpha yz + \frac{(1 - x^2 - y^2 + z^2)y}{2\sqrt{x^2+y^2}} F(x, y, z) \\ z' &= \pi\alpha(1 - x^2 - y^2 + z^2) - z\sqrt{x^2+y^2} F(x, y, z). \end{aligned} \tag{40}$$

It is easily checked that the right hand side of (40) can be continuously extended to $\mathbb{R}^3$ (indeed, since $|1 - r^2 + z^2| \leq 3\sqrt{(r-1)^2 + z^2}$ for $|r - 1|$ and $|z|$ sufficiently small, $(1 - r^2 + z^2)\ln\sqrt{(r-1)^2 + z^2} \to 0$ as $r \to 1$, $z \to 0$ and therefore on $\mathbb{S}_0 \cup \mathbb{R}_0$ we come back to (38)), so the equation (40) is well defined on $\mathbb{R}^3$. The existence and uniqueness of solutions (40) we will prove later by applying a change of coordinates (solutions are defined for all $t \in \mathbb{R}$, except for the line $\mathbb{R}_0$). Denote by $(\Psi_t)_{t\in\mathbb{R}}$ the flow corresponding to (40). We have already noticed that $x' = -2\pi y,\ y' = 2\pi x, z' = 0$ on $\mathbb{S}_0$ and $x' = 0,\ y' = 0, z' = \pi\alpha(1+z^2)$ on $\mathbb{R}_0$. Therefore $\mathbb{S}_0$ is $(\Psi_t)_{t\in\mathbb{R}}$-invariant and $\Psi_t(x,y,0) = \Psi_t(x+iy,0) = (e^{2\pi it}(x+iy),0)$ on $\mathbb{S}_0$. Moreover, $(\Psi_t)_{t\in\mathbb{R}}$ is a local flow on $\mathbb{R}_0$ and

$$\Psi_t(0,0,z) = (0,0,\tan(\pi\alpha t + \arctan z)) \text{ for } t \in \left(-\frac{1}{2\alpha},\frac{1}{2\alpha}\right) - \frac{\arctan z}{\pi\alpha}.$$

We will show that $(\Psi_t)_{t\in\mathbb{R}}$ acts on $\mathbb{R}^3 \setminus (\mathbb{S}_0 \cup \mathbb{R}_0)$ indeed as a flow. We will use a hyperbolic polar coordinates on the hyperbolic half-plane $\{(z,r) : r > 0, z \in \mathbb{R}\}$ given by

$$z + ir \mapsto \frac{i(z+ir)+1}{(z+ir)+i} = ie^{-e^s}e^{2\pi i\omega}$$

together with the usual polar coordinates $x = r\cos 2\pi\theta$, $y = r\sin 2\pi\theta$. It results in toral coordinates $(\omega,\theta,s) \in \mathbb{T}\times\mathbb{T}\times\mathbb{R}$ of $\mathbb{R}^3 \setminus (\mathbb{S}_0 \cup \mathbb{R}_0)$ given by

$$\begin{aligned}
x &= \left(\frac{-2(e^{-e^s}\cos 2\pi\omega - 1)}{(e^{-e^s}\sin 2\pi\omega)^2 + (e^{-e^s}\cos 2\pi\omega - 1)^2} - 1\right)\cos 2\pi\theta,\\
y &= \left(\frac{-2(e^{-e^s}\cos 2\pi\omega - 1)}{(e^{-e^s}\sin 2\pi\omega)^2 + (e^{-e^s}\cos 2\pi\omega - 1)^2} - 1\right)\sin 2\pi\theta,\\
z &= \frac{-2e^{-e^s}\sin 2\pi\omega}{(e^{-e^s}\sin 2\pi\omega)^2 + (e^{-e^s}\cos 2\pi\omega - 1)^2}.
\end{aligned}$$

Denote by $\Upsilon : \mathbb{T}\times\mathbb{T}\times\mathbb{R} \to \mathbb{R}^3 \setminus (\mathbb{S}_0 \cup \mathbb{R}_0)$ the map establishing the change of coordinates. The inverse change of coordinates is given by

$$\begin{aligned}
2\pi\omega &= \arg(x^2+y^2+z^2-1-2zi),\\
2\pi\theta &= \arg(x+yi),\\
s &= \ln\ln\sqrt{\frac{(\sqrt{x^2+y^2}+1)^2+z^2}{(\sqrt{x^2+y^2}-1)^2+z^2}};
\end{aligned}$$

indeed,

$$e^{-e^s} = \left|\frac{i(z+ir)+1}{(z+ir)+i}\right| = \sqrt{\frac{(r-1)^2+z^2}{(r+1)^2+z^2}},$$

$$2\pi\omega = \arg\frac{z+i(r-1)}{z+(r+1)} = \arg(r^2+z^2-1-2zi).$$

Setting additionally $u = e^s$ we have

$$\begin{aligned}
x' &= -2\pi y + 2\pi\alpha xz + \frac{(1-r^2+z^2)x}{2r}u\psi(\omega,\theta),\\
y' &= 2\pi x + 2\pi\alpha yz + \frac{(1-r^2+z^2)y}{2r}u\psi(\omega,\theta),\\
z' &= \pi\alpha(1-r^2+z^2) - zru\psi(\omega,\theta).
\end{aligned} \tag{41}$$

It follows that

$$r' = \frac{xx'+yy'}{r} = 2\pi\alpha rz + \frac{1-r^2+z^2}{2}u\psi(\omega,\theta), \tag{42}$$

and

$$xy' - yx' = 2\pi(x^2+y^2).$$

Since $x = r\cos 2\pi\theta$, $y = r\sin 2\pi\theta$, we have

$$2\pi\theta' = \frac{x}{r}\left(\frac{y}{r}\right)' - \frac{y}{r}\left(\frac{x}{r}\right)' = \frac{xy' - yx'}{x^2+y^2} = 2\pi. \tag{43}$$

Next note that

$|x^2+y^2+z^2-1-2zi|^2 = (r^2+z^2-1)^2 + (2z)^2 = ((r+1)^2+z^2)((r-1)^2+z^2)$,

and hence

$$\begin{aligned} r^2+z^2-1 &= \sqrt{((r+1)^2+z^2)((r-1)^2+z^2)}\cos 2\pi\omega, \\ -2z &= \sqrt{((r+1)^2+z^2)((r-1)^2+z^2)}\sin 2\pi\omega. \end{aligned}$$

It follows that (similarly as in (43))

$$2\pi\omega' = \frac{(-2z)'(r^2+z^2-1) - (r^2+z^2-1)'(-2z)}{((r+1)^2+z^2)((r-1)^2+z^2)}.$$

Moreover, by (41) and (42),

$$\begin{aligned} (r^2+z^2-1)'2z - (2z)'(r^2+z^2-1) &= 2(2zrr' + (1-r^2+z^2)z') \\ &= 2\pi\alpha(4z^2r^2 + (1-r^2+z^2)^2) = 2\pi\alpha((r+1)^2+z^2)((r-1)^2+z^2), \end{aligned}$$

so $2\pi\omega' = 2\pi\alpha$. Since $u = \frac{1}{2}\left(\ln((r+1)^2+z^2) - \ln((r-1)^2+z^2)\right)$, we have

$$u' = \frac{r'(r+1)+z'z}{(r+1)^2+z^2} - \frac{r'(r-1)+z'z}{(r-1)^2+z^2} = \frac{2(1-r^2+z^2)r' - 4rzz'}{((r+1)^2+z^2)((r-1)^2+z^2)}.$$

Moreover, by (41) and (42),

$$\begin{aligned} 2(1-r^2+z^2)r' - 4rzz' &= ((z^2-r^2+1)^2 + 4(rz)^2)u\psi(\omega,\theta) \\ &= ((r+1)^2+z^2)((r-1)^2+z^2)u\psi(\omega,\theta), \end{aligned}$$

hence $u' = u\psi(\omega,\theta)$. Therefore

$$s' = u'/u = \psi(\omega,\theta),$$

hence in the new coordinates the differential equation (40) takes the form

$$\omega' = \alpha, \quad \theta' = 1, \quad s' = \psi(\omega,\theta)$$

and we return to the scheme from Section 8. Denote by $(\Phi_t)_{t\in\mathbb{R}}$ the corresponding flow on $\mathbb{T}\times\mathbb{T}\times\mathbb{R}$, then

$$\Phi_t(\omega,\theta,s) = (\omega + t\alpha, \theta + t, s + \int_0^t \psi(\omega+\tau\alpha,\theta+\tau)d\tau).$$

Let α be an arbitrary irrational number and let $\varphi : \mathbb{T}\to\mathbb{R}$ stand for the function constructed in Section 5. Choose a continuous function $\psi : \mathbb{T}^2 \to \mathbb{R}$ for which $\varphi(\omega) = \int_0^1 \psi(\omega+\tau\alpha,\tau)d\tau$. Assume that for $(\omega,\theta,s)\in\mathbb{T}\times\mathbb{T}\times\mathbb{R}$ we have $\omega - \alpha\theta \in D^{s_-+}$. Then $\varphi^{(n)}(\omega-\alpha\theta)\to+\infty$ as $n\to+\infty$, so

$$\begin{aligned} s_n :&= s + \int_0^n \psi(\omega+\tau\alpha,\theta+\tau)d\tau \\ &= s + \varphi^{(n)}(\omega-\alpha\theta) + \int_0^\theta \left(\psi(\omega+(\tau+n-\theta)\alpha,\tau) - \psi(\omega+(\tau-\theta)\alpha,\tau)\right)d\tau \\ &\to +\infty. \end{aligned}$$

Note that if $(x,y,z) = \Upsilon(\omega,\theta,s)$ then

$$x+iy = \frac{1-e^{-2e^s}}{e^{-2e^s} - 2e^{-e^s}\cos 2\pi\omega + 1}e^{2\pi i\theta}, \qquad z = \frac{-2e^{-e^s}\sin 2\pi\omega}{e^{-2e^s} - 2e^{-e^s}\cos 2\pi\omega + 1}.$$

Since $e^{-2e^{s_n}} \to 0$, setting $\omega_n = \omega + n\alpha$ we obtain

$$\begin{aligned}\Psi_n(x,y,z) &= \Psi_n \circ \Upsilon(\omega,\theta,s) = \Upsilon \circ \Phi_n(\omega,\theta,s) = \Upsilon(\omega_n,\theta+n,s_n) = \Upsilon(\omega_n,\theta,s_n)\\ &= \left(\frac{(1-e^{-2e^{s_n}})e^{2\pi i\theta}}{e^{-2e^{s_n}}-2e^{-e^{s_n}}\cos 2\pi\omega_n+1}, \frac{-2e^{-e^{s_n}}\sin 2\pi\omega_n}{e^{-2e^{s_n}}-2e^{-e^{s_n}}\cos 2\pi\omega_n+1}\right)\\ &\to (e^{2\pi i\theta},0) \in \mathbb{S}_0.\end{aligned}$$

It follows that the ω–limit set of (x,y,z) is equal to $\mathbb{S}_0$.

Now assume that $\omega - \alpha\theta \in D^{s_{-}-}$ and $\omega \neq 0$. Then

$$\begin{aligned}s_n :&= s + \int_0^{n/\alpha} \psi(\omega+\tau\alpha,\theta+\tau)d\tau\\ &= s + \varphi^{(\lfloor\theta+n/\alpha\rfloor)}(\omega-\alpha\theta)\\ &\quad + \int_0^{\{\theta+n/\alpha\}} \psi\left(\omega+(\tau-\theta+\lfloor\theta+n/\alpha\rfloor)\alpha,\tau\right)d\tau - \int_0^{\theta} \psi\left(\omega+(\tau-\theta)\alpha,\tau\right)d\tau\\ &\to -\infty\end{aligned}$$

as $n \to +\infty$, so $e^{-2e^{s_n}} \to 1$. Setting $\theta_n = \theta + n/\alpha$ we obtain

$$\begin{aligned}\Psi_{n/\alpha}(x,y,z) &= \Upsilon \circ \Phi_{n/\alpha}(\omega,\theta,s) = \Upsilon(\omega+n,\theta_n,s_n) = \Upsilon(\omega,\theta_n,s_n)\\ &= \left(\frac{(1-e^{-2e^{s_n}})e^{2\pi i\theta_n}}{e^{-2e^{s_n}}-2e^{-e^{s_n}}\cos 2\pi\omega+1}, \frac{-2e^{-e^{s_n}}\sin 2\pi\omega}{e^{-2e^{s_n}}-2e^{-e^{s_n}}\cos 2\pi\omega+1}\right)\\ &\to \left(0, \frac{-\sin 2\pi\omega}{1-\cos 2\pi\omega}\right) = (0,\tan(\pi\omega-\pi/2)) \in \mathbb{R}_0.\end{aligned}$$

It follows that the ω–limit set of (x,y,z) is equal to $\mathbb{R}_0$.

Let $A_+ := \mathbb{S}_0$ and $A_- := \mathbb{R}_0$. Similar arguments to those above show that if $\omega - \alpha\theta \in D^{s_-s_+}$ then the α–limit set of (x,y,z) is A_{s_-}. Recall that the set $D^{s_-s_+}$ is invariant under the rotation by α, so it is dense. Thus

$$\{\Upsilon(\omega,\theta,s) : (\omega,\theta,s) \in \mathbb{T}\times\mathbb{T}\times\mathbb{R},\ \omega-\alpha\theta \in D^{s_-s_+}\}$$

is dense in $\mathbb{R}^3$. Moreover, the Hausdorff dimension of this set is no smaller than $\dim_H D_{s_-s_+} + 2$. In view of Theorem 9 we have the following.

Theorem 13. *For every irrational α there exists a continuous function $\psi : \mathbb{T}^2 \to \mathbb{R}$ such that the flow $(\Psi_t)_{t\in\mathbb{R}}$ corresponding to (40) is transitive and for each $s_-, s_+ \in \{-,+\}$ the set of points such that the ω–limit set is equal to A_{s_+} and the α–limit set is equal to A_{s_-} is dense. Moreover, for almost every α the Hausdorff dimension of each such set is no smaller than $5/2$.*

8.2. **Differential equations on $\mathbb{S}^3$.** In this section we will deal with continuous (or even Hölder continuous) perturbations of the completely integrable system $z_1' = iaz_1$, $z_2' = ibz_2$ $(a,b \in \mathbb{R}\setminus\{0\})$ on $\mathbb{C}\times\mathbb{C} \simeq \mathbb{R}^4$. Let us consider the system of differential equations

$$\begin{aligned}z_1' &= iaz_1 - F(z_1,z_2)\overline{z_2}\\ z_2' &= ibz_2 + F(z_1,z_2)\overline{z_1},\end{aligned} \tag{44}$$

where $F : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ is a continuous function $\mathbb{R}_+$-homogeneous of degree 1 on each coordinate, i.e. $F(t_1z_1,t_2z_2) = t_1t_2F(z_1,z_2)$ for all $t_1,t_2 > 0$, and such that $\overline{z_1z_2}F(z_1,z_2) \in \mathbb{R}$. Denote by $(\Psi_t)_{t\in\mathbb{R}}$ the associated flow on $\mathbb{C}\times\mathbb{C}$. The existence of $(\Psi_t)_{t\in\mathbb{R}}$ will be shown as a byproduct. Note that $|z_1|^2+|z_2|^2$ is a first integral for (44). Indeed,

$$\frac{d}{dt}|z_1|^2 = 2\Re z_1'\overline{z_1} = 2\Re(ia|z_1|^2 - F(z_1,z_2)\overline{z_1z_2}) = -2F(z_1,z_2)\overline{z_1z_2} \tag{45}$$

and

$$\frac{d}{dt}|z_2|^2 = 2\Re z_2'\overline{z_2} = 2\Re(ia|z_2|^2 + F(z_1,z_2)\overline{z_1 z_2}) = 2F(z_1,z_2)\overline{z_1 z_2}, \tag{46}$$

so $\frac{d}{dt}(|z_1|^2+|z_2|^2)=1$. Therefore,

$$\mathbb{S}^3 = \{(z_1,z_2)\in\mathbb{C}\times\mathbb{C} : |z_1|^2+|z_2|^2=1\}$$

is (Φ_t)–invariant and we confine ourselves to the study of $(\Psi_t)_{t\in\mathbb{R}}$ on $\mathbb{S}^3$. Let us consider

$$\widetilde{F}:\mathbb{C}\setminus\{0\}\times\mathbb{C}\setminus\{0\}\to\mathbb{R},\ \ \widetilde{F}(z_1,z_2)=\frac{F(z_1,z_2)}{z_1z_2}=\frac{F(z_1,z_2)\overline{z_1z_2}}{|z_1|^2|z_2|^2}\in\mathbb{R}.$$

By assumptions, $\widetilde{F}$ is continuous and $\widetilde{F}(z_1,z_2)=\widetilde{F}(z_1/|z_1|,z_2/|z_2|)$. Let

$$\psi:\mathbb{T}\times\mathbb{T}\to\mathbb{R},\ \ \psi(\omega,\theta)=\widetilde{F}(e^{2\pi i\omega},e^{2\pi i\theta}),$$

so ψ is continuous. Moreover, note that each continuous function $\psi:\mathbb{T}\times\mathbb{T}\to\mathbb{R}$ determines a continuous function $F:\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ which is $\mathbb{R}_+$-homogeneous of degree 1 on each coordinate and such that $\overline{z_1z_2}F(z_1,z_2)\in\mathbb{R}$ as follows

$$F(z_1,z_2)=z_1z_2\psi(\omega,\theta) \text{ whenever } z_1=|z_1|e^{2\pi i\omega},\ z_2=|z_2|e^{2\pi i\theta}.$$

If additionally ψ is γ-Hölder continuous then $F:\mathbb{S}^3\to\mathbb{C}$ is γ-Hölder continuous as well (see Proposition 19 in Appendix C). Let

$$\mathbb{S}^1_- := \{(z_1,z_2)\in\mathbb{S}^3 : |z_1|=1, z_2=0\} \text{ and } \mathbb{S}^1_+ := \{(z_1,z_2)\in\mathbb{S}^3 : z_1=0, |z_2|=1\}.$$

Observe that $\mathbb{S}^1_+$, $\mathbb{S}^1_-$ are invariant sets and

$$\Psi(z_1,0)=(e^{iat}z_1,0) \text{ and } \Psi(0,z_2)=(0,e^{ibt}z_2).$$

Let us consider a new coordinates $(\omega,\theta,s)\in\mathbb{T}\times\mathbb{T}\times\mathbb{R}$ of $\mathbb{S}^3\setminus(\mathbb{S}^1_+\cup\mathbb{S}^1_-)$ given by

$$(z_1,z_2)=\Upsilon(\omega,\theta,s)=(e^{2\pi i\omega}\cos\arctan e^s, e^{2\pi i\theta}\sin\arctan e^s).$$

Then

$$e^{2\pi i\omega}=z_1/|z_1|,\ e^{2\pi i\theta}=z_2/|z_2| \text{ and } |z_1|=\cos\arctan e^s,\ \ |z_2|=\sin\arctan e^s.$$

Thus

$$\frac{|z_2|}{|z_1|}=\frac{\sin\arctan e^s}{\cos\arctan e^s}=e^s,$$

so

$$s=\ln|z_2|-\ln|z_1|=\frac{1}{2}(\ln|z_2|^2-\ln|z_1|^2).$$

By (45) and (46), it follows that

$$\begin{aligned}\frac{ds}{dt}&=\frac{1}{2}\left(\frac{1}{|z_2|^2}\frac{d}{dt}|z_2|^2-\frac{1}{|z_1|^2}\frac{d}{dt}|z_1|^2\right)=F(z_1,z_2)\overline{z_1z_2}\left(\frac{1}{|z_2|^2}+\frac{1}{|z_1|^2}\right)\\&=F(z_1,z_2)\frac{\overline{z_1z_2}}{|z_1|^2|z_2|^2}=\widetilde{F}(z_1,z_2)=\widetilde{F}(z_1/|z_1|,z_2/|z_2|)=\psi(\omega,\theta).\end{aligned}$$

Moreover,

$$\begin{aligned}2\pi\omega'&=\Im\frac{z_1'}{z_1}=\Im\frac{iaz_1-F(z_1,z_2)\overline{z_2}}{z_1}=\Im\left(ia-\frac{F(z_1,z_2)\overline{z_1z_2}}{|z_1|^2}\right)=a,\\2\pi\theta'&=\Im\frac{z_2'}{z_2}=\Im\frac{ibz_2-F(z_1,z_2)\overline{z_1}}{z_2}=\Im\left(ib+\frac{F(z_1,z_2)\overline{z_1z_2}}{|z_2|^2}\right)=b.\end{aligned}$$

Hence if $a=2\pi\alpha$ and $b=2\pi$ then in the new coordinates the differential equation (44) on $\mathbb{S}^3\setminus(\mathbb{S}^1_+\cup\mathbb{S}^1_-)$ takes the form

$$\omega'=\alpha,\ \ \theta'=1,\ \ s'=\psi(\omega,\theta),$$

and we return to the scheme from Section 8 as well. Therefore if $(z_1, z_2) = \Upsilon(\omega, \theta, s) \in \mathbb{S}^3$ then $\Psi_t(z_1, z_2)$ equals

$$\left(e^{2\pi i(\omega+t\alpha)} \cos\arctan e^{s+\int_0^t \psi(\omega+\alpha\tau,\theta+\tau)d\tau}, e^{2\pi i(\theta+t)} \sin\arctan e^{s+\int_0^t \psi(\omega+\alpha\tau,\theta+\tau)d\tau}\right).$$

Recall that $\cos\arctan e^s \to 0$ as $s \to +\infty$ and $\sin\arctan e^s \to 0$ as $s \to -\infty$. It follows that if $\int_0^t \psi(\omega + \alpha\tau, \theta + \tau)d\tau \to s_+\infty$ as $t \to +\infty$, then the ω–limit set of (z_1, z_2) is equal to $\mathbb{S}^1_{s_+}$. Moreover, if $\int_0^t \psi(\omega + \alpha\tau, \theta + \tau)d\tau \to s_-\infty$ as $t \to -\infty$, then the α–limit set of (z_1, z_2) is equal to $\mathbb{S}^1_{s_-}$.

Now as a consequence of results from Sections 2-6 we have the following theorem which demonstrates possible coexistence of different behaviours for solutions of (44).

Theorem 14. *For every irrational α there exists a continuous function $F : \mathbb{S}^3 \to \mathbb{R}$ such that the corresponding flow $(\Psi_t)_{t\in\mathbb{R}}$ on $\mathbb{S}^3$ is transitive and for each $s_-, s_+ \in \{-, +\}$ the set $HC_{s_-s_+}$ of points such that the ω–limit set is equal to $\mathbb{S}^1_{s_+}$ and the α–limit set is equal to $\mathbb{S}^1_{s_-}$ is dense. If $\alpha \in DC(a)$ for some $a \geq 1$ then for every $0 < \gamma < 1/((1 + a)a)$ the function F can be chosen γ-Hölder continuous.*

Moreover, for almost every α the Hausdorff dimension of each set $HC_{s_-s_+}$ is no smaller than $5/2$.

Appendix A. Open problems

If a continuous function $\varphi : \mathbb{T} \to \mathbb{R}$ with zero mean has bounded variation then the uniform Denjoy-Koksma inequality holds, i.e. $|\varphi^{(q_n)}(x)| \leq \operatorname{Var}\varphi$ for all $x \in \mathbb{T}$, and as we have already noticed, this implies the absence of discrete orbits for T_φ. Now assume that $\varphi : \mathbb{T} \to \mathbb{R}$ is continuous and it satisfies only $\widehat{\varphi}(n) = \mathrm{O}(1/|n|)$. As it has been proved in [2], φ fulfills an L^2–Denjoy-Koksma inequality, in particular the sequence $(\varphi^{(q_n)})_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{T})$. However, we are not aware of any direct argument based on the L^2–Denjoy-Koksma inequality which shows the absence of discrete orbits.

Problem 1. Does there exist an irrational rotation $Tx = x + \alpha$ and a continuous function $\varphi : \mathbb{T} \to \mathbb{R}$ with $\widehat{\varphi}(n) = \mathrm{O}(1/|n|)$ such that the cylindrical transformation T_φ is Besicovitch?

In Section 3, for α satisfying a Diophantine condition we have constructed a γ–Hölder continuous function $\varphi : \mathbb{T} \to \mathbb{R}$ such that the corresponding cylindrical transformation is Besicovitch. However γ was smaller than $1/2$.

Problem 2. Does there exist a γ–Hölder continuous function $\varphi : \mathbb{T} \to \mathbb{R}$ with $\gamma \geq 1/2$ such that T_φ is Besicovitch? Can we construct Hölder continuous Besicovitch cylindrical transformations over "very" Liouville rotations, i.e. when we assume that the sequence of denominators (q_n) increases very rapidly?

We estimated from below the Hausdorff dimension of the sets $D^{s_-s_+}(\alpha, \varphi)$ by estimating from below the Hausdorff dimension of a subset $F^{s_-s_+}$. However, under the condition (29), the Hausdorff dimension of the latter set is not bigger than $1/2$.

Problem 3. Is $1/2$ an upper bound for the Hausdorff dimension of $D(\alpha, \varphi)$ or can we find α and φ so that $\dim_H(D(\alpha, \varphi)) > 1/2$? Is there any relationship between $\dim_H D(\alpha, \varphi)$ and the Hölder exponent of φ or the type of α while estimating the Hausdorff dimension from above?

It would be also interesting to decide whether there can exist a Besicovitch transformation in the most extremal sense of the definition.

Problem 4. Does there exist a cylindrical transformation which is Besicovitch and such that each orbit is either dense or discrete?

There are some natural questions concerning smooth Besicovitch cylindrical transformations (the problem below being only a sample).

Problem 5. Does there exist a C^∞ (or even analytic) Besicovitch cylindrical transformation over a minimal rotation on higher dimensional tori?

Notice that in the present paper we did not decide whether a C^1 Besicovitch transformation exists over a two-dimensional minimal rotation.

Remark 12. Note that we can view points whose orbits are discrete as special points which are not recurrent. The results of this paper can hence be seen as a contribution toward a better understanding of the problem how big the set of non-recurrent points can be, both in the compact (as in Section 8.2) or non-compact case.

Remark 13. We have not been able to decide whether or not constructions from Sections 2–7 lead to ergodic cocycles. Notice however that the cocycle $\varphi : \mathbb{T} \to \mathbb{R}$ from Section 4 is a measurable coboundary. Indeed, $\|f_k - M_k\|_{L^1} \le 2q_k\delta_k M_k \le 2\frac{\log q_k}{q_{k+1}}$, so the serious $f = \sum_{k=1}^{\infty}(f_k - M_k)$ converges in L^1, and hence $\varphi(x) = f(x+\alpha) - f(x)$ for a.e. $x \in \mathbb{T}$. Moreover, some further modifications of the construction from Section 5 lead also to cocycles which are measurable coboundaries.

Appendix B. Minimality of rotations on tori

Set $A = 16$, $B = \lceil 4^a A\rceil$ and $C = 2B$. Let $S = \left((A_n^{(j)}, B_n^{(j)})\right)_{n\ge1,1\le j\le d}$ be a sequence such that $(A_n^{(j)}, B_n^{(j)})$ is equal to $(4, C)$ or (A, B). Denote by $\mathfrak{C}(a, S)$ the set of all $(\alpha_1, \ldots, \alpha_d) \in \mathbb{T}^d$ such that

$$A_n^{(j)}(q_n^{(j-1)})^a \le q_n^{(j)} \le B_n^{(j)}(q_n^{(j-1)})^a \text{ for all } n \ge 1,\ 1 \le j \le d, \tag{47}$$

with the notation $q_n^{(0)} = q_{n-1}^{(d)}$. For every subset $U \subset \{1, \ldots, d\}$ denote by S^U the sequence $\left((\widetilde{A}_n^{(j)}, \widetilde{B}_n^{(j)})\right)_{n\ge1,1\le j\le d}$ such that

$$(\widetilde{A}_n^{(j)}, \widetilde{B}_n^{(j)}) = \begin{cases} (4, C) & \text{if } \quad j \in U \\ (A_n^{(j)}, B_n^{(j)}) & \text{if } \quad j \notin U. \end{cases}$$

Let $\underline{S}$ be the constant sequence with $(A_n^{(j)}, B_n^{(j)}) = (A, B)$. Then $\mathfrak{C}(a, \underline{S}^{\{1,\ldots,d\}}) = \mathfrak{C}(a)$.

Lemma 15. *The set $\mathfrak{C}(a, \underline{S}) \subset \mathbb{T}^d$ uncountable.*

Proof. Let $[0; a_1^{(j)}, a_2^{(j)}, \ldots]$ stand for the continued fraction expansion of α_j for $j = 1, \ldots, d$. We will use the notation $a_n^{(0)} := a_{n-1}^{(d)}$. We will construct sequences of partial quotients $(a_n^{(j)})_{n=1}^{\infty}$, $j = 1, \ldots, d$ inductively. In the first step choose any natural $A \le a_1^{(1)} \le B$. Then $q_1^{(1)} = a_1^{(1)}$ fulfills (47) for $(n, j) = (1, 1)$. In the inductive step suppose that for a pair of natural numbers (n, l) with $1 \le l \le d$ all partial quotients $a_k^{(j)}$ for $k < n$, $1 \le j \le d$, and $a_n^{(j)}$ for $0 \le j < l$ are already chosen so that the corresponding denominators satisfy the following inequalities

$$A(q_k^{(j-1)})^a \le q_k^{(j)} \le B(q_k^{(j-1)})^a \tag{48}$$

for all pairs (k, j) such that $1 \le k < n$, $1 \le j \le d$ or $k = n$, $1 \le j < l$. Now we can choose $a_n^{(l)} \in \mathbb{N}$ so that

$$A(q_n^{(l-1)})^a \le a_n^{(l)} q_{n-1}^{(l)} + q_{n-2}^{(l)} \le B(q_n^{(l-1)})^a. \tag{49}$$

Indeed, by (48), $q^{(l)}_{n-2} \le q^{(l)}_{n-1} \le q^{(l-1)}_n/A \le (q^{(l-1)}_n)^a/16$, so we can find at least two numbers $a^{(l)}_n$ satisfying (49). Since there are at least two choices for $a^{(l)}_n$ at each step of the construction, the set $\mathfrak{C}(a, \underline{S})$ is uncountable. $\square$

Lemma 16. *Fix $1 \le l \le d$ and let $S = \left((A^{(j)}_n, B^{(j)}_n)\right)_{n\ge1, 1\le j\le d}$ be a sequence such that $(A^{(l)}_n, B^{(l)}_n) = (A, B)$ for $n \ge 1$. If $\overline{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathfrak{C}(a, S)$ then the set*

$$\{\alpha \in \mathbb{T} : (\alpha_1, \ldots, \alpha_{l-1}, \alpha, \alpha_{l+1}, \ldots, \alpha_d) \in \mathfrak{C}(a, S^{\{l\}})\}$$

is uncountable.

Proof. By assumption, for each $n \ge 1$

$$\begin{aligned} A(q^{(l-1)}_n)^a &\le q^{(l)}_n \le B(q^{(l-1)}_n)^a, \\ A^{(l+1)}_n(q^{(l)}_n)^a &\le q^{(l+1)}_n \le B^{(l+1)}_n(q^{(l)}_n)^a, \end{aligned} \tag{50}$$

and hence, by the latter inequality,

$$\frac{(q^{(l+1)}_n)^{1/a}}{(B^{(l+1)}_n)^{1/a}} \le q^{(l)}_n \le \frac{(q^{(l+1)}_n)^{1/a}}{(A^{(l+1)}_n)^{1/a}}. \tag{51}$$

Let

$$I_n := \left[\max\left(4(q^{(l-1)}_n)^a, \frac{(q^{(l+1)}_n)^{1/a}}{(B^{(l+1)}_n)^{1/a}}\right), \min\left(C(q^{(l-1)}_n)^a, \frac{(q^{(l+1)}_n)^{1/a}}{(A^{(l+1)}_n)^{1/a}}\right)\right].$$

By the definition of A, B, C and (50), (51), we have

$$\begin{aligned} C(q^{(l-1)}_n)^a - 4(q^{(l-1)}_n)^a &\ge 3q^{(l-1)}_n, \\ \frac{(q^{(l+1)}_n)^{1/a}}{(A^{(l+1)}_n)^{1/a}} - \frac{(q^{(l+1)}_n)^{1/a}}{(B^{(l+1)}_n)^{1/a}} &= \frac{(q^{(l+1)}_n)^{1/a}}{(A^{(l+1)}_n)^{1/a}}\left(1 - \frac{(A^{(l+1)}_n)^{1/a}}{(B^{(l+1)}_n)^{1/a}}\right) \\ &\ge q^{(l)}_n(1 - (A/B)^{1/a}) \ge \frac{3}{4}q^{(l)}_n, \\ \frac{(q^{(l+1)}_n)^{1/a}}{(A^{(l+1)}_n)^{1/a}} - 4(q^{(l-1)}_n)^a &\ge q^{(l)}_n(1 - 4/A) = \frac{3}{4}q^{(l)}_n, \\ C(q^{(l-1)}_n)^a - \frac{(q^{(l+1)}_n)^{1/a}}{(B^{(l+1)}_n)^{1/a}} &\ge q^{(l)}_n(C/B - 1) = q^{(l)}_n. \end{aligned}$$

Since $q^{(l-1)}_n \ge q^{(l+1)}_{n-1}$ and $q^{(l)}_n \ge 4q^{(l+1)}_{n-1}$, it follows that $|I_n| \ge 3q^{(l+1)}_{n-1}$.

Now we construct an irrational number $\alpha = [0; a_1, a_2, \ldots]$ so that

$$(\alpha_1, \ldots, \alpha_{l-1}, \alpha, \alpha_{l+1}, \ldots, \alpha_d) \in \mathfrak{C}(a, S^{\{l\}}).$$

We construct the sequence (a_n) of partial quotients of α inductively. Since $|I_1| \ge 3q^{(l+1)}_0 = 3$, we can choose $a_1 \in I_1$. As $q_1 = a_1 \in I_1$, we have

$$4(q^{(l-1)}_1)^a \le q_1 \le C(q^{(l-1)}_1)^a,\ A^{(l+1)}_1 q^a_1 \le q^{(l+1)}_1 \le B^{(l+1)}_1 q^a_1.$$

In the n–th step suppose that $a_1, \ldots, a_{n-1}$ are already selected so that

$$4(q^{(l-1)}_k)^a \le q_k \le C(q^{(l-1)}_k)^a,\ A^{(l+1)}_k q^a_k \le q^{(l+1)}_k \le B^{(l+1)}_k q^a_k$$

for $1 \le k < n$. It follows that

$$|I_n| \ge 3q^{(l+1)}_{n-1} \ge 3q_{n-1}.$$

Therefore there are at least two natural numbers a_n such that $a_n q_{n-1} + q_{n-2} \in I_n$. Thus

$$4(q^{(l-1)}_n)^a \le q_n \le C(q^{(l-1)}_n)^a,\ A^{(l+1)}_n q^a_n \le q^{(l+1)}_n \le B^{(l+1)}_n q^a_n.$$

It follows that for each α constructed in this way we have

$$(\alpha_1,\dots,\alpha_{l-1},\alpha,\alpha_{l+1},\dots,\alpha_d)\in\mathfrak{C}(a,S^{\{l\}}).$$

Moreover, there are at least two choices for a_n at each step of the construction, so the set $\mathfrak{C}(a,S^{\{l\}})$ is uncountable. □

Theorem 17. *There are uncountably many $\overline{\alpha}=(\alpha_1,\dots,\alpha_d)\in\mathfrak{C}(a)$ such that $\alpha_1,\dots,\alpha_d,1$ are independent over $\mathbb{Q}$.*

Proof. For each $l=1,\dots,d$ set

$$\mathbb{Z}^{d,l}=\{\overline{m}=(m_1,\dots,m_d)\in\mathbb{Z}^d: m_l\neq 0, m_j=0 \text{ for all } l<j\leq d\}.$$

Let $(\overline{\alpha}[l])_{l=0}^d$ be a sequence in $\mathbb{T}^d$ defined inductively as follows:

(i) $\overline{\alpha}[0]\in\mathfrak{C}(a,\underline{S})$;

(ii) for each $1\leq l\leq d$ let $\overline{\alpha}[l]$ be an element of $\mathfrak{C}(a,\underline{S}^{\{1,\dots,l\}})$ such that $\overline{\alpha}[l]_j=\overline{\alpha}[l-1]_j$ for all $j\neq l$ and $\langle\overline{m},\overline{\alpha}[l]\rangle\notin\mathbb{Z}$ for all $\overline{m}\in\mathbb{Z}^{d,l}$.

The existence of $\overline{\alpha}[0]$ follows directly from Lemma 15. The existence of uncountably many $\overline{\alpha}[l]$ follows from Lemma 16 applied to $\overline{\alpha}=\overline{\alpha}[l-1]$ and $S=\underline{S}^{\{1,\dots,l-1\}}$.

Let $\overline{\alpha}=(\alpha_1,\dots,\alpha_d):=\overline{\alpha}[d]$. Then $\alpha_1,\dots,\alpha_d,1$ are rationally independent. Indeed, let $\overline{m}\in\mathbb{Z}^d$ be a nonzero vector and let l be the largest index for which $m_l\neq 0$. Then $\langle\overline{m},\overline{\alpha}[d]\rangle=\langle\overline{m},\overline{\alpha}[l]\rangle$. Since $\overline{m}\in\mathbb{Z}^{d,l}$, we conclude that $\langle\overline{m},\overline{\alpha}\rangle=\langle\overline{m},\overline{\alpha}[d]\rangle=\langle\overline{m},\overline{\alpha}[l]\rangle\notin\mathbb{Z}$. Moreover,

$$(\alpha_1,\dots,\alpha_d)=\overline{\alpha}[d]\in\mathfrak{C}(a,\underline{S}^{\{1,\dots,d\}})=\mathfrak{C}(a).$$

At each step of the construction we had uncountably many choices, so the set of constructed vectors $\overline{\alpha}$ is uncountable as well. □

Appendix C. Hölder continuity

Let $f:\mathbb{T}\to\mathbb{R}$ be a continuous function. Set $\mathbb{D}=\{z\in\mathbb{C}:|z|\leq 1\}$. Denote by $\widehat{f}:\mathbb{D}\to\mathbb{C}$ the continuous function given by $\widehat{f}(z)=zf(\omega)$ whenever $z=|z|e^{2\pi i\omega}$.

Lemma 18. *Suppose f is a γ-Hölder continuous function $(0<\gamma\leq 1)$ with $|f(\omega)-f(\omega')|\leq M\|\omega-\omega'\|^\gamma$. Then $\widehat{f}$ is γ-Hölder continuous and $|\widehat{f}(z)-\widehat{f}(z')|\leq 2(\|f\|_{C^0}+M)|z-z'|^\gamma$.*

Proof. Set $C=\|f\|_{C^0}$. Note that for all $z=|z|e^{2\pi i\omega}$, $z'=|z'|e^{2\pi i\omega'}$ we have

$$\begin{aligned}|z-z'|^2&=|z|^2+|z'|^2-2\Re z\overline{z'}=(|z|-|z'|)^2+2|z||z'|\Re(1-e^{2\pi i(\omega-\omega')})\\&=(|z|-|z'|)^2+4|z||z'|\sin^2(\pi\|\omega-\omega'\|)\geq 4|z||z'|\sin^2(\pi\|\omega-\omega'\|).\end{aligned}$$

In view of (15), it follows that

$$|z-z'|\geq 4\sqrt{|z||z'|}\|\omega-\omega'\|. \tag{52}$$

Suppose that $|z|\leq|z'|$, then $|z|\leq\sqrt{|z||z'|}$, and hence

$$\begin{aligned}|\widehat{f}(z)-\widehat{f}(z')|&=|zf(\omega)-z'f(\omega')|\leq|z-z'||f(\omega')|+|z||f(\omega)-f(\omega')|\\&\leq C|z-z'|+M\sqrt{|z||z'|}\|\omega-\omega'\|^\gamma.\end{aligned}$$

Since $z,z'\in\mathbb{D}$, we have $|z-z'|\leq 2^{1-\gamma}|z-z'|^\gamma$ and $\sqrt{|z||z'|}\leq\sqrt{|z||z'|}^\gamma$, so

$$\begin{aligned}|\widehat{f}(z)-\widehat{f}(z')|&\leq C2^{1-\gamma}|z-z'|^\gamma+M\sqrt{|z||z'|}^\gamma\|\omega-\omega'\|^\gamma\\&\leq(2^{1-\gamma}C+M)\left(|z-z'|+\sqrt{|z||z'|}\|\omega-\omega'\|\right)^\gamma.\end{aligned}$$

In view of (52), it follows that

$$|\widehat{f}(z)-\widehat{f}(z')|\leq(2^{1-\gamma}C+M)2^\gamma|z-z'|^\gamma\leq 2(C+M)|z-z'|^\gamma.$$

□

Let $\psi : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ be a continuous function. Denote by $F : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ the function given by $F(z_1, z_2) = z_1 z_2 \psi(\omega, \theta)$ whenever $z_1 = |z_1| e^{2\pi i \omega}$, $z_2 = |z_2| e^{2\pi i \theta}$.

Proposition 19. *If ψ is γ-Hölder continuous for some $0 < \gamma \leq 1$ then F is γ-Hölder continuous.*

Proof. Let $C = \|\psi\|_{C^0}$ and let $M \geq 0$ be such that $|\psi(\omega, \theta) - \psi(\omega', \theta')| \leq M(\|\omega - \omega'\| + \|\theta - \theta'\|)^\gamma$. If $z_1 = |z_1| e^{2\pi i \omega}$, $z_2 = |z_2| e^{2\pi i \theta}$, $z_1' = |z_1'| e^{2\pi i \omega'}$, $z_2' = |z_2'| e^{2\pi i \theta'}$ then, by Lemma 18,

$$|z_1 \psi(\omega, \theta') - z_1' \psi(\omega', \theta')| \leq 2(C + M)|z_1 - z_1'|^\gamma, \text{ uniformly in } \theta'$$
$$|z_2 \psi(\omega, \theta) - z_2' \psi(\omega, \theta')| \leq 2(C + M)|z_2 - z_2'|^\gamma, \text{ uniformly in } \omega.$$

Moreover,

$$\begin{aligned}
&|F(z_1, z_2) - F(z_1', z_2')| = |z_1 z_2 \psi(\omega, \theta) - z_1' z_2' \psi(\omega', \theta')| \\
&\leq |z_2'||z_1 \psi(\omega, \theta') - z_1' \psi(\omega', \theta')| + |z_1||z_2 \psi(\omega, \theta) - z_2' \psi(\omega, \theta')| \\
&\leq |z_1 \psi(\omega, \theta') - z_1' \psi(\omega', \theta')| + |z_2 \psi(\omega, \theta) - z_2' \psi(\omega, \theta')| \\
&\leq 2(C + M) \left(|z_1 - z_1'|^\gamma + |z_2 - z_2'|^\gamma\right) \leq 4(C + M) \left(|z_1 - z_1'| + |z_2 - z_2'|\right)^\gamma .
\end{aligned}$$

□

References

[1] J. Aaronson, *An introduction to infinite ergodic theory*, Mathematical Surveys and Monographs, 50, AMS, Providence, RI, 1997.

[2] J. Aaronson, M. Lemańczyk, C. Mauduit, H. Nakada, *Koksma's inequality and group extensions of Kronecker transformations*, in Algorithms, fractals, and dynamics (Okayama/Kyoto, 1992), 27–50, Plenum, New York, 1995.

[3] G. Atkinson, *A class of transitive cylinder transformations*, J. London Math. Soc. (2) 17 (1978), 263–270.

[4] J. Auslander, *Minimal flows and their extensions*, North-Holland Mathematics Studies, 153. Notas de Matemática, 122. North-Holland Publishing Co., Amsterdam, 1988.

[5] A.S. Besicovitch, *A problem on topological transformations of the plane. II*, Proc. Cambridge Philos. Soc. 47 (1951), 38–45.

[6] N.C. Bernardes, *On the set of points with a dense orbit*, Proc. Amer. Math. Soc. 128 (2000), 3421–3423.

[7] K. Falconer, *Fractal geometry. Mathematical foundations and applications*, John Wiley & Sons, Inc., Hoboken, NJ, 2003.

[8] W.H. Gottschalk, G.A. Hedlund, *Topological Dynamics*, Amer. Math. Soc. Colloq. Publ. 36, Amer. Math. Soc., Providence, RI, 1955.

[9] M.R. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Publ. Math. IHES **49** (1979), 5-234.

[10] A.Ya. Khinchin, *Continued fractions*, The University of Chicago Press, Chicago-London, 1964.

[11] L. Kuipers, H. Niederreiter, *Uniform distribution of sequences*, Pure and Applied Mathematics, Wiley-Interscience, New York-London-Sydney, 1974.

[12] J. Kwiatkowski, A. Siemaszko, *Discrete orbits for topologically transitive cylindrical transformations*, Discrete Contin. Dyn. Syst. 27 (2010), 945–961.

[13] P. Le Calvez, J-C. Yoccoz, *Un théorème d'indice pour les homéomorphismes du plan au voisinage d'un point fixe*, Annals Math. (2) 146 (1997), 241–293.

[14] M. Lemańczyk, M.K. Mentzen, *Topological ergodicity of real cocycles over minimal rotations*, Monatsh. Math. 134 (2002), 227–246.

[15] S. Matsumoto, M. Shishikura, *Minimal sets of certain annular homeomorphisms*, Hiroshima Math. J. 32 (2002), 207–215.

[16] M.K. Mentzen, A. Siemaszko, *Cylinder cocycle extensions of minimal rotations on monothetic groups*, Colloq. Math. 101 (2004), 75–88.

[17] H. Poincaré, *On curves defined by differential equations* (in Russian), Ogiz, Moscow, 1947.

[18] K. Schmidt, *Cocycles on ergodic transformation groups*, Macmillan Lectures in Mathematics, Vol. 1. Macmillan Company of India, Delhi, 1977.

[19] J.-Ch. Yoccoz, *Centralisateurs et conjugaison différentiable des difféomorphismes du cercle*, Petits diviseurs en dimension 1, Astérisque No. 231 (1995), 89–242.

K. Frączek, Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland

M. Lemańczyk, Institute of Mathematics of Polish Academy of Sciences, ul. Śniadeckich 8, 00-950 Warszawa, Poland and Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland

E-mail address: `fraczek@mat.umk.pl, mlem@mat.umk.pl`