

ON TYPICALITY OF TRANSLATION FLOWS WHICH ARE DISJOINT WITH THEIR INVERSE

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ABSTRACT. In this paper we prove that translation structures for which the corresponding vertical translation flow is weakly mixing and disjoint with its inverse, form a G_δ -dense set in every non-hyperelliptic connected component of the moduli space \mathcal{M} . This is in contrast to hyperelliptic case, where for every translation structure the associated vertical flow is isomorphic to its inverse. To prove the main result, we study limits of the off-diagonal 3-joinings of special representations of vertical translation flows. Moreover, we construct a locally defined continuous embedding of the moduli space into the space of measure-preserving flows to obtain the G_δ -condition.

1. INTRODUCTION

Let M be an orientable compact connected topological surface, and Σ be a finite set of singular points. On M we can consider a *translation structure* ζ , i.e. an atlas on $M \setminus \Sigma$ such that every transition transformation is a translation. Every translation surface can be viewed as a polygon with pairwise parallel sides of the same length which are glued together (gluing is made by a translation). Parameters given by the sides of such polygons establish a parametrization of the so-called *moduli space* \mathcal{M} and yield a topology on \mathcal{M} . To each translation structure ζ we associate the corresponding Lebesgue measure λ_ζ on M . Moreover, for every direction we consider the flow which acts by translation in that direction with unit speed. Such translation flows preserve λ_ζ . In this paper we are interested in vertical translation flows. It is worth to mention that the study of directional flows on translation surfaces originates from problems concerning billiard flows on rational polygons (see [7],[14]).

In [17] the authors give a complete characterisation of connected components of the moduli space; all of them are orbifolds. On each connected component C we consider an action of $SL_2(\mathbb{R})$ which is derived from the linear action of $SL_2(\mathbb{R})$ on polygons. Moreover, there is a Lebesgue measure ν_C on C which is invariant under this action. Let \mathcal{M}_1 be the set of $\zeta \in \mathcal{M}$ such that $\lambda_\zeta(M) = 1$ and let $C_1 := C \cap \mathcal{M}_1$. We also consider a measure ν_{C_1} on C_1 which is a projectivization of ν_C . This measure is finite and invariant under the action of $SL_2(\mathbb{R})$ on C_1 . In fact, the action of $SL_2(\mathbb{R})$ is ergodic with respect to this measure (see [18] and [22]). This gives an opportunity to use the ergodic theory to study dynamical properties of vertical flows on almost all translations structures. In particular, it was used to prove that the sets of translation structures for which the vertical translation flow is ergodic (see [18]), and further is weakly mixing (see [3]) are of full measure in both C and C_1 . At the same time, there are no mixing translation flows (see [13]). In this paper we are interested in translation structures for which the corresponding vertical translation flow is disjoint with its inverse, which is a stronger notion than being not reversible. Recall that a measure preserving flow $\{T_t\}_{t \in \mathbb{R}}$ on (X, μ) is reversible, if there exists an involution $\theta : X \rightarrow X$ which preserves μ and

$$\theta \circ T_{-t} = T_t \circ \theta \text{ for all } t \in \mathbb{R}.$$

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Our result concerns the topological typicality of the desired property rather than measure-theoretical. As a by-product, we give a method to show that the set of translation structures for which the associated vertical translation flows satisfy any property which is G_δ in the space of measure preserving flows, is also a G_δ -set. Among these properties are for instance weak mixing, ergodicity and rigidity (see [12]).

In the classification of connected components given in [17] we distinguish so called hyperelliptic components. For every hyperelliptic component C there exists an involution $\theta : M \rightarrow M$ such that for every $\zeta \in C$ it is given in local coordinates by the formula $z \mapsto -z + c$ for some $c \in \mathbb{C}$. In particular, the vertical flow on (M, ζ) is reversible; it is isomorphic with its inverse by the involution θ (see remark 2.8). In contrast, in this paper we show that on non-hyperelliptic components of the moduli space the set of translation structures for which the vertical flow is disjoint with its inverse is topologically large. It is expressed by the following theorem.

Theorem 1.1. *Let C be a non-hyperelliptic connected component of the moduli space of translation structures. Then the set of translation structures whose vertical flow is weakly mixing and disjoint with its inverse is a G_δ -dense set in C .*

It is also worth to mention that on non-hyperelliptic components we can also find a non-trivial set of translation structures for which the vertical flow is reversible.

Proposition 1.2. *Let C be a non-hyperelliptic connected component of the moduli space of translation structures. Then the set of translation structures whose vertical flow is reversible is dense in C .*

Recall that a measure-preserving flow $\{T_t\}_{t \in \mathbb{R}}$ on a standard Borel probability space (X, \mathcal{B}, μ) is disjoint with its inverse if the only $(T_t \times T_{-t})$ -invariant probability measure on $X \times X$, which projects on each coordinate as μ is the product measure $\mu \otimes \mu$. In [9] the authors developed techniques to prove non-isomorphism of a flow T^f to its inverse that are based on studying the weak closure of off-diagonal 3-self-joinings. Moreover, in [4] the authors improved those techniques to show that a large class of special flows over interval exchange transformations and under piecewise absolutely continuous functions have the property of being non-isomorphic with their inverse. The idea of detecting non-isomorphism of a dynamical system and its inverse by studying the weak closure of off-diagonal 3-self-joinings was introduced by Ryzhikov in [20]. In this paper we prove that techniques mentioned earlier can be used to detect disjointness of a vertical flow with its inverse.

To prove the G_δ condition, we use the result of Danilenko and Ryzhikov from [5] (which derives from a version for automorphisms given in [6]), where they proved that the flows with the property of being disjoint with their inverse form a G_δ -dense set in the space of measure preserving flows. To use their result we construct a locally defined continuous embedding of the moduli space into the space of measure preserving flows. To show the density condition, we largely rely on the proof of Lemma 14 in [8].

In Section 2 we give a general background concerning joinings, interval exchange transformations, space of measure preserving flows, translation flows and moduli spaces. In particular, we give some tools needed to prove the continuity of a map with values in the space of flows and we state some connections between the moduli space and interval exchange transformations.

In Section 3 we introduce a criterion of disjointness of two flows by researching the weak limits of certain 3-self-joinings. This is a direct improvement of the criteria stated in [9] and [4] as we show that these are actually criteria of two weakly mixing flows being disjoint. Furthermore, we state a criterion of a flow being weakly mixing, which also uses weak limits of 3-self-joinings as a tool.

In Section 4 we state combinatorial conditions on translation structures which are later used in proving the density condition in our main theorem. To be precise, we show that our results apply to every non-hyperelliptic component of the moduli space.

In Section 5 we show that on a given translation surface (M, ζ) there exists $\varepsilon_\zeta > 0$ such that for every absolutely continuous measure μ on M which has density f satisfying $\int_M |f(x) - 1| dx <$

ε_ζ there exists a homeomorphism $H : M \rightarrow M$ such that $H_*\mu$ is the Lebesgue measure with H depending continuously on f . The results of section 5 were inspired by the works of Moser in [19] and Goffman, Pedrick in [11].

In Section 6 we use the results presented in the previous section to construct a continuous embedding of each connected component of the moduli space into the space of measure-preserving flows. The embedding is defined locally, but we also show that this is enough to transfer the G_δ condition.

Finally, in Section 7 we first state a result which is a conclusion from the previous sections, that in every connected component of the moduli space the set of translation structures whose associated vertical translation flow is disjoint with its inverse, is a G_δ set. In the remainder of this section we use the results from [4] to show that the criteria introduced in Section 3 can be used for a dense set of translation structures in every non-hyperelliptic component, which leads to the proof of Theorem 1.1. As a by-product we get the proof of Proposition 1.2.

2. PRELIMINARIES

We will now give some details regarding interval exchange transformations, joinings of dynamical systems and some basic information about moduli space.

2.1. Joinings. In this subsection we give some definitions which are stated for standard Borel probability spaces. However these definitions can be easily extended to standard Borel spaces with finite measures. Though here we state definitions for probability spaces, in the remainder of this paper we will freely use them in case when the measure is finite and not necessarily a probability. In particular we say that the measure preserving flows $\mathcal{T} = \{T_t\}_{t \in \mathbb{R}}$ on (X, \mathcal{B}, μ) and $\mathcal{S} = \{S_t\}_{t \in \mathbb{R}}$ on (Y, \mathcal{C}, ν) are *isomorphic*, if there exists a measurable $F : (X, \mathcal{B}) \rightarrow (Y, \mathcal{C})$ such that

$$T_t = F^{-1} \circ S_t \circ F \text{ for } t \in \mathbb{R} \quad \text{and} \quad F_*\mu = \frac{\mu(X)}{\nu(Y)}\nu.$$

Let $K > 0$ be a natural number and for $1 \leq i \leq K$ let $\mathcal{T}^i = \{T_t^i\}_{t \in \mathbb{R}}$ be a measure preserving flow acting on a standard Borel probability space $(X^i, \mathcal{B}^i, \mu^i)$. We say that a measure λ on $(X^1 \times \dots \times X^K, \mathcal{B}^1 \otimes \dots \otimes \mathcal{B}^K)$ is a *K-joining* if it is $\mathcal{T}^1 \times \dots \times \mathcal{T}^K$ -invariant and it projects on X^i as μ^i for each $i = 1, \dots, K$. We denote by $J(\mathcal{T}^1, \dots, \mathcal{T}^K)$ the set of all joinings of \mathcal{T}^i for $i = 1, \dots, K$ and by $J^e(\mathcal{T}^1, \dots, \mathcal{T}^K)$ the subset of ergodic joinings. If for $i = 1, \dots, K$, $(X^i, \mathcal{B}^i, \mu^i, \mathcal{T}^i)$ are copies of the same flow, then we say that λ is a *K-self joining*. We denote the set of *K-self joinings* of a flow \mathcal{T} by $J_K(\mathcal{T})$ and ergodic *K-self joinings* by $J_K^e(\mathcal{T})$. If $\mathcal{T}^1, \dots, \mathcal{T}^K$ are ergodic, then the following remarks hold.

Remark 2.1. $J(\mathcal{T}^1, \dots, \mathcal{T}^K)$ is a compact simplex in the space of all $\mathcal{T}^1 \times \dots \times \mathcal{T}^K$ -invariant measures and its set of extremal points ($J(\mathcal{T}^1, \dots, \mathcal{T}^K)$) equals $J^e(\mathcal{T}^1, \dots, \mathcal{T}^K)$.

Remark 2.2 (Ergodic decomposition). For each $\lambda \in J(\mathcal{T}^1, \dots, \mathcal{T}^K)$ there exists a unique probability measure κ on $J^e(\mathcal{T}^1, \dots, \mathcal{T}^K)$ such that

$$\lambda = \int_{J^e(\mathcal{T}^1, \dots, \mathcal{T}^K)} \rho d\kappa(\rho).$$

Assume that $K = 2$. Note that $\mu^1 \otimes \mu^2 \in J(\mathcal{T}^1, \mathcal{T}^2)$. We say that the flows \mathcal{T}^1 and \mathcal{T}^2 are *disjoint in the sense of Furstenberg* (or simply disjoint) if the product measure is the only joining between them.

Remark 2.3. If two flows are disjoint, then they have no common factor. In particular, they are not isomorphic.

The notions of joinings and disjointness can be rewritten for automorphisms instead of flows. Then we also have the following well-known observation.

Remark 2.4. If (X, \mathcal{B}, μ, T) is an ergodic automorphism and $(Y, \mathcal{C}, \nu, Id)$ is the identity then T and Id are disjoint.

Let $\phi : (X^1, \mathcal{B}^1, \mu^1, \mathcal{T}^1) \rightarrow (X^2, \mathcal{B}^2, \mu^2, \mathcal{T}^2)$ be an isomorphism. It is easy to see that $\mu_\phi^1 := (Id \times \phi)_* \mu^1$ is a joining of \mathcal{T}^1 and \mathcal{T}^2 . We say that μ_ϕ^1 is a *graph joining*. We have the following remark.

Remark 2.5. Let $\lambda \in J(\mathcal{T}^1, \mathcal{T}^2)$ and let $\Pi \subseteq \mathcal{B}^1$ be a family of measurable sets. Let $\phi : (X^1, \mathcal{B}^1, \mu^1, \mathcal{T}^1) \rightarrow (X^2, \mathcal{B}^2, \mu^2, \mathcal{T}^2)$ be an isomorphism. Then the following are equivalent:

- (1) $\lambda(A \times B) = \mu^1(A \cap \phi^{-1}(B))$ for all $A \in \Pi$ and $B \in \phi(\Pi)$;
- (2) $\lambda(A \times X \triangle X \times \phi A) = 0$ for every $A \in \Pi$;
- (3) $\lambda(A \times \phi A^c) = \lambda(A^c \times \phi A) = 0$ for every $A \in \Pi$.

Consider graph joinings between two identical flows $(X, \mathcal{B}, \mu, \mathcal{T})$. If $\phi = T_{-t}$ for some $t \in \mathbb{R}$ then we say that μ_ϕ is a *2-off-diagonal joining* and we denote it by μ_t . In other words for $A, B \in \mathcal{B}$ we have

$$\mu_t(A \times B) = \mu(A \cap T_t B) = \mu(T_{-t} A \cap B).$$

This definition is easily extended to higher dimensions, namely a *K-off-diagonal joining* $\mu_{t_1, \dots, t_{K-1}}$ is a *K-joining* given by the formula

$$(1) \quad \mu_{t_1, \dots, t_{K-1}}(A_1 \times \dots \times A_K) = \mu(T_{-t_1} A_1 \cap \dots \cap T_{-t_{K-1}} A_{K-1} \cap A_K),$$

for all $A_1, \dots, A_K \in \mathcal{B}$.

Let $\mathcal{P}(\mathbb{R}^{K-1})$ be the set of Borel probability measures on \mathbb{R}^{K-1} . For every $P \in \mathcal{P}(\mathbb{R}^{K-1})$ we consider the *K-integral joining* given by

$$\begin{aligned} & \left(\int_{\mathbb{R}^{K-1}} \mu_{t_1, \dots, t_{K-1}} dP(t_1, \dots, t_{K-1}) \right) (A_1 \times \dots \times A_K) \\ & := \int_{\mathbb{R}^{K-1}} \mu_{t_1, \dots, t_{K-1}}(A_1 \times \dots \times A_K) dP(t_1, \dots, t_{K-1}), \end{aligned}$$

where $A_1, \dots, A_K \in \mathcal{B}$.

A *Markov operator* $\Psi : L^2(X, \mathcal{B}, \mu) \rightarrow L^2(X, \mathcal{B}, \mu)$ is a linear operator which satisfies

- (1) it is a contraction, that is $\|\Psi f\|_2 \leq \|f\|_2$ for every $f \in L^2(X, \mathcal{B}, \mu)$;
- (2) $f \geq 0 \Rightarrow \Psi(f) \geq 0$;
- (3) $\Psi(\mathbf{1}) = \mathbf{1} = \Psi^*(\mathbf{1})$.

With every 2-self joining $\lambda \in J_2(\mathcal{T})$, we can associate a Markov operator $\Psi(\lambda) : L^2(X, \mathcal{B}, \mu) \rightarrow L^2(X, \mathcal{B}, \mu)$ such that

$$(2) \quad \lambda(A \times B) = \int_X \Psi(\lambda)(\chi_A) \chi_B d\mu \text{ for any } A, B \in \mathcal{B}.$$

Denote by $\mathcal{J}(\mathcal{T})$ the set of all Markov operators which commute with the Koopman operator associated with \mathcal{T} . It appears that if we consider weak-* topology on $\mathcal{J}(\mathcal{T})$, then (2) defines an affine homeomorphism $\Psi : J_2(\mathcal{T}) \rightarrow \mathcal{J}(\mathcal{T})$. For more information about joinings and Markov operators we refer to [10].

Consider the affine continuous map $\Pi_{1,3} : J_3(\mathcal{T}) \rightarrow J_2(\mathcal{T})$ given by

$$(3) \quad \Pi_{1,3}(\lambda)(A \times B) := \lambda(A \times X \times B) \text{ for any } A, B \in \mathcal{B}.$$

In other words $\Pi_{1,3}(\lambda)$ is the projection of the joining λ on the first and third coordinates. Analogously, we define $\Pi_{2,3}$, the projection on the second and third coordinates. Since $J_2(\mathcal{T})$ and $\mathcal{J}(\mathcal{T})$ are affinely homeomorphic, we can consider the affine continuous maps $\Psi \circ \Pi_{i,3} : J_3(\mathcal{T}) \rightarrow \mathcal{J}(\mathcal{T})$ for $i = 1, 2$. Note that for any $t, s \in \mathbb{R}$ we have

$$(4) \quad \Psi \circ \Pi_{1,3}(\mu_{t,s}) = T_{-t}, \text{ and } \Psi \circ \Pi_{2,3}(\mu_{t,s}) = T_{-s}.$$

For $i \in \{1, 2\}$ let $\sigma_i : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the projection on the i -th coordinate. Then for every $P \in \mathcal{P}(\mathbb{R}^2)$, we also have

$$(5) \quad \begin{aligned} \Pi_{i,3} \left(\int_{\mathbb{R}^2} \mu_{-t,-s} dP(t,s) \right) &= \int_{\mathbb{R}^2} \Pi_{i,3}(\mu_{-t,-s}) dP(t,s) \text{ and} \\ \Psi \circ \Pi_{i,3} \left(\int_{\mathbb{R}^2} \mu_{-t,-s} dP(t,s) \right) &= \int_{\mathbb{R}} T_t d((\sigma_i)_* P)(t), \end{aligned}$$

for $i = 1, 2$.

2.2. Special flows. Let (X, \mathcal{B}, μ) be a standard Borel probability space. Let $T : X \rightarrow X$ be an ergodic μ -preserving automorphism. Let $f \in L^1([0, 1])$ be positive and for any $n \in \mathbb{Z}$ consider

$$f^{(n)}(x) := \begin{cases} \sum_{i=0}^{n-1} f(T^i x) & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \\ -\sum_{i=n}^{-1} f(T^i x) & \text{if } n \leq -1. \end{cases}$$

Define $X^f := ((x, r); x \in X, 0 \leq r < f(x))$ and on X^f consider the measure $\mu \otimes \text{Leb}|_{X^f}$. The special flow $T^f = \{T_t^f\}_{t \in \mathbb{R}}$ is the measure preserving flow acting on X^f by the formula

$$T_t^f(x, r) := (T^n x, r + t - f^{(n)}(x)),$$

where $n \in \mathbb{Z}$ is unique, such that $f^{(n)}(x) \leq r + t < f^{(n+1)}(x)$. We say that f is the *roof function* and T is the *base* of the special flow. In view of Ambrose Representation Theorem (see [1]), every ergodic flow is measure theoretically isomorphic to a special flow. Such a special flow is called a *special representation* of the flow. In this paper we deal with special flows whose roof functions are piecewise continuous and whose bases are interval exchange transformations. We always assume that roof functions are right-continuous and that the left limits exist. If a piecewise continuous bounded function f has a discontinuity at x , then the *jump at x* is the number $d := f(x) - \lim_{y \rightarrow x^-} f(y)$.

2.3. Space of flows. Let $(X, \mathcal{B}(X), \mu)$ be a standard Borel probability space. By $\text{Flow}(X)$ we denote the set of all measure preserving flows on X . Let $\mathcal{T} = \{T_t\}_{t \in \mathbb{R}} \in \text{Flow}(X)$, $A \in \mathcal{B}(X)$ and $\varepsilon > 0$. Let

$$U(\mathcal{T}, A, \varepsilon) := \{S = \{S_t\}_{t \in \mathbb{R}} \in \text{Flow}(X); \sup_{t \in [-1, 1]} \mu(T_t A \Delta S_t A) < \varepsilon\}.$$

It appears that the family of sets of the above form gives a subbase of a topology, and $\text{Flow}(X)$ endowed with this topology is a Polish space.

Let (Y, d) be a metric space. It follows that a map $F : Y \rightarrow \text{Flow}(X)$ is continuous if for any $y \in Y$ and $A \in \mathcal{B}(X)$ we have

$$(6) \quad \text{for any } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that } d(y, z) < \delta \Rightarrow F(z) \in U(F(y), A, \varepsilon).$$

By using the fact that for any $A_1, B_1, A_2, B_2 \in \mathcal{B}(X)$ we have

$$A_1 \Delta B_1 = A_1^c \Delta B_1^c \text{ and } (A_1 \cup B_1) \Delta (A_2 \cup B_2) \subseteq (A_1 \Delta A_2) \cup (B_1 \Delta B_2),$$

we can prove that the set of all $A \in \mathcal{B}(X)$, for which for every $\varepsilon > 0$ there exists δ_A such that (6) is satisfied, form an algebra. By using the triangle inequality

$$\mu(A \Delta B) \leq \mu(A \Delta C) + \mu(B \Delta C) \text{ for } A, B, C \in \mathcal{B}(X),$$

we can prove that this algebra is closed under taking the countable union of increasing family of sets and thus, it is a σ -algebra. Hence it is enough to check (6) for a family of sets which generates $\mathcal{B}(X)$.

All non-atomic standard Borel probability spaces are measure theoretically isomorphic (see Theorem 3.4.23 in [21]). Let $(X_1, \mathcal{B}(X_1), \mu_1)$ and $(X_2, \mathcal{B}(X_2), \mu_2)$ be standard Borel non-atomic

probability spaces and let $H : X_1 \rightarrow X_2$ be some isomorphism. Then $\text{Flow}(X_1)$ and $\text{Flow}(X_2)$ can be identified by a homeomorphism $\phi : \text{Flow}(X_1) \rightarrow \text{Flow}(X_2)$ given by the formula

$$\phi(\mathcal{T}) := H \circ \mathcal{T} \circ H^{-1}.$$

Remark 2.6. To prove that $F : (Y, d) \rightarrow \text{Flow}(X_1)$ is continuous, we can instead prove that $\phi \circ F : (Y, d) \rightarrow \text{Flow}(X_2)$ is continuous. In other words, we need to prove that for every $y \in Y$ and $A \in \mathcal{D} \subset \mathcal{B}(X_2)$, where \mathcal{D} generates $\mathcal{B}(X_2)$, we have

for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d(y, z) < \delta \Rightarrow \phi \circ F(z) \in U(\phi \circ F(y), A, \varepsilon)$.

2.4. Interval exchange transformations. Let \mathcal{A} be an alphabet of d elements. Let now $\epsilon \in \{0, 1\}$ and let $\pi_\epsilon : \mathcal{A} \rightarrow \{1, \dots, d\}$ be bijections. We will now consider a permutation π as a pair $\{\pi_0, \pi_1\}$ where $\pi_0(\alpha)$ corresponds to the position of letter α before permutation, while $\pi_1(\alpha)$ defines the position of α after permutation. We say that a permutation π is *irreducible* if there is no $1 \leq k < d$ such that

$$\pi_1 \circ \pi_0^{-1}(\{1, \dots, k\}) = \{1, \dots, k\}.$$

In this paper we will only deal with irreducible permutations, so this assumption will usually be omitted. We say that the permutation is *symmetric* if

$$\pi_1(\alpha) = d + 1 - \pi_0(\alpha) \text{ for every } \alpha \in \mathcal{A}.$$

Note that a symmetric permutation is always irreducible.

The intervals that we will now consider are always left-side closed and right-side open unless told otherwise. Let I be an interval equipped with its Borel σ -algebra and Lebesgue measure Leb . Without losing generality, we can assume that the left endpoint of I is 0. Let $\{I_\alpha\}_{\alpha \in \mathcal{A}}$ be a partition of I into d intervals, where I_α has length $\lambda_\alpha \geq 0$. We will denote $\lambda := \{\lambda_\alpha\}_{\alpha \in \mathcal{A}}$ the *length vector* and obviously we have $|\lambda| := \sum_{\alpha \in \mathcal{A}} \lambda_\alpha = \text{Leb}(I)$. The *interval exchange transformation (IET)* $T_{\pi, \lambda} : I \rightarrow I$ is the automorphism which permutes intervals I_α according to the permutation π . Let now $\Omega_\pi := [(\Omega_\pi)_{\alpha\beta}]_{\alpha, \beta \in \mathcal{A}}$ be the $d \times d$ matrix given by the following formula

$$(7) \quad (\Omega_\pi)_{\alpha\beta} := \begin{cases} +1 & \text{if } \pi_0(\alpha) < \pi_0(\beta) \text{ and } \pi_1(\alpha) > \pi_1(\beta); \\ -1 & \text{if } \pi_0(\alpha) > \pi_0(\beta) \text{ and } \pi_1(\alpha) < \pi_1(\beta); \\ 0 & \text{otherwise.} \end{cases}$$

We will say that Ω_π is the *translation matrix* of $T_{\pi, \lambda}$. The name of the matrix is derived from the fact that $T_{\pi, \lambda}$ acts on an interval I_α as a translation by number $\sum_{\beta \in \mathcal{A}} (\Omega_\pi)_{\alpha\beta} \lambda_\beta$.

Let ∂I_α be the left endpoint of I_α . We say that the IET $T_{\pi, \lambda}$ satisfies *Keane's condition* if

$$T_{\pi, \lambda}^m(\partial I_\alpha) = \partial I_\beta \text{ for } m > 0 \text{ implies } \alpha = \pi_1^{-1}(1), \beta = \pi_0^{-1}(1) \text{ and } m = 1.$$

It is easy to see that $T_{\pi, \lambda}$ satisfies Keane's condition whenever λ is a rationally independent vector (that is there is no nontrivial integer linear combination of numbers λ_α , which will give a rational number).

Denote by $S_{\mathcal{A}}^0$ the set of all irreducible permutations of \mathcal{A} . We may consider the space $S_{\mathcal{A}}^0 \times \mathbb{R}_{\geq 0}^{\mathcal{A}}$ of all IETs of d intervals. Define the operator $R : S_{\mathcal{A}}^0 \times \mathbb{R}_{\geq 0}^{\mathcal{A}} \rightarrow S_{\mathcal{A}}^0 \times \mathbb{R}_{\geq 0}^{\mathcal{A}}$, such that $R(\pi, \lambda) = R(T_{\pi, \lambda})$ is the first return map of $T_{\pi, \lambda}$ to the interval $[0, |\lambda| - \min\{\lambda_{\pi_0^{-1}(d)}, \lambda_{\pi_1^{-1}(d)}\}]$. The operator R is called the *Rauzy-Veech induction* (or *righthand side Rauzy-Veech induction*). The *Rauzy classes* are the minimal subsets of $S_{\mathcal{A}}^0$ which are invariant under the induced action of R on $S_{\mathcal{A}}^0$.

Let

$$l : \{1, \dots, d\} \rightarrow \{1, \dots, d\} \text{ be given by } l(i) = d + 1 - i.$$

The function l acts on $S_{\mathcal{A}}^0$ by mapping $\{\pi_0, \pi_1\}$ onto $\{l \circ \pi_0, l \circ \pi_1\}$. The *extended Rauzy classes* are minimal subsets of $S_{\mathcal{A}}^0$ which are invariant under R and action of l . We have the following result.

Theorem 2.7 (Rauzy). *Any Rauzy class of permutations of $d \geq 2$ elements contains at least one permutation π such that*

$$\pi_1 \circ \pi_0^{-1}(1) = d \quad \text{and} \quad \pi_1 \circ \pi_0^{-1}(d) = 1.$$

2.5. Translation surfaces and moduli space. Let M be an orientable compact and connected topological surface of genus $g \geq 1$. Let $\Sigma := \{A_1, \dots, A_s\}$ be a finite subset of singular points in M . Let $\kappa := (\kappa_1, \dots, \kappa_s)$ be a vector of positive integers satisfying $\sum_{i=1}^s \kappa_i = 2g - 2$. A *translation structure* on M is a maximal atlas ζ on $M \setminus \Sigma$ of charts by open sets of \mathbb{C} such that any coordinate change between charts is a translation and for each $1 \leq i \leq s$ there exists a neighbourhood $V_i \subset M$ of A_i , a neighbourhood $W_i \subset \mathbb{C}$ of 0 and a ramified covering $\pi : (V_i, A_i) \rightarrow (W_i, 0)$ of degree $\kappa_i + 1$ such that each injective restriction of π is a chart of ζ . On (M, ζ) we can consider a holomorphic 1-form which in the local coordinates can be written as dz . We will denote this form also by ζ . We identify the associated 2-form $\frac{i}{2}\zeta \wedge \bar{\zeta}$ with the Lebesgue measure λ_ζ on M . Moreover, the quadratic form $|\zeta|^2$ yields a Riemannian metric (M, ζ) . By d_ζ we denote the distance given by this metric. We also consider on (M, ζ) a vertical translation flow, denoted by $\mathcal{T}^\zeta = \{\mathcal{T}_t^\zeta\}_{t \in \mathbb{R}}$, which in local coordinates is a unit speed flow in the vertical direction. The flow \mathcal{T}^ζ preserves λ_ζ and thus can be viewed as an element of $\text{Flow}(M, \lambda_\zeta)$.

In the set of all translation structures on M we identify the structures ζ_1 and ζ_2 if there exists a homeomorphism $H : M \rightarrow M$ which sends singular points of ζ_1 onto singular points of ζ_2 and $\zeta_1 = H^*\zeta_2$. In terms of local coordinates, H is locally a translation. This is an equivalence relation and its equivalence classes form the *moduli space* denoted by $\text{Mod}(M)$. The moduli space can be divided into subsets called *strata* $\mathcal{M}(M, \Sigma, \kappa) = \mathcal{M}(\kappa)$, for which the vector of degrees of singularities is given by κ . It appears that each such stratum $\mathcal{M}(M, \Sigma, \kappa)$ is a complex orbifold (see [22]) and has a finite number of connected components (see [17]). On \mathcal{M} we can consider an action of $SL(2, \mathbb{R})$. It is given by composing the charts of a translation surface with a given linear map. The strata are invariant under the action of $SL(2, \mathbb{R})$. It is worth noting that in particular for every $\theta \in \mathbb{R}/\mathbb{Z}$ we can apply the rotation r_θ by θ to the translation structure and almost every angle θ yields no saddle connection of a vertical flow.

Let $\pi = (\pi_0, \pi_1)$ be a permutation of the alphabet \mathcal{A} of $d > 1$ elements and let $\lambda \in \mathbb{R}_{\geq 0}^{\mathcal{A}}$. Consider also $\tau \in \mathbb{R}^{\mathcal{A}}$ such that for each $1 \leq k < d$ we have

$$\sum_{\{\alpha; \pi_0(\alpha) \leq k\}} \tau_\alpha > 0 \quad \text{and} \quad \sum_{\{\alpha; \pi_1(\alpha) \leq k\}} \tau_\alpha < 0.$$

Moreover we require that

$$\pi_i(\alpha) = \pi_i(\beta) + 1 \wedge \lambda_\alpha = \lambda_\beta = 0 \Rightarrow \tau_\alpha \cdot \tau_\beta > 0 \quad \text{for all } i = 0, 1 \text{ and } \alpha, \beta \in \mathcal{A}.$$

For a fixed permutation π , we denote by Θ_π the set of triples (π, λ, τ) satisfying the above conditions.

Consider the polygonal curve in \mathbb{C} obtained by connecting the points 0 and $\sum_{i \leq k} (\lambda_{\pi_0^{-1}(i)} + i\tau_{\pi_0^{-1}(i)})$ for $k = 1, \dots, d$, using the line segments. Analogously we can consider the polygonal curve obtained by connecting the points 0 and $\sum_{i \leq k} (\lambda_{\pi_1^{-1}(i)} + i\tau_{\pi_1^{-1}(i)})$ for $k = 1, \dots, d$. These two polygonal curves define a polygon with d pairs of parallel sides. By identifying those sides we obtain a translation surface M , with Σ being the set of vertices of the polygon (some of them may be identified). We denote by $M(\pi, \lambda, \tau)$ the translation structure given by (π, λ, τ) .

It appears that whenever \mathcal{T}^ζ admits no saddle-connections, ζ can be viewed as $M(\pi, \lambda, \tau)$ for some $(\pi, \lambda, \tau) \in \Theta_\pi$, with π being some permutation (see *e.g.* [23]). Moreover we can consider $(\pi, \lambda, \tau) \in \Theta_\pi$ as local coordinates in the neighbourhood of such ζ in the corresponding stratum. Since almost every rotation yields no saddle-connections, to obtain local coordinates in the neighbourhood of ζ for which \mathcal{T}^ζ has a saddle connection, we can use the rotation to obtain local coordinates around rotated form and then rotate it back.

Kontsevich and Zorich in [17] gave a complete characterization of connected components of strata in the moduli space. In particular, they showed that each stratum $\mathcal{M}(2g - 2)$ and

$\mathcal{M}(g-1, g-1)$, where g is the genus of the surface, contains exactly one so-called *hyperelliptic* connected component, which we denote by $\mathcal{M}^{hyp}(2g-2)$ and $\mathcal{M}^{hyp}(g-1, g-1)$ respectively. For every hyperelliptic component $C \subset \mathcal{M}$, there exists an involution $\phi : M \rightarrow M$ such that for every $\zeta \in C$ we have $\phi^*\zeta = -\zeta$. In particular we have the following remark.

Remark 2.8. For every hyperelliptic connected component $C \subset \mathcal{M}$ and for every $\zeta \in C$, the vertical flow on (M, ζ) is isomorphic with its inverse.

It appears that the connected components of the moduli space can be described by the Rauzy classes of permutations. Let us recall first the notion of non-degeneracy, as introduced by Veech. We say that a permutation $\pi = \{\pi_0, \pi_1\}$ of \mathcal{A} is *degenerate* if one of the following conditions is satisfied:

$$(8) \quad \pi_1 \circ \pi_0^{-1}(j+1) = \pi_1 \circ \pi_0^{-1}(j) + 1 \text{ for some } 1 \leq j < d;$$

$$(9) \quad \pi_1 \circ \pi_0^{-1}(\pi_0 \circ \pi_1^{-1}(d) + 1) = \pi_1 \circ \pi_0^{-1}(d) + 1$$

$$(10) \quad \pi_0 \circ \pi_1^{-1}(1) - 1 = \pi_0 \circ \pi_1^{-1}(\pi_1 \circ \pi_0^{-1}(1) - 1)$$

$$(11) \quad \pi_0 \circ \pi_1^{-1}(d) = \pi_0 \circ \pi_1^{-1}(1) - 1 \text{ and } \pi_1 \circ \pi_0^{-1}(d) = \pi_1 \circ \pi_0^{-1}(1) - 1.$$

Otherwise the permutation is called *non-degenerate*. The property of non-degeneracy is invariant under the action of the Rauzy-Veech induction. The importance of this notion is given by the following theorem.

Theorem 2.9 (Veech). *The extended Rauzy classes of nondegenerate permutations are in one-to-one correspondence with the connected components of the strata in the moduli space.*

In view of the above theorem, for each genus $g \geq 2$, the hyperelliptic components $\mathcal{M}^{hyp}(2g-2)$ and $\mathcal{M}^{hyp}(g-1, g-1)$ correspond to the extended Rauzy classes of symmetric permutations of $2g$ and $2g-1$ elements respectively.

Remark 2.10. Connected components which are associated with extended Rauzy graphs of permutations of $d \leq 5$ elements are hyperelliptic.

For a given extended Rauzy class \mathcal{R} , let $C_{\mathcal{R}}$ be its associated connected component of the moduli space. Then for any $\pi \in \mathcal{R}$ the map $M : \Theta_{\pi} \rightarrow C_{\mathcal{R}}$ given by $(\pi, \lambda, \tau) \mapsto M(\pi, \lambda, \tau)$ is continuous and the range of the map M is dense in $C_{\mathcal{R}}$. Moreover, recall that, due to Theorem 2.7, for every connected component of the moduli space we can find a permutation $\bar{\pi}$ belonging to the corresponding extended Rauzy class, satisfying $\bar{\pi}_1 \circ \bar{\pi}_0^{-1}(d) = 1$ and $\bar{\pi}_0 \circ \bar{\pi}_1^{-1}(d) = 1$. Hence, to prove that some condition is satisfied for a dense set of translation structures in $C_{\mathcal{R}}$, it is enough to prove that it holds for translation structures, whose associated polygonal parameters belong to a dense subset of $\Theta_{\bar{\pi}}$.

Let \mathcal{R} be any extended Rauzy class. Let us consider a transformation $\tilde{R} : \bigcup_{\pi \in \mathcal{R}} \Theta_{\pi} \mapsto \bigcup_{\pi \in \mathcal{R}} \Theta_{\pi}$ called a *polygonal Rauzy Veech induction* (or *righthand side polygonal Rauzy Veech induction*) which yields different parameters of a translation surface.

Let $\pi \in \mathcal{R}$ and let $(\pi, \lambda, \tau) \in \Theta_{\pi}$. Assume that $\lambda_{\pi_0^{-1}(d)} \neq \lambda_{\pi_1^{-1}(d)}$. If $\lambda_{\pi_0^{-1}(d)} < \lambda_{\pi_1^{-1}(d)}$, then for any $a \in \mathcal{A}$ define

$$\tilde{\pi}_0(a) := \begin{cases} \pi_0(a) & \text{if } \pi_0(a) \leq \pi_0(\pi_1^{-1}(d)); \\ \pi_0(\pi_1^{-1}(d)) + 1 & \text{if } \pi_0(a) = d; \\ \pi_0(a) + 1 & \text{if } \pi_0(\pi_1^{-1}(d)) < \pi_0(a) \leq d-1, \end{cases}$$

$$\tilde{\pi}_1(a) := \pi_1(a),$$

$$\tilde{\lambda}_a := \begin{cases} \lambda_{\pi_1^{-1}(d)} - \lambda_{\pi_0^{-1}(d)} & \text{if } \pi_1(a) = d; \\ \lambda_a & \text{otherwise,} \end{cases}$$

$$\tilde{\tau}_a := \begin{cases} \tau_{\pi_1^{-1}(d)} - \tau_{\pi_0^{-1}(d)} & \text{if } \pi_1(a) = d; \\ \tau_a & \text{otherwise.} \end{cases}$$

Analogously, if $\lambda_{\pi_0^{-1}(d)} > \lambda_{\pi_1^{-1}(d)}$, we define

$$\begin{aligned}\tilde{\pi}_0(a) &:= \pi_0(a), \\ \tilde{\pi}_1(a) &:= \begin{cases} \pi_1(a) & \text{if } \pi_1(a) \leq \pi_1(\pi_0^{-1}(d)); \\ \pi_1(\pi_0^{-1}(d)) + 1 & \text{if } \pi_1(a) = d; \\ \pi_1(a) + 1 & \text{if } \pi_1(\pi_0^{-1}(d)) < \pi_1(a) \leq d - 1, \end{cases} \\ \tilde{\lambda}_a &:= \begin{cases} \lambda_{\pi_0^{-1}(d)} - \lambda_{\pi_1^{-1}(d)} & \text{if } \pi_0(a) = d; \\ \lambda_a & \text{otherwise,} \end{cases} \\ \tilde{\tau}_a &:= \begin{cases} \tau_{\pi_0^{-1}(d)} - \tau_{\pi_1^{-1}(d)} & \text{if } \pi_0(a) = d; \\ \tau_a & \text{otherwise.} \end{cases}\end{aligned}$$

We define \tilde{R} by setting $\tilde{R}(\pi, \lambda, \tau) := (\tilde{\pi}, \tilde{\lambda}, \tilde{\tau})$. It is defined almost everywhere on $\bigcup_{\pi \in \mathcal{R}} \Theta_\pi$ and if $M(\pi, \lambda, \tau)$ admits no saddle connection, it can be iterated indefinitely. Similarly, we can also define a left hand side polygonal Rauzy Veech induction. Note that the polygons derived from (π, λ, τ) and $(\tilde{\pi}, \tilde{\lambda}, \tilde{\tau})$ represent the same translation surface, i.e. $M(\pi, \lambda, \tau) = M(\tilde{\pi}, \tilde{\lambda}, \tilde{\tau})$. Indeed, the latter is obtained from (π, λ, τ) by cutting out the triangle formed by the last top side and the last bottom side and gluing it to a side of a polygon which is identified with one of the two sides forming the triangle.

Every $\zeta \in C_\pi$ which does not have vertical saddle-connections can be represented as $M(\pi, \lambda, \tau)$, for some $(\pi, \lambda, \tau) \in \Theta_\pi$. We can consider the metric on the neighbourhood of $M(\pi, \lambda, \tau)$ on $\mathcal{M}(M, \Sigma, \kappa)$ given by

$$d_{Mod}((\pi, \lambda', \tau'), (\pi, \lambda'', \tau'')) := \sum_{a \in \mathcal{A}} (|\lambda'_a - \lambda''_a| + |\tau'_a - \tau''_a|).$$

If ζ admits vertical saddle-connections, we can apply r_θ for some $\theta \in \mathbb{R}/\mathbb{Z}$, so that $r_\theta^* \zeta$ does not have vertical saddle-connections and then define a metric in the neighbourhood of ζ .

For any $\zeta = M(\pi, \lambda, \tau) \in \mathcal{M}(M, \Sigma, \kappa)$ we can consider a special representation of the vertical flow on (M, ζ) . The basis of this special flow is the IET $T_{\pi, \lambda}$ and the roof function h is positive and constant over exchanged intervals. Hence h can be considered as a vector $(h_a)_{a \in \mathcal{A}} \in \mathbb{R}_{>0}^{\mathcal{A}}$, where h_a is the value of h over the exchanged interval labelled by a . The vector h is given by the formula

$$(12) \quad h = -\Omega_\pi \tau,$$

where Ω_π is the translation matrix of (π, λ) . This gives rise to new local coordinates of the moduli space. In particular, the polygonal Rauzy-Veech induction receives a new form. Namely, if $\lambda_{\pi_0^{-1}(d)} \neq \lambda_{\pi_1^{-1}(d)}$ then $\tilde{R}(\pi, \lambda, h) = (\tilde{\pi}, \tilde{\lambda}, \tilde{h})$, where the formulas for $\tilde{\pi}$ and $\tilde{\lambda}$ remain unchanged and for any $a \in \mathcal{A}$ we take

$$\tilde{h}_a := \begin{cases} h_{\pi_0^{-1}(d)} + h_{\pi_1^{-1}(d)} & \text{if } a = \pi_0^{-1}(d) \text{ and } \lambda_{\pi_0^{-1}(d)} < \lambda_{\pi_1^{-1}(d)}; \\ h_{\pi_0^{-1}(d)} + h_{\pi_1^{-1}(d)} & \text{if } a = \pi_1^{-1}(d) \text{ and } \lambda_{\pi_0^{-1}(d)} > \lambda_{\pi_1^{-1}(d)}; \\ h_a & \text{otherwise.} \end{cases}$$

3. CONSEQUENCES OF LIMIT JOININGS

In this section, we formulate a criterion for two flows to be disjoint, and a criterion for a flow to be weakly mixing. Both criteria rely on the properties of the weak limit of some sequence of 3-off diagonal joinings.

For every measure $\lambda \in \mathcal{P}(X \times Y)$, we denote by $\lambda|_X$ and $\lambda|_Y$ the projections of λ on X and Y respectively, that is for every measurable subsets $A \subseteq X$ and $B \subseteq Y$ we have

$$\lambda|_X(A) = \lambda(A \times Y) \quad \text{and} \quad \lambda|_Y(B) = \lambda(X \times B).$$

Let $\mathcal{T} = \{T_t\}_{t \in \mathbb{R}}$ and $\mathcal{S} = \{S_t\}_{t \in \mathbb{R}}$ be weakly mixing flows acting on standard Borel spaces (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) respectively.

Lemma 3.1. *Let $\lambda \in J^e(\mathcal{T}, \mathcal{S})$. Let $\rho \in J_2^e(\mathcal{T} \times \mathcal{S}, \lambda)$, which is defined on $X_1 \times Y_1 \times X_2 \times Y_2$ with $X_1 = X_2 = X$ and $Y_1 = Y_2 = Y$. Assume that for some $r, r' \in \mathbb{R}$ we have $\rho|_{X_1 \times X_2} = \mu_{T_r}$ and $\rho|_{Y_1 \times Y_2} = \nu_{S_{r'}}$. If $r \neq r'$ then $\lambda = \mu \otimes \nu$.*

Proof. First we prove that $\lambda = (T_r \times S_{r'})_* \lambda$. We show that (3) in Remark 2.5 is satisfied for the π -system of product sets and the isomorphism $\phi := T_{-r} \times S_{-r'}$ between $(X_1 \times Y_1, \lambda)$ and $(X_2 \times Y_2, \lambda)$. In other words, for every $A \in \mathcal{B}$ and $B \in \mathcal{C}$ we have

$$\rho(A \times B \times (T_{-r} \times S_{-r'})(A \times B)^c) = \rho((A \times B)^c \times (T_{-r} \times S_{-r'})(A \times B)) = 0.$$

Indeed, recall that μ_r and $\nu_{r'}$ are graph joinings of \mathcal{T} and \mathcal{S} given by T_{-r} and $S_{-r'}$ respectively. By Remark 2.5 this implies that for every $A \in \mathcal{B}$ and $B \in \mathcal{C}$ we have

$$\mu_r(A \times T_{-r}A^c) = 0 \quad \text{and} \quad \nu_{r'}(B \times T_{-r}B^c) = 0.$$

Thus we obtain

$$\begin{aligned} \rho(A \times B \times (T_{-r} \times S_{-r'})(A \times B)^c) &= \rho(A \times B \times T_{-r}A^c \times S_{-r'}B) \\ &+ \rho(A \times B \times T_{-r}A^c \times S_{-r'}B^c) + \rho(A \times B \times T_{-r}A \times S_{-r'}B^c) \\ &\leq 2\rho(A \times Y \times T_{-r}A^c \times Y) + \rho(X \times B \times X \times S_{-r'}B^c) \\ &= 2\mu_r(A \times T_{-r}A^c) + \nu_{r'}(B \times S_{-r'}B^c) = 0 \end{aligned}$$

and

$$\begin{aligned} \rho((A \times B)^c \times T_{-r}A \times S_{-r'}B) &= \rho(A^c \times B \times T_{-r}A \times S_{-r'}B) \\ &+ \rho(A^c \times B^c \times T_{-r}A \times S_{-r'}B) + \rho(A \times B^c \times T_{-r}A \times S_{-r'}B) \\ &\leq 2\rho(A^c \times Y \times T_{-r}A \times Y) + \rho(X \times B^c \times X \times S_{-r'}B) \\ &= 2\mu_r(A^c \times T_{-r}A) + \nu_{r'}(B^c \times S_{-r'}B) = 0. \end{aligned}$$

Hence we have proved that (3) in Remark 2.5 is satisfied for the π -system of product sets. Since $\rho \in J_2^e(\mathcal{T} \times \mathcal{S}, \lambda)$, in view of (2) in Remark 2.5 we get

$$\begin{aligned} \lambda(A \times B) &= \rho(A \times B \times X \times Y) = \rho(X \times Y \times T_{-r}A \times S_{-r'}B) \\ &= \lambda(T_{-r}A \times S_{-r'}B) = (T_r \times S_{r'})_* \lambda(A \times B), \end{aligned}$$

for all $A \in \mathcal{B}$ and $B \in \mathcal{C}$. Since the π -system of product sets generates $\mathcal{B} \otimes \mathcal{C}$, we get that the measures λ and $(T_r \times S_{r'})_* \lambda$ are equal. By the $(\mathcal{T} \times \mathcal{S})$ -invariance of λ , we have that λ is $(Id \times S_{r-r'})$ -invariant. By weak mixing of \mathcal{S} , $S_{r-r'}$ is ergodic whenever $r \neq r'$. Since Id is disjoint with every ergodic transformation (see Remark 2.4), we get $\lambda = \mu \otimes \nu$. \square

Proposition 3.2. *Assume that for some real sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ we have*

$$\mu_{a_n, b_n} \rightarrow (1 - \alpha) \int_{\mathbb{R}^2} \mu_{-t, -u} dP(t, u) + \alpha \xi_1,$$

and

$$\nu_{a_n, b_n} \rightarrow (1 - \alpha') \int_{\mathbb{R}^2} \nu_{-t, -u} dQ(t, u) + \alpha' \xi_2,$$

for some $0 \leq \alpha, \alpha' < 1$, measures $P, Q \in \mathcal{P}(\mathbb{R}^2)$ and $\xi_1 \in J_3(\mathcal{T})$, $\xi_2 \in J_3(\mathcal{S})$. Assume moreover, that there exists a set $B \in \mathcal{B}(\mathbb{R}^2)$, such that

$$(13) \quad (1 - \alpha)P(B) - (1 - \alpha')Q(B) > \alpha'.$$

Then \mathcal{T} and \mathcal{S} are disjoint.

Remark 3.3. The above proposition can be also proven in higher dimensional case, that is when we consider limits of joinings of higher rank.

Proof of Proposition 3.2. Let $\xi_1 = \int_{J_3^e(\mathcal{T})} \rho^{\mathcal{T}} d\kappa_1(\rho^{\mathcal{T}})$ and $\xi_2 = \int_{J_3^e(\mathcal{S})} \rho^{\mathcal{S}} d\kappa_2(\rho^{\mathcal{S}})$ be the ergodic decompositions of ξ_1 and ξ_2 respectively. Let also \mathcal{A}_1 be the set of 3-off-diagonal joinings in $J_3^e(\mathcal{T})$ and \mathcal{A}_2 be the set of 3-off-diagonal joinings in $J_3^e(\mathcal{S})$. In view of Souslin theorem the sets \mathcal{A}_1 and \mathcal{A}_2 are measurable. We can assume that $\kappa_1(\mathcal{A}_1) = \kappa_2(\mathcal{A}_2) = 0$. Indeed, let $\beta := 1 - \kappa_1(\mathcal{A}_1) \geq 0$ and $\beta' := 1 - \kappa_2(\mathcal{A}_2) \geq 0$. Then

$$\xi_1 = (1 - \beta) \int_{\mathbb{R}^2} \mu_{-t, -u} dP'(t, u) + \beta \xi'_1$$

and

$$\xi_2 = (1 - \beta') \int_{\mathbb{R}^2} \nu_{-t, -u} dQ'(t, u) + \beta' \xi'_2,$$

where $\xi'_1 \in J_3(\mathcal{T})$ and $\xi'_2 \in J_3(\mathcal{S})$ do not have 3-off-diagonal joinings in their ergodic decomposition. Then

$$\begin{aligned} \mu_{a_n, b_n} &\rightarrow (1 - \alpha) \int_{\mathbb{R}^2} \mu_{-t, -u} dP(t, u) + \alpha \left((1 - \beta) \int_{\mathbb{R}^2} \mu_{-t, -u} dP'(t, u) + \beta \xi'_1 \right) \\ &= (1 - \alpha\beta) \int_{\mathbb{R}^2} \mu_{-t, -u} d \left(\frac{1 - \alpha}{1 - \alpha\beta} P + \frac{\alpha(1 - \beta)}{1 - \alpha\beta} P' \right) + \alpha\beta \xi'_1 \\ &= (1 - \alpha\beta) \int_{\mathbb{R}^2} \mu_{-t, -u} d\bar{P} + \alpha\beta \xi'_1, \end{aligned}$$

where $\bar{P} = \frac{1 - \alpha}{1 - \alpha\beta} P + \frac{\alpha(1 - \beta)}{1 - \alpha\beta} P'$. Analogously

$$\nu_{a_n, b_n} \rightarrow (1 - \alpha'\beta') \int_{\mathbb{R}^2} \nu_{-t, -u} d\bar{Q} + \alpha'\beta' \xi'_2,$$

where $\bar{Q} = \frac{1 - \alpha'}{1 - \alpha'\beta'} Q + \frac{\alpha'(1 - \beta')}{1 - \alpha'\beta'} Q'$. Then for the set B satisfying (13) we have

$$\begin{aligned} &(1 - \alpha\beta)\bar{P}(B) - (1 - \alpha'\beta')\bar{Q}(B) \\ &= (1 - \alpha)P(B) + \alpha(1 - \beta)P'(B) - (1 - \alpha')Q(B) - \alpha'(1 - \beta')Q'(B) \\ &> \alpha' + \alpha(1 - \beta)P'(B) - \alpha'(1 - \beta')Q'(B) \geq \alpha' - \alpha'(1 - \beta') = \alpha'\beta'. \end{aligned}$$

It is enough then, to replace P, Q by \bar{P}, \bar{Q} and α, α' by $\alpha\beta, \alpha'\beta'$ respectively.

Let $\lambda \in J^e(\mathcal{T}, \mathcal{S})$. We show that $\lambda = \mu \otimes \nu$. Consider the sequence $\{\lambda_{a_n, b_n}\}_{n \in \mathbb{N}}$ in $J_3^e(\mathcal{T} \times \mathcal{S}, \lambda)$. By the compactness of $J_3(\mathcal{T} \times \mathcal{S}, \lambda)$ we have that $\lambda_{a_n, b_n} \rightarrow \eta$ weakly in $J_3(\mathcal{T} \times \mathcal{S}, \lambda)$, up to taking a subsequence. Moreover, by assumptions we have

$$\eta|_{X_1 \times X_2 \times X_3} = (1 - \alpha) \int_{\mathbb{R}^2} \mu_{-t, -u} dP(t, u) + \alpha \xi_1$$

and

$$\eta|_{Y_1 \times Y_2 \times Y_3} = (1 - \alpha') \int_{\mathbb{R}^2} \nu_{-t, -u} dQ(t, u) + \alpha' \xi_2.$$

Let $h^{\mathcal{T}} : \mathbb{R}^2 \rightarrow \mathcal{A}_1$ and $h^{\mathcal{S}} : \mathbb{R}^2 \rightarrow \mathcal{A}_2$ be given by $h^{\mathcal{T}}(t, u) := \mu_{-t, -u}$ and $h^{\mathcal{S}}(t, u) := \nu_{-t, -u}$. Then

$$(14) \quad \eta|_{X_1 \times X_2 \times X_3} = \int_{J_3^e(\mathcal{T})} \rho^{\mathcal{T}} d((1 - \alpha)h_*^{\mathcal{T}} P + \alpha\kappa_1)(\rho^{\mathcal{T}}),$$

and

$$(15) \quad \eta|_{Y_1 \times Y_2 \times Y_3} = \int_{J_3^e(\mathcal{S})} \rho^{\mathcal{S}} d((1 - \alpha')h_*^{\mathcal{S}} Q + \alpha'\kappa_2)(\rho^{\mathcal{S}}).$$

Let now $\eta = \int_{J_3^e(\mathcal{T} \times \mathcal{S}, \lambda)} \psi d\kappa(\psi)$ be the ergodic decomposition of η . Then we have

$$\eta|_{X_1 \times X_2 \times X_3} = \int_{J_3^e(\mathcal{T} \times \mathcal{S}, \lambda)} \psi|_{X_1 \times X_2 \times X_3} d\kappa(\psi),$$

and

$$\eta|_{Y_1 \times Y_2 \times Y_3} = \int_{J_3^e(\mathcal{T} \times \mathcal{S}, \lambda)} \psi|_{Y_1 \times Y_2 \times Y_3} d\kappa(\psi).$$

Since $\psi \in J_3^e(\mathcal{T} \times \mathcal{S})$, we have $\psi|_{X_1 \times X_2 \times X_3} \in J_3^e(\mathcal{T})$ and $\psi|_{Y_1 \times Y_2 \times Y_3} \in J_3^e(\mathcal{S})$. Consider $\Omega^{\mathcal{T}} : J_3^e(\mathcal{T} \times \mathcal{S}, \lambda) \rightarrow J_3^e(\mathcal{T})$ and $\Omega^{\mathcal{S}} : J_3^e(\mathcal{T} \times \mathcal{S}, \lambda) \rightarrow J_3^e(\mathcal{S})$ given by

$$\Omega^{\mathcal{T}}(\psi) = \psi|_{X_1 \times X_2 \times X_3} \quad \text{and} \quad \Omega^{\mathcal{S}}(\psi) = \psi|_{Y_1 \times Y_2 \times Y_3}.$$

We have

$$\eta|_{X_1 \times X_2 \times X_3} = \int_{J_3^e(\mathcal{T})} \rho^{\mathcal{T}} d(\Omega_*^{\mathcal{T}} \kappa)(\rho^{\mathcal{T}}),$$

and

$$\eta|_{Y_1 \times Y_2 \times Y_3} = \int_{J_3^e(\mathcal{S})} \rho^{\mathcal{S}} d(\Omega_*^{\mathcal{S}} \kappa)(\rho^{\mathcal{S}}).$$

By comparing this with (14) and (15) and using the uniqueness of ergodic decomposition we obtain that

$$(16) \quad \Omega_*^{\mathcal{T}} \kappa = (1 - \alpha)h_*^{\mathcal{T}} P + \alpha\kappa_1 \quad \text{and} \quad \Omega_*^{\mathcal{S}} \kappa = (1 - \alpha')h_*^{\mathcal{S}} Q + \alpha'\kappa_2.$$

Let now

$$\mathcal{A} := \{\psi \in J_3^e(\mathcal{T} \times \mathcal{S}, \lambda) : \exists t, u, t', u' \in \mathbb{R}, (t, u) \neq (t', u'), \\ \psi|_{X_1 \times X_2 \times X_3} = \mu_{-t, -u}, \psi|_{Y_1 \times Y_2 \times Y_3} = \nu_{-t', -u'}\}.$$

We now show that $\kappa(\mathcal{A}) > 0$. For any measurable subsets $C \subset J_3^e(\mathcal{T})$ and $D \subset J_3^e(\mathcal{S})$ denote by $C \bar{\times} D$ the set of all $\psi \in J_3^e(\mathcal{T} \times \mathcal{S}, \lambda)$ such that $\psi|_{X_1 \times X_2 \times X_3} \in C$ and $\psi|_{Y_1 \times Y_2 \times Y_3} \in D$.

Assume that $\kappa(\mathcal{A}) = 0$. Let B be the set satisfying (13). If $(t, u) \in B$ then by the definition of $h^{\mathcal{T}}$ and $h^{\mathcal{S}}$ we have $\mu_{-t, -u} \in h^{\mathcal{T}}(B)$ and $\nu_{-t, -u} \in h^{\mathcal{S}}(B)$. Moreover $\kappa(\mathcal{A}) = 0$ and $h^{\mathcal{T}}(B) \bar{\times} (\mathcal{A}_2 \setminus h^{\mathcal{S}}(B)) \subset \mathcal{A}$ yield

$$(17) \quad \kappa(h^{\mathcal{T}}(B) \bar{\times} \mathcal{A}_2) = \kappa(h^{\mathcal{T}}(B) \bar{\times} h^{\mathcal{S}}(B)).$$

Note that $\kappa_1(h^{\mathcal{T}}(B)) \leq \kappa_1(\mathcal{A}_1) = 0$. Hence, (16) and (17) implies

$$(18) \quad \begin{aligned} (1 - \alpha)P(B) &= (1 - \alpha)h_*^{\mathcal{T}} P(h^{\mathcal{T}}(B)) = [(1 - \alpha)h_*^{\mathcal{T}} P + \alpha\kappa_1](h^{\mathcal{T}}(B)) \\ &= \Omega_*^{\mathcal{T}} \kappa(h^{\mathcal{T}}(B)) = \kappa(h^{\mathcal{T}}(B) \bar{\times} J_3^e(\mathcal{S})) \\ &= \kappa(h^{\mathcal{T}}(B) \bar{\times} \mathcal{A}_2) + \kappa(h^{\mathcal{T}}(B) \bar{\times} (J_3^e(\mathcal{S}) \setminus \mathcal{A}_2)) \\ &= \kappa(h^{\mathcal{T}}(B) \bar{\times} h^{\mathcal{S}}(B)) + \kappa(h^{\mathcal{T}}(B) \bar{\times} (J_3^e(\mathcal{S}) \setminus \mathcal{A}_2)). \end{aligned}$$

Analogously we also obtain

$$(19) \quad (1 - \alpha')Q(B) = (1 - \alpha')h_*^{\mathcal{S}} Q(h^{\mathcal{S}}(B)) = \kappa(h^{\mathcal{T}}(B) \bar{\times} h^{\mathcal{S}}(B)) + \kappa((J_3^e(\mathcal{T}) \setminus \mathcal{A}_1) \bar{\times} h^{\mathcal{S}}(B)).$$

Moreover, in view of (16) we get

$$\begin{aligned} \kappa(h^{\mathcal{T}}(B) \bar{\times} (J_3^e(\mathcal{S}) \setminus \mathcal{A}_2)) &\leq \kappa(J_3^e(\mathcal{T}) \bar{\times} (J_3^e(\mathcal{S}) \setminus \mathcal{A}_2)) \\ &= \Omega_*^{\mathcal{S}} \kappa(J_3^e(\mathcal{S}) \setminus \mathcal{A}_2) = \alpha' \kappa_2(J_3^e(\mathcal{S}) \setminus \mathcal{A}_2) = \alpha'. \end{aligned}$$

Since $(1 - \alpha)P(B) - (1 - \alpha')Q(B) > \alpha'$, by subtracting (18) and (19) we obtain

$$\begin{aligned} \alpha' &< (1 - \alpha)h_*^{\mathcal{T}} P(h^{\mathcal{T}}(B)) - (1 - \alpha')h_*^{\mathcal{S}} Q(h^{\mathcal{S}}(B)) \\ &= (\kappa(h^{\mathcal{T}}(B) \bar{\times} h^{\mathcal{S}}(B)) + \kappa(h^{\mathcal{T}}(B) \bar{\times} (J_3^e(\mathcal{S}) \setminus \mathcal{A}_2))) \\ &\quad - (\kappa(h^{\mathcal{T}}(B) \bar{\times} h^{\mathcal{S}}(B)) + \kappa((J_3^e(\mathcal{T}) \setminus \mathcal{A}_1) \bar{\times} h^{\mathcal{S}}(B))) \\ &= \kappa(h^{\mathcal{T}}(B) \bar{\times} (J_3^e(\mathcal{S}) \setminus \mathcal{A}_2)) - \kappa((J_3^e(\mathcal{T}) \setminus \mathcal{A}_1) \bar{\times} h^{\mathcal{S}}(B)) \leq \alpha', \end{aligned}$$

which is a contradiction. This yields $\kappa(\mathcal{A}) > 0$ and hence \mathcal{A} is non-empty. Therefore, there exists $\psi \in \mathcal{A} \subset J_3^e(\mathcal{T} \times \mathcal{S}, \lambda)$ such that $\psi|_{X_1 \times X_2 \times X_3} = \mu_{t, u}$ and $\psi|_{Y_1 \times Y_2 \times Y_3} = \nu_{t', u'}$ with $(t, u) \neq (t', u')$.

Assume that $t \neq t'$ (the case when $u \neq u'$ is analogous). Then $\phi := \Pi_{1,3}(\psi) \in J_2^e(\mathcal{T} \times \mathcal{S}, \lambda)$ satisfies

$$\phi|_{X_1 \times X_3} = \mu_t \quad \text{and} \quad \phi|_{Y_1 \times Y_3} = \nu_{t'}.$$

Thus, by Lemma 3.1, $\lambda = \mu \otimes \nu$. □

The above criterion strengthens the results obtained in [4], that is the flows described in this paper are not only non-isomorphic with their inverses, but also disjoint. To prove the main result of this paper, we use the following simplified version of Proposition 3.2.

Corollary 3.4. *Let $\mathcal{T} = \{T_t\}_{t \in \mathbb{R}}$ and $\mathcal{S} = \{S_t\}_{t \in \mathbb{R}}$ be weakly mixing flows acting on the standard Borel spaces (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) respectively. Assume that for some real sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ we have*

$$\mu_{a_n, b_n} \rightarrow \int_{\mathbb{R}^2} \mu_{-t, -u} dP(t, u) \quad \text{and} \quad \nu_{a_n, b_n} \rightarrow \int_{\mathbb{R}^2} \nu_{-t, -u} dQ(t, u),$$

for some measures $P, Q \in \mathcal{P}(\mathbb{R}^2)$. If $P \neq Q$, then \mathcal{T} and \mathcal{S} are disjoint.

Let $\xi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by $\xi(t, u) := t - 2u$. The following result gives a condition on limit joinings which imply weak mixing of a flow.

Proposition 3.5. *Let $\mathcal{T} = \{T_t\}_{t \in \mathbb{R}}$ be an ergodic flow on (X, \mathcal{B}, μ) and assume that there exists a real increasing sequence $\{b_n\}_{n \in \mathbb{N}}$, a real number $\rho \in [0, 1)$ and a probability measure $P \in \mathcal{P}(\mathbb{R}^2)$ such that*

$$(20) \quad \mu_{2b_n, b_n} \rightarrow (1 - \rho) \int_{\mathbb{R}^2} \mu_{-t, -u} dP(t, u) + \rho\psi,$$

for some $\psi \in J_3(\mathcal{T})$. If P is not supported on an affine lattice in \mathbb{R}^2 then \mathcal{T} is weakly mixing. In particular, if there exist two rationally independent real numbers d_1 and d_2 such that d_1, d_2 and 0 are atoms of ξ_*P , then the flow \mathcal{T} is weakly mixing.

Proof. Assume that P is not supported on an affine lattice and the flow \mathcal{T} is not weakly mixing. Then there exists a non-zero function $f \in L^2(X, \mu)$ and $a \in \mathbb{R} \setminus \{0\}$ such that

$$(21) \quad \forall t \in \mathbb{R}, \quad f \circ T_t = e^{-2\pi i a t} f.$$

Recall that $\sigma_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ denotes the projection on the first coordinate. By applying $\Psi \circ \Pi_{1,3}$ (see (3)) to (20) and using (4) and (5), we obtain

$$T_{2b_n} \rightarrow (1 - \rho) \int_{\mathbb{R}} T_t dP_1(t) + \rho\Psi_1,$$

where $P_1 := (\sigma_1)_*P$ and Ψ_1 is a Markov operator. Let $\langle \cdot, \cdot \rangle$ be the scalar product on $L^2(X, \mu)$. By (21), we get

$$\|f\|^2 = |\langle f, f \rangle| = |\langle f, e^{-2\pi i a t} f \rangle| = |\langle f, f \circ T_t \rangle| = |\langle f, f \circ T_{2b_n} \rangle|$$

for every $n \in \mathbb{N}$. As $n \rightarrow \infty$, we get

$$\|f\|^2 = |\langle f, f \circ T_{2b_n} \rangle| = \left| \left\langle f, (1 - \rho) \int_{\mathbb{R}} f \circ T_t dP_1(t) + \rho\Psi_1(f) \right\rangle \right|.$$

On the other hand by the fact that Markov operator is a contraction, we get

$$\begin{aligned}
& \left| \left\langle f, (1 - \rho) \int_{\mathbb{R}} f \circ T_t dP_1(t) + \rho \Psi_1(f) \right\rangle \right| \\
& \leq (1 - \rho) \left| \left\langle f, \int_{\mathbb{R}} f \circ T_t dP_1(t) \right\rangle \right| + \rho |\langle f, \Psi_1(f) \rangle| \\
& \leq (1 - \rho) \left| \int_{\mathbb{R}} \langle f, f \circ T_t \rangle dP_1(t) \right| + \rho \|f\|^2 \\
& = (1 - \rho) \|f\|^2 \left| \int_{\mathbb{R}} e^{-2\pi i a t} dP_1(t) \right| + \rho \|f\|^2
\end{aligned}$$

Thus we get

$$\left| \int_{\mathbb{R}} e^{-2\pi i a t} dP_1(t) \right| = 1$$

that is

$$\int_{\mathbb{R}} e^{-2\pi i a t} dP_1(t) = e^{-2\pi i b} \quad \text{for some } b \in \mathbb{R}.$$

It follows that

$$\int_{\mathbb{R}} e^{-2\pi i (a t - b)} dP_1(t) = 1.$$

This implies

$$P_1(\{t \in \mathbb{R}; at - b \in \mathbb{Z}\}) = 1.$$

Consider now $P_2 := (\sigma_2)_* P$. Analogously, by applying $\Psi \circ \Pi_{2,3}$ to (20), we get

$$P_2(\{u \in \mathbb{R}; au - c \in \mathbb{Z}\}) = 1 \quad \text{for some } c \in \mathbb{R}.$$

Combining the two above results, we finally obtain

$$(22) \quad P(\{(t, u) \in \mathbb{R}^2; a(t, u) - (b, c) \in \mathbb{Z}^2\}) = 1,$$

which is a contradiction with our assumption. Thus if P is not supported on an affine lattice then the flow \mathcal{T} is weakly mixing.

Suppose now that $\xi_* P$ has atoms at points $0, d_1$ and d_2 . Assume again that \mathcal{T} is not weakly mixing and that $e^{2\pi i a}$, $a \neq 0$, is an eigenvalue. By the definition of ξ , the lines $(x, \frac{1}{2}(x - d_i))$ for $i = 1, 2$ and $(x, \frac{1}{2}x)$ have positive measure P . This together with (22) yields $x_0, x_1, x_2 \in \mathbb{R}$, such that

$$\begin{aligned}
a(x_0, \frac{1}{2}x_0) - (b, c) & \in \mathbb{Z}^2, \\
a(x_1, \frac{1}{2}(x_1 - d_1)) - (b, c) & \in \mathbb{Z}^2, \\
a(x_2, \frac{1}{2}(x_2 - d_2)) - (b, c) & \in \mathbb{Z}^2.
\end{aligned}$$

This implies

$$\begin{aligned}
a(x_1 - x_0, \frac{1}{2}(x_1 - x_0) - \frac{1}{2}d_1) & \in \mathbb{Z}^2, \\
a(x_2 - x_0, \frac{1}{2}(x_2 - x_0) - \frac{1}{2}d_2) & \in \mathbb{Z}^2.
\end{aligned}$$

By applying ξ to the above, we get that $ad_1 \in \mathbb{Z}$ and $ad_2 \in \mathbb{Z}$. Since $a, d_1, d_2 \neq 0$, we get that here $(ad_1)d_2 - (ad_2)d_1 = 0$ is a non-trivial integer combination of d_1 and d_2 . By the rational independence of d_1 and d_2 this yields $a = 0$. This is a contradiction, hence \mathcal{T} is weakly mixing. \square

4. ACCEPTABLE PERMUTATIONS

In this section, we establish a technical result concerning a particular non-degenerate permutation, which plays a key role in proving that our main result applies to all non-hyperelliptic connected components of the moduli space. In particular, in view of the Remark 2.10, we assume that the alphabet we consider has $d \geq 6$ elements. Recall that in every Rauzy class we can fix a non-degenerate permutation $\pi = \{\pi_0, \pi_1\}$ satisfying

$$(23) \quad \pi_1 \circ \pi_0^{-1}(1) = d \quad \text{and} \quad \pi_1 \circ \pi_0^{-1}(d) = 1.$$

We have the following theorem.

Proposition 4.1. *In every Rauzy class corresponding to a non-hyperelliptic connected component of the moduli space \mathcal{M} , there exists a permutation $\pi = \{\pi_0, \pi_1\}$ satisfying (23) such that there exist distinct symbols $\alpha_1, \alpha_2, \gamma_1, \gamma_2 \in \mathcal{A} \setminus \{\pi_0^{-1}(1), \pi_0^{-1}(d)\}$ satisfying the three following properties*

$$(24) \quad \begin{aligned} \Omega_{\alpha_1\alpha_2} &= \Omega_{\alpha_2\alpha_1} = 0, \\ \Omega_{\alpha_1\gamma_2}\Omega_{\alpha_2\gamma_1} &= 0 \\ \Omega_{\alpha_1\gamma_1}\Omega_{\alpha_2\gamma_2} &\neq 0, \end{aligned}$$

where $\Omega := \Omega_\pi$ is the associated translation matrix.

Proof. Let $\pi = \{\pi_0, \pi_1\}$ be a non-degenerate permutation satisfying (23) that belongs to a Rauzy class associated with a non-hyperelliptic connected component. Then it is not symmetric, hence its translation matrix Ω contains zero entries outside the diagonal. Indeed, assume contrary to our claim that

$$\pi_0(\alpha) < \pi_0(\beta) \Leftrightarrow \pi_1(\alpha) > \pi_1(\beta) \quad \text{for all } \alpha, \beta \in \mathcal{A}.$$

Then for every $\alpha \in \mathcal{A}$

$$\pi_1(\alpha) = \#\{\beta \in \mathcal{A}; \pi_1(\beta) < \pi_1(\alpha)\} + 1 = \#\{\beta \in \mathcal{A}; \pi_0(\beta) > \pi_0(\alpha)\} + 1 = d - \pi_0(\alpha) + 1.$$

Hence π is a symmetric permutation.

We need to consider two cases separately.

Case 1. Assume first that there exists a symbol $\alpha \in \mathcal{A}$ such that for all symbols $\beta \in \mathcal{A}$ with $1 < \pi_0(\beta) < d$ we have

$$\pi_0(\delta) < \pi_0(\alpha) \Leftrightarrow \pi_1(\delta) < \pi_1(\alpha)$$

that is

$$(25) \quad \Omega_{\alpha\beta} = 0 \quad \text{for all } \beta \in \mathcal{A} \setminus \{\pi_0^{-1}(1), \pi_0^{-1}(d)\}.$$

Since π is non-degenerate, there exist symbols α_1, γ_1 such that

$$1 < \pi_0(\alpha_1) < \pi_0(\gamma_1) < \pi_0(\alpha) \quad \text{and} \quad 1 < \pi_1(\gamma_1) < \pi_1(\alpha_1) < \pi_1(\alpha).$$

Otherwise, π satisfies (9) and hence, it is degenerate. Similarly, there exist symbols α_2, γ_2 such that

$$d > \pi_0(\alpha_2) > \pi_0(\gamma_2) > \pi_0(\alpha) \quad \text{and} \quad d > \pi_1(\gamma_2) > \pi_1(\alpha_2) > \pi_1(\alpha).$$

Otherwise, π satisfies (10) and it is again degenerate. Thus we have

$$\Omega_{\alpha_1\alpha_2} = \Omega_{\alpha_2\alpha_1} = \Omega_{\alpha_1\gamma_2} = \Omega_{\alpha_2\gamma_1} = 0 \quad \text{and} \quad \Omega_{\alpha_1\gamma_1} = 1 \quad \text{and} \quad \Omega_{\alpha_2\gamma_2} = -1,$$

which is the desired property. Hence $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ are the desired symbols.

Case 2. Assume now, that there are no symbols satisfying (25). Since there are zeroes outside the diagonal in Ω_π , there exist two distinct symbols $\alpha_1, \alpha_2 \in \mathcal{A}$ such that $\Omega_{\alpha_1\alpha_2} = \Omega_{\alpha_2\alpha_1} = 0$.

Case 2a. Suppose first that the rows of Ω_π corresponding to α_1 and α_2 are not identical. Then there exists a symbol γ such that $\Omega_{\alpha_1\gamma} \neq 0$ and $\Omega_{\alpha_2\gamma} = 0$ or $\Omega_{\alpha_2\gamma} \neq 0$ and $\Omega_{\alpha_1\gamma} = 0$. Assume that the first case holds (the second is done analogously) and set $\gamma_1 := \gamma$. Note that $\gamma_1 \in \mathcal{A} \setminus \{\pi_0^{-1}(1), \pi_0^{-1}(d)\}$. Since α_2 does not satisfy (25), there exist two $\gamma_2 \in \mathcal{A} \setminus \{\pi_0^{-1}(1), \pi_0^{-1}(d)\}$, $\gamma_2 \neq \gamma_1$, such that $\Omega_{\alpha_2\gamma_2} \neq 0$. Thus we obtain (24).

Case 2b. Suppose that the rows of the matrix Ω corresponding to indices α_1, α_2 are identical. Then there are no indices γ_1, γ_2 such that $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ satisfy (24). We show that there is a different set of symbols satisfying (24). Note that all symbols β such that

$$(26) \quad \pi_0(\alpha_1) < \pi_0(\beta) < \pi_0(\alpha_2)$$

satisfy

$$(27) \quad \pi_1(\alpha_1) < \pi_1(\beta) < \pi_1(\alpha_2).$$

Otherwise only one of the entries $\Omega_{\alpha_1\beta}$ and $\Omega_{\alpha_2\beta}$ would be non-zero. In other words all symbols $\beta \in \mathcal{A}$ satisfying (26) satisfy $\Omega_{\alpha_1\beta} = \Omega_{\alpha_2\beta} = 0$. Observe that there exist two different symbols $\hat{\alpha}_1, \gamma_1 \in \mathcal{A}$ satisfying (26) such that $\Omega_{\hat{\alpha}_1\gamma_1} \neq 0$. Otherwise the permutation π satisfies (8) and it is degenerate. Since α_1, α_2 do not satisfy (25) and the corresponding rows are identical, there exists a symbol $\gamma_2 \in \mathcal{A}$ such that $\Omega_{\alpha_1\gamma_2} = \Omega_{\alpha_2\gamma_2} \neq 0$. It follows that $\Omega_{\hat{\alpha}_1\gamma_2} = \Omega_{\alpha_2\gamma_2} \neq 0$. We have

$$\Omega_{\hat{\alpha}_1\alpha_2} = \Omega_{\alpha_2\hat{\alpha}_1} = \Omega_{\alpha_2\gamma_1} = 0 \quad \text{and} \quad \Omega_{\hat{\alpha}_1\gamma_1} \neq 0 \quad \text{and} \quad \Omega_{\alpha_2\gamma_2} \neq 0.$$

Hence, $\hat{\alpha}_1, \alpha_2, \gamma_1, \gamma_2$ are the desired symbols. \square

Corollary 4.2. *If π is a nonsymmetric and nondegenerate permutation satisfying (23), and $\tau \in \mathbb{R}^{\mathcal{A}}$ is a rationally independent vector, then there exist $\alpha_1, \alpha_2 \in \mathcal{A}$ such that $\Omega_{\alpha_1\alpha_2} = \Omega_{\alpha_2\alpha_1} = 0$ and for each $i = 1, 2$ the numbers*

$$(\Omega\tau)_{\alpha_2} - (\Omega\tau)_{\alpha_1}, \quad \text{and} \quad (\Omega\tau)_{\alpha_i} - ((\Omega\tau)_{\pi_0^{-1}(1)} + (\Omega\tau)_{\pi_0^{-1}(d)})$$

are rationally independent.

Proof. We prove the case when $i = 1$. If $i = 2$, the proof goes along the same lines. Consider symbols $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ given by Proposition 4.1. We have $\Omega_{\alpha_1\alpha_2} = \Omega_{\alpha_2\alpha_1} = 0$. Assume that there exist integers p and q such that

$$\begin{aligned} 0 &= p((\Omega\tau)_{\alpha_2} - (\Omega\tau)_{\alpha_1}) + q((\Omega\tau)_{\alpha_1} - ((\Omega\tau)_{\pi_0^{-1}(1)} + (\Omega\tau)_{\pi_0^{-1}(d)})) \\ &= \sum_{\beta \in \mathcal{A}} (-q\Omega_{\pi_0^{-1}(1)\beta} - q\Omega_{\pi_0^{-1}(d)\beta} + (q-p)\Omega_{\alpha_1\beta} + p\Omega_{\alpha_2\beta})\tau_\beta. \end{aligned}$$

By rational independence of τ , this yields

$$-q\Omega_{\pi_0^{-1}(1)\beta} - q\Omega_{\pi_0^{-1}(d)\beta} + (q-p)\Omega_{\alpha_1\beta} + p\Omega_{\alpha_2\beta} = 0,$$

for every $\beta \in \mathcal{A}$. Since

$$\Omega_{\pi_0^{-1}(1)\beta} = 1 \text{ for } \beta \in \mathcal{A} \setminus \{\pi_0^{-1}(1)\} \text{ and } \Omega_{\pi_0^{-1}(d)\beta} = -1 \text{ for } \beta \in \mathcal{A} \setminus \{\pi_0^{-1}(d)\},$$

we have

$$(q-p)\Omega_{\alpha_1\beta} + q\Omega_{\alpha_2\beta} = 0 \text{ for } \beta \in \mathcal{A} \setminus \{\pi_0^{-1}(1), \pi_0^{-1}(d)\}$$

Since by (24) the matrix

$$\begin{bmatrix} \Omega_{\alpha_1\gamma_1} & \Omega_{\alpha_1\gamma_2} \\ \Omega_{\alpha_2\gamma_1} & \Omega_{\alpha_2\gamma_2} \end{bmatrix}$$

is triangular and has non-zero entries on the main diagonal, it follows that

$$p = q = 0,$$

which proves the rational independence. \square

5. THE MEASURES ON THE SURFACE

In this section we will deal with measures on a given surface (M, Σ) which are absolutely continuous with respect to the Lebesgue measure. We want to prove that, if the density of such measure is bounded and close enough to the constant function 1 in L^1 , then there is an explicit way to construct a homeomorphism which pushes this measure to the Lebesgue measure. The computation given below is partially inspired by the paper of Moser [19]. We will need the following auxiliary lemma.

Lemma 5.1. *Let $x, y \in \mathbb{R}^2$ be two points on the plane and let \overline{xy} be the segment with endpoints at x and y . Let H_1, H_2 be two affine transformations on \mathbb{R}^2 . If $H_1(x) = H_2(x)$ and $H_1(y) = H_2(y)$, then $H_1|_{\overline{xy}} = H_2|_{\overline{xy}}$. Moreover, for each noncollinear triples $x_1, x_2, x_3 \in \mathbb{R}^2$ and $y_1, y_2, y_3 \in \mathbb{R}^2$ there exists a unique invertible affine transformation H such that $H(x_i) = y_i$ for $i = 1, 2, 3$.*

We will now prove lemmas which give a construction of a homeomorphism of an isosceles right triangle which pushes forward any given absolutely continuous measure whose density satisfies some conditions to the Lebesgue measure. For any affine transformation $G: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ define $\text{lin}(G) = DG$ as the matrix determining its linear part, D denotes the derivative. Moreover for any real 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have the following formula for the operator norm

$$(28) \quad \|M\| := \sqrt{\frac{a^2 + b^2 + c^2 + d^2 + \sqrt{(a^2 + b^2 + c^2 + d^2)^2 - 4(\det(M))^2}}{2}}.$$

In particular, $\|\text{lin}(G)\|$ is a Lipschitz constant of G .

On the space of homeomorphisms $\text{Hom}(X)$ of a compact metric space X , we will consider the standard metric $d_{\text{Hom}}(H, G) := \max\{\sup_{x \in X} d(H(x), G(x)), \sup_{x \in X} d(H^{-1}(x), G^{-1}(x))\}$, where d is the metric on X .

Throughout this section we will heavily depend on the following construction. Let $0 < a < 1$, and let V be the triangle in \mathbb{R}^2 with vertices in points $(0, a), (a, 0), (0, -a)$. Let V_1 and V_2 be the triangles whose vertices are $(0, a), (a, 0), (0, 0)$ and $(0, 0), (a, 0), (0, -a)$ respectively. Let $0 \leq h < 1$ and $0 < \epsilon < 1$. Let $y(h) := (\epsilon a(1 - h), ha)$. Consider the triangles given by the following sets of vertices:

- $C_1 = C_1(h, \epsilon)$ given by $\{(0, 0), (0, a), y(h, \epsilon)\}$;
- $C_2 = C_2(h, \epsilon)$ given by $\{(a, 0), (0, a), y(h, \epsilon)\}$;
- $C_3 = C_3(h, \epsilon)$ given by $\{(0, 0), (\epsilon a, 0), y(h, \epsilon)\}$;
- $C_4 = C_4(h, \epsilon)$ given by $\{(a, 0), (\epsilon a, 0), y(h, \epsilon)\}$;
- $C_5 = C_5(h, \epsilon)$ given by $\{(0, 0), (0, -a), (\epsilon a, 0)\}$;
- $C_6 = C_6(h, \epsilon)$ given by $\{(a, 0), (0, -a), (\epsilon a, 0)\}$.

Let $\hat{h} := \frac{h}{h+1} \geq 0$. Consider the point $\hat{y}(h, \epsilon) = (\epsilon a - \hat{h}\epsilon a, -\hat{h}a)$. Consider also the triangles

- $\hat{C}_1 = \hat{C}_1(h, \epsilon)$ given by $\{(0, 0), (0, a), (\epsilon a, 0)\}$;
- $\hat{C}_2 = \hat{C}_2(h, \epsilon)$ given by $\{(a, 0), (0, a), (\epsilon a, 0)\}$;
- $\hat{C}_3 = \hat{C}_3(h, \epsilon)$ given by $\{(0, 0), (\epsilon a, 0), \hat{y}(h, \epsilon)\}$;
- $\hat{C}_4 = \hat{C}_4(h, \epsilon)$ given by $\{(a, 0), (\epsilon a, 0), \hat{y}(h, \epsilon)\}$;
- $\hat{C}_5 = \hat{C}_5(h, \epsilon)$ given by $\{(0, 0), (0, -a), \hat{y}(h, \epsilon)\}$;
- $\hat{C}_6 = \hat{C}_6(h, \epsilon)$ given by $\{(a, 0), (0, -a), \hat{y}(h, \epsilon)\}$.

By the definition of h and \hat{h} we have

$$(29) \quad \begin{aligned} \frac{\text{Leb}(C_1)}{\text{Leb}(\hat{C}_1)} &= \frac{\text{Leb}(C_2)}{\text{Leb}(\hat{C}_2)} = 1 - h \quad \text{and} \\ \frac{\text{Leb}(C_3)}{\text{Leb}(\hat{C}_3)} &= \frac{\text{Leb}(C_4)}{\text{Leb}(\hat{C}_4)} = \frac{\text{Leb}(C_5)}{\text{Leb}(\hat{C}_5)} = \frac{\text{Leb}(C_6)}{\text{Leb}(\hat{C}_6)} = 1 + h. \end{aligned}$$

Define $H(h, \epsilon) : V \rightarrow V$ as a piecewise affine homeomorphism such that

- (30) (i) $H(h, \epsilon)(C_i) = \hat{C}_i$, $H(h, \epsilon)|_{C_i}$ is affine for $i = 1, \dots, 6$;
(ii) $H(h, \epsilon)$ fixes $(0, 0)$, $(0, a)$, $(0, -a)$, $(a, 0)$,
(iii) $H(h, \epsilon)(y) = (a\epsilon, 0)$ and $H(h, \epsilon)(a\epsilon, 0) = \hat{y}$.

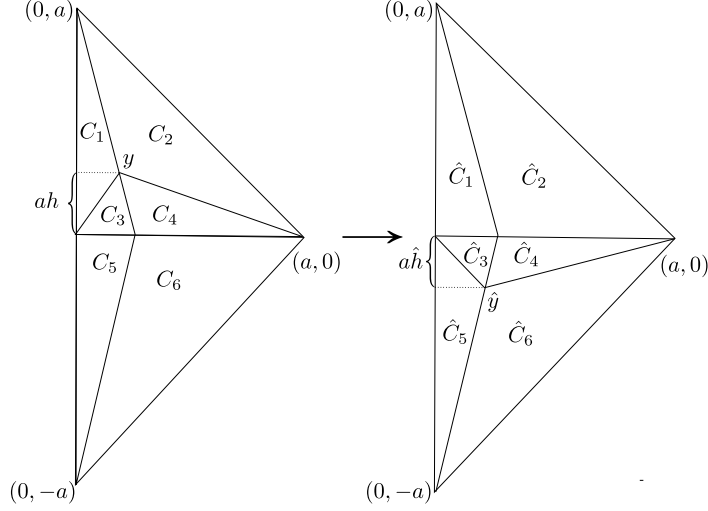


FIGURE 1. The division of V into triangles and the map $H(h, \epsilon)$.

Note that by Lemma 5.1, $H(h, \epsilon)$ is well defined everywhere on V and also $H(h, \epsilon)|_{\partial V} = Id|_{\partial V}$. Moreover

$$\begin{aligned} \text{lin}(H(h, \epsilon)|_{C_1}) &:= \begin{bmatrix} 1 + \frac{h}{1-h} & 0 \\ -\frac{h}{\epsilon(1-h)} & 1 \end{bmatrix}; & \text{lin}(H(h, \epsilon)|_{C_2}) &:= \begin{bmatrix} 1 - \frac{\epsilon h}{(1-h)(1-\epsilon)} & \frac{-\epsilon h}{(1-h)(1-\epsilon)} \\ \frac{h}{(1-h)(1-\epsilon)} & 1 + \frac{h}{(1-h)(1-\epsilon)} \end{bmatrix}; \\ \text{lin}(H(h, \epsilon)|_{C_3}) &:= \begin{bmatrix} 1 - \frac{h}{1+h} & \frac{2\epsilon}{1+h} \\ \frac{-h}{\epsilon(1+h)} & 1 - \frac{2h}{1+h} \end{bmatrix}; & \text{lin}(H(h, \epsilon)|_{C_4}) &:= \begin{bmatrix} 1 + \frac{\epsilon h}{(1+h)(1-\epsilon)} & \frac{\epsilon(2+h-2\epsilon)}{(1+h)(1-\epsilon)} \\ \frac{h}{(1+h)(1-\epsilon)} & 1 - \frac{h}{(1+h)(1-\epsilon)} \end{bmatrix}; \\ \text{lin}(H(h, \epsilon)|_{C_5}) &:= \begin{bmatrix} 1 - \frac{h}{1+h} & 0 \\ -\frac{h}{\epsilon(1+h)} & 1 \end{bmatrix}; & \text{lin}(H(h, \epsilon)|_{C_6}) &:= \begin{bmatrix} 1 + \frac{\epsilon h}{(1+h)(1-\epsilon)} & \frac{-\epsilon h}{(1+h)(1-\epsilon)} \\ \frac{h}{(1+h)(1-\epsilon)} & 1 - \frac{h}{(1+h)(1-\epsilon)} \end{bmatrix}. \end{aligned}$$

By (29) we have

$$\det(\text{lin}(H(h, \epsilon)|_{C_1})) = \det(\text{lin}(H(h, \epsilon)|_{C_2})) = \frac{1}{1-h} \geq 1,$$

and

$$\begin{aligned} \det(\text{lin}(H(h, \epsilon)|_{C_3})) &= \det(\text{lin}(H(h, \epsilon)|_{C_4})) \\ &= \det(\text{lin}(H(h, \epsilon)|_{C_5})) = \det(\text{lin}(H(h, \epsilon)|_{C_6})) = \frac{1}{1+h} \leq 1. \end{aligned}$$

It is also worth noting that $(0, 0)$ is fixed by the affine maps $H(h, \epsilon)|_{C_1}$, $H(h, \epsilon)|_{C_3}$ and $H(h, \epsilon)|_{C_5}$, while $(a, 0)$ is a common fixed point for the transformations $H(h, \epsilon)|_{C_2}$, $H(h, \epsilon)|_{C_4}$ and $H(h, \epsilon)|_{C_6}$.

We can also define $H(h, \epsilon) : V \rightarrow V$ for $-1 < h \leq 0$. Let $J : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the reflection across the x -axis. Note that $JV = V$, $JV_1 = V_2$ and $JV_2 = V_1$. Now define $\hat{h} := \frac{h}{1+|h|}$,

$$(31) \quad \begin{aligned} C_i(h, \epsilon) &:= J(C_i(-h, \epsilon)), \quad \hat{C}_i(h, \epsilon) := J(\hat{C}_i(-h, \epsilon)) \text{ and} \\ H(h, \epsilon) &:= J \circ H(-h, \epsilon) \circ J. \end{aligned}$$

For $i = 1, \dots, 6$ we have

$$\text{lin}(H(h, \epsilon)|_{C_i}) = J \circ \text{lin}(H(-h, \epsilon)|_{C_i}) \circ J.$$

Since J is an isometry, we also obtain

$$\det(\text{lin}(H(h, \epsilon)|_{C_1})) = \det(\text{lin}(H(h, \epsilon)|_{C_2})) = \frac{1}{1+h} \geq 1,$$

and

$$\begin{aligned} \det(\text{lin}(H(h, \epsilon)|_{C_3})) &= \det(\text{lin}(H(h, \epsilon)|_{C_4})) \\ &= \det(\text{lin}(H(h, \epsilon)|_{C_5})) = \det(\text{lin}(H(h, \epsilon)|_{C_6})) = \frac{1}{1-h} \leq 1. \end{aligned}$$

Hence in general for $-1 < h < 1$ we have

$$(32) \quad \det(\text{lin}(H(h, \epsilon)|_{C_1})) = \det(\text{lin}(H(h, \epsilon)|_{C_2})) = \frac{1}{1-|h|} \geq 1,$$

and

$$(33) \quad \begin{aligned} \det(\text{lin}(H(h, \epsilon)|_{C_3})) &= \det(\text{lin}(H(h, \epsilon)|_{C_4})) \\ &= \det(\text{lin}(H(h, \epsilon)|_{C_5})) = \det(\text{lin}(H(h, \epsilon)|_{C_6})) = \frac{1}{1+|h|} \leq 1. \end{aligned}$$

Lemma 5.2. *For any fixed $\epsilon > 0$ and for every $h_1, h_2 \in (-\frac{1}{2}, \frac{1}{2})$ we have*

$$(34) \quad d_{\text{Hom}}(H(h_1, \epsilon), H(h_2, \epsilon)) \leq \frac{20a}{\epsilon} |h_2 - h_1|.$$

Proof. We first prove that

$$(35) \quad \|H(h_1, \epsilon)(x) - H(h_2, \epsilon)(x)\| \leq \frac{20a}{\epsilon} |h_2 - h_1|,$$

for every $x \in V$. Indeed, assume that h_1 and h_2 are non-negative numbers and $h_1 \geq h_2$. Consider the triangle W_1 with vertices $(0, 0), y(h_1, \epsilon), y(h_2, \epsilon)$ and the triangle W_2 given by the points $(a, 0), y(h_1, \epsilon), y(h_2, \epsilon)$. Assume that $x \in V \setminus (W_1 \cup W_2)$. Then $x \in C_i(h_1, \epsilon) \Leftrightarrow x \in C_i(h_2, \epsilon)$. This implies that both $H(h_1, \epsilon)$ and $H(h_2, \epsilon)$ act on x by affine transformations whose linear parts are of the same form. Since for $i = 1, \dots, 6$ the affine maps $H(h_1, \epsilon)|_{C_i(h_1, \epsilon)}$ and $H(h_2, \epsilon)|_{C_i(h_2, \epsilon)}$ share a common fixed point,

we get that

$$H(h_1, \epsilon)(x) - H(h_2, \epsilon)(x) = \text{lin}(H(h_1, \epsilon)|_{C_i(h_1, \epsilon)})x - \text{lin}(H(h_2, \epsilon)|_{C_i(h_2, \epsilon)})x.$$

By using the formula (28) for each $i = 1, \dots, 6$ we get

$$\|\text{lin}(H(h_1, \epsilon)|_{C_i(h_1, \epsilon)}) - \text{lin}(H(h_2, \epsilon)|_{C_i(h_2, \epsilon)})\| \leq \frac{10}{\epsilon} (h_1 - h_2).$$

Since the above norm is the operator norm for $H(h_1, \epsilon) - H(h_2, \epsilon)$ (which is a linear transformation), this implies that

$$\sup_{x \in V \setminus (W_1 \cup W_2)} \|H(h_1, \epsilon)(x) - H(h_2, \epsilon)(x)\| \leq \frac{10}{\epsilon} (h_1 - h_2) \|x\| < \frac{20a}{\epsilon} (h_1 - h_2).$$

Next note that $W_1 = C_3(h_1, \epsilon) \cap C_1(h_2, \epsilon)$ and $W_2 = C_4(h_1, \epsilon) \cap C_2(h_2, \epsilon)$. We now prove that for $x \in W_1 \cup W_2$ we also have $\|H(h_1, \epsilon)(x) - H(h_2, \epsilon)(x)\| \leq \frac{20a}{\epsilon} (h_1 - h_2)$. Suppose that $x \in W_1$; the proof for $x \in W_2$ is analogous. Consider the segment $I_x \subset W_1$ with endpoints on the segments $\overline{(0, 0), y(h_1, \epsilon)}$ and $\overline{(0, 0), y(h_2, \epsilon)}$ such that $x \in I_x$ and which is parallel to $\overline{y(h_1, \epsilon), y(h_2, \epsilon)}$. Then

$$(36) \quad |I_x| \leq \|y(h_1, \epsilon) - y(h_2, \epsilon)\| = a\sqrt{1 + \epsilon}(h_1 - h_2) < 2a(h_1 - h_2).$$

Note that I_x divides the intervals $\overline{(0,0), y(h_1, \epsilon)}$ and $\overline{(0,0), y(h_2, \epsilon)}$ with the same ratio. Since affine transformations do not change the ratio of the lengths of collinear segments and

$$H(h_1, \epsilon)(\overline{(0,0), y(h_1, \epsilon)}) = H(h_2, \epsilon)(\overline{(0,0), y(h_2, \epsilon)}) = \overline{(0,0), (0, \epsilon a)},$$

it follows that the segments $H(h_1, \epsilon)(I_x)$ and $H(h_2, \epsilon)(I_x)$ share a common endpoint in $\overline{(0,0), (0, \epsilon a)}$. Using again the conservation of the ratio of the lengths of collinear segments by affine transformations, we get

$$\frac{|H(h_1, \epsilon)(I_x)|}{|I_x|} = \frac{|H(h_1, \epsilon)(\overline{y(h_1, \epsilon), y(h_2, \epsilon)})|}{|\overline{y(h_1, \epsilon), y(h_2, \epsilon)}|} = \frac{|H(h_1, \epsilon)(\overline{y(h_1, \epsilon), (\epsilon a, 0)})|}{|\overline{y(h_1, \epsilon), (\epsilon a, 0)}|} = \frac{1}{1 + h_1} \leq 1,$$

and

$$\frac{|H(h_2, \epsilon)(I_x)|}{|I_x|} = \frac{|H(h_2, \epsilon)(\overline{y(h_1, \epsilon), y(h_2, \epsilon)})|}{|\overline{y(h_1, \epsilon), y(h_2, \epsilon)}|} = \frac{|H(h_2, \epsilon)(\overline{(0, a), y(h_2, \epsilon)})|}{|\overline{(0, a), y(h_2, \epsilon)}|} = \frac{1}{1 - h_2} \leq 2.$$

As $H(h_1, \epsilon)(x) \in H(h_1, \epsilon)(I_x)$ and $H(h_2, \epsilon)(x) \in H(h_2, \epsilon)(I_x)$, we obtain

$$\begin{aligned} \|H(h_1, \epsilon)(x) - H(h_2, \epsilon)(x)\| &\leq |H(h_1, \epsilon)(I_x)| + |H(h_2, \epsilon)(I_x)| \leq 3|I_x| \\ &< 6a(h_2 - h_1) < \frac{20}{\epsilon}a(h_1 - h_2). \end{aligned}$$

By proceeding analogously for $h_2 \geq h_1$ we prove that

$$\|H(h_1, \epsilon)(x) - H(h_2, \epsilon)(x)\| \leq \frac{20}{\epsilon}a(h_2 - h_1).$$

The case when h_1 and h_2 are non-positive is analogous. To prove the similar inequality when h_1 and h_2 are of different sign, let $h_0 := 0$. Then $H(h_0, \epsilon) = Id$. Using the previous case we show that

$$\|H(h_2, \epsilon)(x) - x\| \leq \frac{20}{\epsilon}a|h_0 - h_2|,$$

and

$$\|H(h_1, \epsilon)(x) - x\| \leq \frac{20}{\epsilon}a|h_1 - h_0|.$$

Since h_1, h_2 have different sign, the numbers $h_0 - h_2, h_1 - h_0$ are of the same sign. It follows that

$$\|H(h_1, \epsilon)(x) - H(h_2, \epsilon)(x)\| \leq \|H(h_1, \epsilon)(x) - x\| + \|H(h_2, \epsilon)(x) - x\| \leq \frac{20}{\epsilon}a|h_2 - h_1|.$$

By proceeding as in the proof of (35) and replacing $H(h_i, \epsilon)$ by $H^{-1}(h_i, \epsilon)$ for $i = 1, 2$, we can prove that for every $x \in V$ we also have

$$(37) \quad \|H^{-1}(h_2, \epsilon)(x) - H^{-1}(h_1, \epsilon)(x)\| \leq \frac{20}{\epsilon}a|h_2 - h_1|,$$

which concludes the proof of the lemma. \square

Lemma 5.3. *Let V, V_1 and V_2 be the triangles defined above. Let $0 < \hat{\epsilon} < 10^{-8}$, and let $\kappa > 0$. Suppose that $f \in L^1(V)$ satisfies*

$$(38) \quad f > \kappa; \quad \frac{1}{1 + \hat{\epsilon}} < f \text{ or } f < \frac{1}{1 - \hat{\epsilon}}; \quad \int_V f(x)dx = \text{Leb}(V).$$

Let $\mu_f := f dx$. Then there exists a piecewise affine homeomorphism $H_f : V \rightarrow V$ such that

- (i) $(H_f)_* \mu_f(V_i) = \text{Leb}(V_i)$ for $i = 1, 2$;
- (ii) $H_f|_{\partial V} = Id|_{\partial V}$;
- (iii) there exists $-\hat{\epsilon} < h_f < \hat{\epsilon}$ such that $\det(DH_f^{-1})$ is constant on each V_i and is equal to $1 \pm h_f$;
- (iv) the Lipschitz constants of H_f and H_f^{-1} are less than $\frac{5}{4}$;
- (v) the maps $f \mapsto H_f \in \text{Hom}(V)$ and $f \mapsto \det(DH_f^{-1}) \in L^\infty(V)$ are continuous on the set of $f \in L^1(V)$ satisfying (38) for a given κ .

Proof. Since μ_f is an absolutely continuous measure with respect to Leb , there are no segments of positive measure μ_f in V . Hence there exists a unique $-1 < h_f < 1$ and a point $y = y_f = (\sqrt{\hat{\varepsilon}}a(1 - |h_f|), h_f a)$ such that the quadrilateral with vertices $\{(0, a), (0, 0), (a, 0), y\}$ and the quadrilateral with vertices $\{(0, -a), (0, 0), (a, 0), y\}$ have the same measure μ_f equal to $\frac{1}{2} \text{Leb}(V)$.

Consider the triangles $C_i = C_i^f := C_i(h_f, \sqrt{\hat{\varepsilon}})$ for $i = 1, \dots, 6$. By the definition of h_f we have

$$\mu_f(C_1 \cup C_2) = \mu_f(C_3 \cup C_4 \cup C_5 \cup C_6) = \frac{1}{2} \text{Leb}(V).$$

We now evaluate the bounds on h_f . Assume that $f > \frac{1}{1+\hat{\varepsilon}}$. Since $\text{Leb}(V) = a^2$ we have

$$\begin{aligned} \frac{1}{2}a^2 &= \mu_f(C_3 \cup C_4 \cup C_5 \cup C_6) = \int_{C_3 \cup C_4 \cup C_5 \cup C_6} f(x) dx \\ &> \frac{1}{1+\hat{\varepsilon}} \text{Leb}(C_3 \cup C_4 \cup C_5 \cup C_6) = \frac{1}{1+\hat{\varepsilon}} \left(\frac{1}{2}(a + |h_f|a)a \right). \end{aligned}$$

Hence

$$(39) \quad f > \frac{1}{1+\hat{\varepsilon}} \Rightarrow |h_f| < \hat{\varepsilon}.$$

Now assume that $f < \frac{1}{1-\hat{\varepsilon}}$. Then we have

$$\begin{aligned} \frac{1}{2}a^2 &= \mu_f(C_1 \cup C_2) = \int_{C_1 \cup C_2} f(x) dx \\ &< \frac{1}{(1-\hat{\varepsilon})} \text{Leb}(C_1 \cup C_2) = \frac{1}{1-\hat{\varepsilon}} \left(\frac{1}{2}(a - |h_f|a)a \right). \end{aligned}$$

This shows

$$(40) \quad f < \frac{1}{1-\hat{\varepsilon}} \Rightarrow |h_f| < \hat{\varepsilon}.$$

Definition of H_f . Define $H_f := H(h_f, \sqrt{\hat{\varepsilon}})$, a piecewise affine homeomorphism on V . Note that by definition we have $H_f|_{\partial V} = \text{Id}|_{\partial V}$. Moreover

$$(H_f)_* \mu_f(V_1) = \mu_f(C_1 \cup C_2) = \frac{1}{2} \text{Leb}(V) = \text{Leb}(V_1)$$

and

$$(H_f)_* \mu_f(V_2) = \mu_f(C_3 \cup C_4 \cup C_5 \cup C_6) = \frac{1}{2} \text{Leb}(V) = \text{Leb}(V_2).$$

Hence H_f satisfies points (i) and (ii).

Furthermore, by (32) and (33) we have that

$$(41) \quad \det(\text{lin}(H_f|_{C_1})) = \det(\text{lin}(H_f|_{C_2})) = \frac{1}{1-|h_f|} \geq 1,$$

and

$$(42) \quad \begin{aligned} \det(\text{lin}(H_f|_{C_3})) &= \det(\text{lin}(H_f|_{C_4})) \\ \det(\text{lin}(H_f|_{C_5})) &= \det(\text{lin}(H_f|_{C_6})) = \frac{1}{1+|h_f|} \leq 1. \end{aligned}$$

Note that $V_1 = \hat{C}_1 \cup \hat{C}_2$ and $V_2 = \hat{C}_3 \cup \hat{C}_4 \cup \hat{C}_5 \cup \hat{C}_6$ for $h_f \geq 0$ and $V_1 = \hat{C}_3 \cup \hat{C}_4 \cup \hat{C}_5 \cup \hat{C}_6$ and $V_2 = \hat{C}_1 \cup \hat{C}_2$ for $h_f \leq 0$. Hence by (41) and (42) we have

$$(43) \quad \det(\text{lin}(H_f^{-1}|_{V_1})) = 1 - h_f \quad \text{and} \quad \det(\text{lin}(H_f^{-1}|_{V_2})) = 1 + h_f$$

Thus H_f satisfies (iii).

The norm of the linear part. We will now prove that $\|\text{lin}(H_f)|_{C_i}\| < \frac{5}{4}$ for $i = 1, \dots, 6$. Note that each of the matrices $\text{lin}(H_f)|_{C_i}$ is of the form $M = \begin{pmatrix} 1+b & c \\ d & 1+e \end{pmatrix}$, where $|b|, |c|, |d|, |e| < 3\sqrt{\hat{\varepsilon}}$. Hence, using the formula (28) and the fact that $\varepsilon < 10^{-8}$, we obtain that

$$\begin{aligned}
& \|\text{lin}(H_f)|_{C_i}\| < \\
(44) \quad & < \sqrt{\frac{2(1+3\sqrt{\hat{\varepsilon}}) + 2 \cdot 3\sqrt{\hat{\varepsilon}} + \sqrt{(2(1+3\sqrt{\hat{\varepsilon}}) + 2 \cdot 3\sqrt{\hat{\varepsilon}})^2 - 4(\det(\text{lin}(H_f)|_{C_i}))^2}}{2}} \\
& < \sqrt{\frac{2 + 12\sqrt{\hat{\varepsilon}} + 36\hat{\varepsilon} + \sqrt{(2 + 12\sqrt{\hat{\varepsilon}} + 36\hat{\varepsilon})^2 - 4(\frac{1}{1+\hat{\varepsilon}})^2}}{2}} \\
& < \sqrt{1 + 5\sqrt[4]{\hat{\varepsilon}}} < \frac{5}{4}.
\end{aligned}$$

In the same way we prove that $\|\text{lin}(H_f)^{-1}|_{C_i}\| < \frac{5}{4}$. Thus H_f satisfies (iv).

Continuity of $f \mapsto H_f$. Suppose that $f, g \in L^1(V)$ satisfy (38). By Lemma 5.2, we already know that

$$(45) \quad d_{\text{Hom}}(H_f, H_g) \leq \frac{20}{\sqrt{\hat{\varepsilon}}} a |h_f - h_g|.$$

We prove that

$$(46) \quad |h_f - h_g| \leq C \|f - g\|_{L^1},$$

for some constant $C > 0$ depending only on a and κ . Let $\delta := \|f - g\|_{L^1}$.

Case h_f and h_g have the same sign. Assume that $h_f \geq h_g \geq 0$ or $0 \geq h_g \geq h_f$. Then

$$\begin{aligned}
0 &= \mu_f(C_1^f \cup C_2^f) - \mu_g(C_1^g \cup C_2^g) \\
&= \int_{C_1^f \cup C_2^f} f(x) dx - \int_{C_1^g \cup C_2^g} g(x) dx \\
&= \int_{C_1^g \cup C_2^g} (f - g)(x) dx - \int_{(C_1^g \cup C_2^g) \setminus (C_1^f \cup C_2^f)} g(x) dx \\
&\leq \delta - |h_f - h_g| \frac{a\kappa}{2},
\end{aligned}$$

and hence

$$|h_f - h_g| \leq \frac{2\delta}{a\kappa}.$$

Thus (46) holds with $C = \frac{2}{a\kappa}$.

Case of h_f, h_g with different sign. Suppose that $h_f \geq 0 \geq h_g$. Then we have

$$\begin{aligned}
0 &= \mu_f(C_1^f \cup C_2^f) - \mu_g(C_3^g \cup C_4^g \cup C_5^g \cup C_6^g) \\
&= \int_{C_1^f \cup C_2^f} f(x) dx - \int_{C_3^g \cup C_4^g \cup C_5^g \cup C_6^g} g(x) dx \\
&= \int_{C_1^f \cup C_2^f} (f - g)(x) dx - \int_{(C_3^g \cup C_4^g \cup C_5^g \cup C_6^g) \setminus (C_1^f \cup C_2^f)} g(x) dx \\
&\leq \delta - |h_f - h_g| \frac{a\kappa}{2}.
\end{aligned}$$

Thus we have

$$0 \leq h_f - h_g \leq \frac{2\delta}{a\kappa},$$

which completes the proof of (46).

By combining (45) and (46) we obtain

$$d_{\text{Hom}}(H_f, H_g) \leq \frac{10}{\sqrt{\hat{\varepsilon}}} |h_f - h_g| < \frac{10}{\sqrt{\hat{\varepsilon}}} C \|f - g\|_{L^1}.$$

This concludes the proof of the continuity of $f \mapsto H_f$. By the formula given in (43), the continuity of the map $f \mapsto h_f$ also implies the continuity of the map $f \mapsto \det(DH_f^{-1}) \in L^\infty(V)$. Thus (v) is proved. \square

Let (X, μ) be a standard metric probability space. For $0 < s_1 < s_2$, define $\mathcal{W}(X, s_1, s_2) \subset L^1(X, \mu)$ by

$$(47) \quad \mathcal{W}(X, s_1, s_2) := \{f \in L^1(X); s_1 < f < s_2; \int_X f d\mu(x) = \mu(X)\}.$$

Let V be the triangle with vertices $(0, -a), (0, a), (a, 0)$, equipped with the (normalized) 2-dimensional Lebesgue measure. We need the following lemma.

Lemma 5.4. *Let $H : \mathcal{W}(V, s_1, s_2) \rightarrow \text{Hom}(V)$ be a continuous map. Assume that there exists $\ell > 0$ such that, for every $f \in \mathcal{W}(V, s_1, s_2)$, the homeomorphism $H(f)^{-1}$ is Lipschitz with constant ℓ . Then the transformation*

$$W(s_1, s_2) \ni f \mapsto f \circ H(f) \in L^1(V)$$

is continuous.

Proof. Take $f \in \mathcal{W}(V, s_1, s_2)$ and $\epsilon > 0$. Let $f_\epsilon : V \rightarrow \mathbb{R}$ be a uniformly continuous function such that $\|f_\epsilon - f\|_{L^1} < \epsilon$. Let $0 < \delta < \epsilon$ be such that

$$(48) \quad \|x - y\| < \delta \Rightarrow |f_\epsilon(x) - f_\epsilon(y)| < \epsilon.$$

Consider $0 < \delta' < \epsilon$ such that for every $g \in W(V, s_1, s_2)$ we have

$$(49) \quad \|f - g\|_{L^1} < \delta' \Rightarrow d_{\text{Hom}}(H(f), H(g)) < \delta,$$

and let $g \in W(V, s_1, s_2)$ be such that $\|f - g\|_{L^1} < \delta'$. Since $H(g)^{-1}$ is Lipschitz with constant ℓ , $H(g)_* \text{Leb}$ is an absolutely continuous measure with density bounded by ℓ^2 . Hence for every $h \in L^1(V)$ we have

$$(50) \quad \|h \circ H(g)\|_{L^1} = \int_V |h \circ H(g)(x)| dx \leq \int_V \ell^2 |h(x)| dx = \ell^2 \|h\|_{L^1}.$$

Then

$$\|f \circ H(f) - g \circ H(g)\|_{L^1} \leq \|f \circ H(f) - f \circ H(g)\|_{L^1} + \|f \circ H(g) - g \circ H(g)\|_{L^1}$$

and, by (50),

$$\|f \circ H(g) - g \circ H(g)\|_{L^1} \leq \ell^2 \|f - g\|_{L^1}.$$

Moreover

$$\begin{aligned} \|f \circ H(f) - f \circ H(g)\|_{L^1} &\leq \|f \circ H(f) - f_\epsilon \circ H(f)\|_{L^1} + \|f_\epsilon \circ H(f) - f_\epsilon \circ H(g)\|_{L^1} \\ &\quad + \|f_\epsilon \circ H(g) - f \circ H(g)\|_{L^1} \\ &\leq 2\ell^2 \|f - f_\epsilon\|_{L^1} + \|f_\epsilon \circ H(f) - f_\epsilon \circ H(g)\|_{L^1}, \end{aligned}$$

where the last inequality comes from (50). By (49) and (48), we have

$$\|f_\epsilon \circ H(f) - f_\epsilon \circ H(g)\|_{L^1} < \epsilon.$$

To sum up we obtain

$$\|f \circ H(f) - g \circ H(g)\|_{L^1} \leq \ell^2 \|f - g\|_{L^1} + 2\ell^2 \|f - f_\epsilon\|_{L^1} + \epsilon \leq (3\ell^2 + 1)\epsilon,$$

which proves the assertion. \square

Remark 5.5. The statement of Lemma 5.4 remains valid if we replace V with any 2-dimensional Riemannian surface M .

Lemma 5.6. *Let $0 < \hat{\epsilon} < 10^{-8}$. Let $f \in \mathcal{W}(V, \frac{1}{1+\hat{\epsilon}}, \frac{1}{1-\hat{\epsilon}})$ and $\mu_f := f dx$. Then there exists a homeomorphism $H_f : V \rightarrow V$, depending continuously on f , such that $(H_f)_* \mu_f = \text{Leb}$ and $H_f|_{\partial V} = \text{Id}|_{\partial V}$.*

Proof. We assume that $a = 1$. The prove for $a \neq 1$ goes along the same lines. Let $f \in \mathcal{W}(V, \frac{1}{1+\varepsilon}, \frac{1}{1-\varepsilon})$. Denote by V_1^1 and V_2^1 the two halves of V which are both isosceles right triangles with $\text{diam}(V_1^1) = \text{diam}(V_2^1) = \sqrt{2}$.

Inductively, for $n \in \mathbb{N}$ define the family $\{V_i^n\}_{i=1}^{2^n}$ of congruent right isosceles triangles which divide V , $V_i^n = V_{2i-1}^{n+1} \cup V_{2i}^{n+1}$ for $i = 1, \dots, 2^n$ and they satisfy

$$(51) \quad \text{diam}(V_n^i) = \frac{1}{\sqrt{2}^{n-2}}.$$

We will construct H_f inductively as a limit of piecewise affine transformations.

In the first step, using Lemma 5.3, we obtain a piecewise affine homeomorphism $H_f^1 : V \rightarrow V$ such that

$$(H_f^1)_* \mu_f(V_1^1) = (H_f^1)_* \mu_f(V_2^1) = \frac{1}{2} \text{Leb}(V) \text{ and } H_f^1|_{\partial V} = \text{Id}|_{\partial V}.$$

Moreover $\det(D(H_f^1)^{-1})$ is constant on each V_1^1 and V_2^1 .

Suppose now that for some $n \in \mathbb{N}$ we have constructed piecewise affine homeomorphisms $H_f^j : V \rightarrow V$ for $j = 1, \dots, n$ such that for all $i = 1, \dots, 2^n$ we have

$$(52) \quad (H_f^n \circ \dots \circ H_f^1)_* \mu_f(V_i^n) = \frac{1}{2^n} \text{Leb}(V) = \text{Leb}(V_i^n) \text{ and } H_f^j|_{\partial V} = \text{Id}|_{\partial V}.$$

Moreover, suppose that $\det(D(H_f^n \circ \dots \circ H_f^1)^{-1})$ is constant on each V_i^n and equals $d_i^n > 0$.

With these assumptions the measure $(H_f^n \circ \dots \circ H_f^1)_* \mu_f$ is absolutely continuous and its density $f_n : V \rightarrow \mathbb{R}_{>0}$ satisfies

$$f_n(x) = d_i^n \cdot f((H_f^n \circ \dots \circ H_f^1)^{-1}x) \text{ if } x \in V_i^n,$$

and by (52)

$$\int_{V_i^n} f_n(x) dx = (H_f^n \circ \dots \circ H_f^1)_* \mu_f(V_i^n) = \text{Leb}(V_i^n).$$

Take any $1 \leq i \leq 2^n$. In view of (47), if $d_i^n < 1$ then

$$f_n(x) < \frac{d_i^n}{1-\varepsilon} < \frac{1}{1-\varepsilon} \text{ for all } x \in V_i^n$$

and if $d_i^n \geq 1$ then

$$f_n(x) > \frac{d_i^n}{1+\varepsilon} \geq \frac{1}{1+\varepsilon} \text{ for all } x \in V_i^n.$$

It follows that $f_n : V_i^n \rightarrow \mathbb{R}_{>0}$ is a positive density satisfying (38) with $\kappa = \frac{d_i^n}{1+\varepsilon}$. Hence we can apply Lemma 5.3 to the triangle V_i^n and the density function $f_n : V_i^n \rightarrow \mathbb{R}_{>0}$, thus obtaining a piecewise affine homeomorphism $H_f^{n+1,i} : V_i^n \rightarrow V_i^n$ such that

$$(53) \quad (H_f^{n+1,i})_* (\mu_{f_n}|_{V_i^n})(V_{2i-1}^{n+1}) = (H_f^{n+1,i})_* (\mu_{f_n}|_{V_i^n})(V_{2i}^{n+1}) = \frac{1}{2} \text{Leb}(V_i^n) = \frac{1}{2^{n+1}} \text{Leb}(V),$$

$$(54) \quad H_f^{n+1,i}|_{\partial V_i^n} = \text{Id}|_{\partial V_i^n},$$

and

$$(55) \quad \det D((H_f^{n+1,i})^{-1}) \text{ is constant on both } V_{2i-1}^{n+1} \text{ and } V_{2i}^{n+1}.$$

Finally we define a piecewise affine homeomorphism $H_f^{n+1} : V \rightarrow V$ given by

$$H_f^{n+1}(x) := H_f^{n+1,i}(x) \text{ whenever } x \in V_i^n.$$

Then $H_f^{n+1}(V_i^n) = V_i^n$ and, by (54), we have $H_f^{n+1}|_{\partial V} = \text{Id}|_{\partial V}$. Moreover, by (53),

$$(H_f^{n+1} \circ \dots \circ H_f^1)_* \mu_f(V_{2i-1}^{n+1}) = (H_f^{n+1} \circ \dots \circ H_f^1)_* \mu_f(V_{2i}^{n+1}) = \frac{1}{2^{n+1}} \text{Leb}(V).$$

In view of (55), $\det(D(H_f^{n+1})^{-1})$ is constant on each V_j^{n+1} for $j = 1, \dots, 2^{n+1}$ and then so is $\det(D(H_f^{n+1} \circ \dots \circ H_f^1)^{-1})$. Thus, we have proved that H_f^{n+1} satisfies the assumptions of the induction.

Note that, by (iii) in Lemma 5.3, we have

$$(56) \quad 1 - \hat{\varepsilon} < \det(D(H_f^n)^{-1}) < 1 + \hat{\varepsilon} \text{ for every } n \in \mathbb{N},$$

and since $(H_f^j)^{-1}$ are piecewise linear homeomorphisms, it follows that

$$(57) \quad (1 - \hat{\varepsilon})^n \leq \det(D(H_f^n \circ \dots \circ H_f^1)^{-1}) \leq (1 + \hat{\varepsilon})^n \text{ almost everywhere}$$

and the above inequalities do not depend on f .

We now show that

$$(58) \quad H_f(x) := \lim_{n \rightarrow \infty} H_f^n \circ \dots \circ H_f^1(x)$$

yields a homeomorphism $H_f : V \rightarrow V$. First note that

$$(59) \quad H_f^m(V_i^n) = V_i^n \text{ for } i = 1, \dots, 2^n \text{ and } m > n.$$

Moreover, by (51) we have

$$(60) \quad \max_{i=1, \dots, 2^n} \text{diam}(V_i^n) \rightarrow 0 \text{ for } n \rightarrow \infty.$$

This implies that $\{H_f^n \circ \dots \circ H_f^1\}_{n \in \mathbb{N}}$ is a Cauchy sequence. Indeed, for any $\epsilon > 0$ by (60) we can choose $N \in \mathbb{N}$ such that $\max_{i=1, \dots, 2^N} \text{diam}(V_i^N) < \epsilon$. Moreover, by (59), for all $m, n \geq N$ we have

$$H_f^n \circ \dots \circ H_f^1(x) \in V_i^N \iff H_f^m \circ \dots \circ H_f^1(x) \in V_i^N.$$

Hence $\|H_f^n \circ \dots \circ H_f^1(x) - H_f^m \circ \dots \circ H_f^1(x)\| < \epsilon$ for all $x \in V$. It follows that the map $H_f : V \rightarrow V$ given by (58) is well defined and the convergence in (58) is uniform. This implies that H_f is continuous. Since $H_f^n|_{\partial V} = Id|_{\partial V}$ for all $n \in \mathbb{N}$, we also have $H_f|_{\partial V} = Id|_{\partial V}$.

Set $W_i^n := (H_f^n \circ \dots \circ H_f^1)^{-1}(V_i^n)$. In view of (59),

$$(61) \quad W_i^m = (H_f^m \circ \dots \circ H_f^1)^{-1}(V_i^n) \text{ for } m > n.$$

Therefore,

$$(62) \quad (H_f^n \circ \dots \circ H_f^1)^{-1}(x) \in W_i^N \iff (H_f^m \circ \dots \circ H_f^1)^{-1}(x) \in W_i^N \text{ if } m, n \geq N.$$

By (iv) in Lemma 5.3, $(H_f^n)^{-1}$ is a Lipschitz automorphism with a Lipschitz constant $\frac{5}{4}$. Thus, by (51), we have

$$\text{diam}(W_i^n) < \text{diam}(V_i^n) \left(\frac{5}{4}\right)^n = 2 \left(\frac{5}{4\sqrt{2}}\right)^n,$$

so

$$(63) \quad \max_{i=1, \dots, 2^n} \text{diam}(W_i^n) \rightarrow 0 \text{ for } n \rightarrow \infty.$$

Using (63) and (62) and repeating the same arguments as for H_f by replacing V_i^n with W_i^n , we obtain that the map $G_f : V \rightarrow V$ given by

$$G_f(x) := \lim_{n \rightarrow \infty} (H_f^n \circ \dots \circ H_f^1)^{-1}(x)$$

is well defined and continuous. We now show that $H_f \circ G_f = Id$ and $G_f \circ H_f = Id$. First note that in view of (61) and the compactness of V_i^n and W_i^n we have $H_f(W_i^n) = V_i^n$ and $G_f(V_i^n) = W_i^n$. Hence $H_f \circ G_f(V_i^n) = V_i^n$ and $G_f \circ H_f(W_i^n) = W_i^n$. Let $\epsilon > 0$ and $N \in \mathbb{N}$ be such that

$$\max_{i=1, \dots, 2^N} \text{diam}(V_i^N) < \epsilon \text{ and } \max_{i=1, \dots, 2^N} \text{diam}(W_i^N) < \epsilon.$$

This implies that

$$\|H_f(G_f(x)) - x\| < \epsilon \text{ and } \|G_f(H_f(x)) - x\| < \epsilon \text{ for every } x \in V.$$

Since ϵ was arbitrary, this shows that $H_f \circ G_f = Id$ and $G_f \circ H_f = Id$. Thus H_f is a homeomorphism.

Note that the family of sets $\{V_i^n; n \in \mathbb{N}, i = 1, \dots, 2^n\}$ generates the Borel σ -algebra on V . Since $(H_f)^{-1}(V_i^n) = W_i^n = (H_f^n \circ \dots \circ H_f^1)^{-1}(V_i^n)$, by (52), we have

$$(H_f)_* \mu_f(V_i^n) = (H_f^n \circ \dots \circ H_f^1)_* \mu_f(V_i^n) = \text{Leb}(V_i^n).$$

It follows that $(H_f)_* \mu_f = \text{Leb}$.

In the remainder of the proof we will show that H_f depends continuously on f . Fix $\epsilon > 0$ and then choose $m \in \mathbb{N}$ such that $2(5/4\sqrt{2})^m < \epsilon/3$. Then

$$\max_{i=1, \dots, 2^m} \text{diam}(V_i^m) < \frac{\epsilon}{3} \quad \text{and} \quad \max_{i=1, \dots, 2^m} \text{diam}(W_i^m) < \frac{\epsilon}{3}.$$

Since $(H_f)^{-1}(V_i^m) = W_i^m = (H_f^m \circ \dots \circ H_f^1)^{-1}(V_i^m)$, it follows that

$$\sup_{x \in V} \|H_f(x) - H_f^m \circ \dots \circ H_f^1(x)\| < \frac{\epsilon}{3} \quad \text{and} \quad \sup_{x \in V} \|(H_f)^{-1}(x) - (H_f^m \circ \dots \circ H_f^1)^{-1}(x)\| < \frac{\epsilon}{3}.$$

Therefore

$$(64) \quad d_{Hom}(H_f, H_f^m \circ \dots \circ H_f^1) < \frac{\epsilon}{3} \quad \text{for every } f \in \mathcal{W}(V, \frac{1}{1+\epsilon}, \frac{1}{1-\epsilon}).$$

By (v) in Lemma 5.3, the maps $f \mapsto H_f^1 \in Hom(V)$ and $f \mapsto \det D(H_f^1)^{-1} \in L^\infty(V)$ are continuous. Suppose now that for $k \geq 1$ we proved that

$$(65) \quad f \mapsto H_f^k \circ \dots \circ H_f^1 \in Hom(V) \quad \text{and} \quad f \mapsto \det D(H_f^k \circ \dots \circ H_f^1)^{-1} \in L^\infty(V)$$

are continuous. We now prove that

$$f \mapsto H_f^{k+1} \circ \dots \circ H_f^1 \quad \text{and} \quad f \mapsto \det D(H_f^{k+1} \circ \dots \circ H_f^1)^{-1}$$

are also continuous. Since for every $i = 1, \dots, k$, H_f^i and $(H_f^i)^{-1}$ are Lipschitz homeomorphisms with constant $\frac{5}{4}$, we get

$$(66) \quad H_f^k \circ \dots \circ H_f^1 \quad \text{and} \quad (H_f^k \circ \dots \circ H_f^1)^{-1} \quad \text{are Lipschitz with constant } (\frac{5}{4})^k.$$

Moreover by (57) we have

$$f_k = \det D(H_f^k \circ \dots \circ H_f^1)^{-1} \cdot \left(f \circ (H_f^k \circ \dots \circ H_f^1)^{-1} \right) \in \mathcal{W}(V, \frac{(1-\epsilon)^k}{1+\epsilon}, \frac{(1+\epsilon)^k}{1-\epsilon}).$$

In view of Lemma 5.3, H_f^{k+1} depends continuously on f_k . By (65) and (66), Lemma 5.4 implies that $f \mapsto f \circ (H_f^k \circ \dots \circ H_f^1)^{-1} \in L^1(V)$ is continuous. Together with (65) this gives the continuity of

$$f \mapsto f_k = f \circ (H_f^k \circ \dots \circ H_f^1)^{-1} \cdot \det D(H_f^k \circ \dots \circ H_f^1)^{-1} \in L^1(V).$$

It follows that H_f^{k+1} depends continuously on f .

Again, since H_f^i and $(H_f^i)^{-1}$ are Lipschitz with constant $\frac{5}{4}$, for any $x \in V$ and $f, g \in \mathcal{W}(V, \frac{1}{1+\epsilon}, \frac{1}{1-\epsilon})$ we have

$$\begin{aligned} & \|H_f^{k+1}(H_f^k \circ \dots \circ H_f^1(x)) - H_g^{k+1}(H_g^k \circ \dots \circ H_g^1(x))\| \\ & \leq \|H_f^{k+1}(H_f^k \circ \dots \circ H_f^1(x)) - H_g^{k+1}(H_f^k \circ \dots \circ H_f^1(x))\| \\ & \quad + \|H_g^{k+1}(H_f^k \circ \dots \circ H_f^1(x)) - H_g^{k+1}(H_g^k \circ \dots \circ H_g^1(x))\| \\ & \leq d_{Hom}(H_f^{k+1}, H_g^{k+1}) + \frac{5}{4} d_{Hom}(H_f^k \circ \dots \circ H_f^1, H_g^k \circ \dots \circ H_g^1) \end{aligned}$$

and similarly

$$\begin{aligned} & \|(H_f^k \circ \dots \circ H_f^1)^{-1}(H_f^{k+1})^{-1}(x) - (H_g^k \circ \dots \circ H_g^1)^{-1}(H_g^{k+1})^{-1}(x)\| \\ & \leq \left(\frac{5}{4}\right)^k d_{Hom}(H_f^{k+1}, H_g^{k+1}) + d_{Hom}(H_f^k \circ \dots \circ H_f^1, H_g^k \circ \dots \circ H_g^1). \end{aligned}$$

This proves the continuous dependence of $H_f^{k+1} \circ \dots \circ H_f^1$ on f . Finally, since H_f^i are piecewise linear homeomorphisms, we have

$$D(H_f^{k+1} \circ \dots \circ H_f^1)^{-1} = D(H_f^{k+1})^{-1} D(H_f^k \circ \dots \circ H_f^1)^{-1} \text{ almost everywhere.}$$

By (v) in Lemma 5.3, $f_k \mapsto \det D(H_f^{k+1})^{-1} \in L^\infty(V)$ depends continuously on f_k . Since $f \mapsto f_k \in L^1(V)$ is continuous, it follows that $f \mapsto \det D(H_f^{k+1})^{-1} \in L^\infty(V)$ is also continuous. The uniform boundaries in (56) and in (57), together with (65), yield the continuity of $f \mapsto \det D(H_f^{k+1} \circ \dots \circ H_f^1)^{-1} \in L^\infty(V)$.

Fix any $f \in \mathcal{W}(V, \frac{1}{1+\hat{\varepsilon}}, \frac{1}{1-\hat{\varepsilon}})$. Then there exists $\delta > 0$ such that for any $g \in \mathcal{W}(V, \frac{1}{1+\hat{\varepsilon}}, \frac{1}{1-\hat{\varepsilon}})$ with $\|f - g\|_{L^1} < \delta$ we have

$$d_{Hom}(H_f^m \circ \dots \circ H_f^1, H_g^m \circ \dots \circ H_g^1) < \frac{\epsilon}{3}.$$

Combining this with (64) we obtain

$$d_{Hom}(H_f, H_g) < \epsilon,$$

which concludes the proof of the continuity of $f \mapsto H_f$. \square

Remark 5.7. Note that the above lemma is also valid for any triangle, since every two triangles are conjugated by an affine map (although the restriction on $\hat{\varepsilon}$ may vary).

The following theorem is the main result of this section.

Theorem 5.8. *Let (M, Σ, ζ) be a translation surface. There exists $\varepsilon_\zeta = \varepsilon > 0$ such that, for all $f \in \mathcal{W}(M, \frac{1}{1+\varepsilon}, \frac{1}{1-\varepsilon})$, there exists a homeomorphism $\mathcal{H}_f : (M, \Sigma) \rightarrow (M, \Sigma)$ such that $(\mathcal{H}_f)_* \mu_f = \lambda_\zeta$, where $\mu_f := f \lambda_\zeta$. Moreover, \mathcal{H}_f depends continuously on f .*

Proof. On (M, Σ) consider a triangulation of $m + 1$ triangles such that the set of vertices of this triangulation contains the set Σ . By connectedness, there is an ordering $\{U_i : 0 \leq i \leq m\}$ of the triangles such that, for every $i = 1, \dots, m$, the triangle U_i has a common edge with some $U_{k(i)}$ for $0 \leq k(i) < i$. Indeed, choose any triangle U_0 from the triangulation. Next choose any neighbouring triangle as U_1 and set $k(1) = 0$. Now suppose that for some $1 \leq \ell \leq m$ we have chosen triangles $\{U_i : 0 \leq i \leq \ell\}$ such that $k(i) < i$ for $1 \leq i \leq \ell$. If $\ell = m$ then the process is over. If $\ell < m$ then choose as $U_{\ell+1}$ any triangle that has a common boundary with $\bigcup_{i=0}^{\ell} U_i$. This triangle exists by connectedness. Finally, let $0 \leq k(\ell+1) \leq \ell$ be such that $U_{\ell+1}$ has a common edge with $U_{k(\ell+1)}$.

For every $i = 1, \dots, m$ consider a small isosceles right triangle $W_i \subset U_i \cup U_{k(i)}$ such that its shortest height lies on the common edge of U_i and $U_{k(i)}$ and $W_i \cap W_j = \emptyset$ whenever $i \neq j$. Furthermore, we assume that each of the triangles W_i is of the same size and we choose a parametrization such that W_i has vertices in points $(0, -a), (0, a)$ and $(a, 0)$, with $(0, -a) \in U_i$ for each $i = 1, \dots, m$ (see Fig. 2).

Let $B = [b_{ij}]_{i,j \in \{1, \dots, m\}}$ be the matrix given by

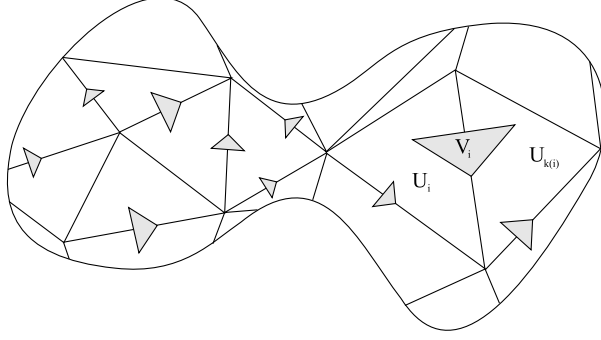
$$b_{ij} = \begin{cases} 1 & \text{if } i = j; \\ -1 & \text{if } i = k(j); \\ 0 & \text{otherwise.} \end{cases}$$

For each positive $f \in \mathcal{W}(M, \frac{1}{1+\varepsilon}, \frac{1}{1-\varepsilon})$, denote by $v^f \in \mathbb{R}^m$ the solution of the following system of linear equations

$$(67) \quad Bv^f = [\lambda_\zeta(U_i) - \mu_f(U_i)]_{i=1, \dots, m}.$$

Then $f \mapsto v^f$ is continuous. Let $\hat{\varepsilon} > 0$ be such that Lemma 5.6 can be applied to any triangle U_i (it exists due to Remark 5.7). Observe that if f is constant equal to 1, then $v^f = (0, \dots, 0)$. By the continuity of $f \mapsto v^f$ we can choose $0 < \varepsilon < \frac{\hat{\varepsilon}}{3}$ such that

$$(68) \quad |v_i^f| < \frac{a^2 \hat{\varepsilon}}{12}$$

FIGURE 2. A triangulation of the surface M and a choice of small triangles.

for every $f \in \mathcal{W}(M, \frac{1}{1+\varepsilon}, \frac{1}{1-\varepsilon})$.

Let $f \in \mathcal{W}(M, \frac{1}{1+\varepsilon}, \frac{1}{1-\varepsilon})$. We now construct a family of piecewise affine homeomorphisms $G_f^i : W_i \rightarrow W_i$, $i = 1, \dots, m$, such that G_f^i depends continuously on f , $G_f^i|_{\partial W_i} = Id$,

$$(69) \quad 1 - \frac{\hat{\varepsilon}}{3} < \det(D(G_f^i)^{-1}(x)) < 1 + \frac{\hat{\varepsilon}}{3} \text{ whenever } D(G_f^i)^{-1}(x) \text{ is well defined,}$$

and finally we require that the homeomorphism $\mathcal{G}_f : M \rightarrow M$ defined by

$$\mathcal{G}_f(x) := \begin{cases} G_f^i(x) & \text{if } x \in W_i \text{ for some } i = 1, \dots, m, \\ x & \text{otherwise,} \end{cases}$$

satisfies

$$(70) \quad (\mathcal{G}_f)_* \mu_f(U_j) = \lambda_\zeta(U_j) \text{ for all } 0 \leq j \leq m.$$

First note that for each $i = 1, \dots, m$ we can choose $-1 < h_f^i < 1$ such that the quadrilateral $Q_f^i \subset W_i$ with vertices in points $(0, 0)$, $(0, -a)$, $(a, 0)$ and $y_f^i := (\sqrt{\varepsilon}a(1 - |h_f^i|), h_f^i a)$ has measure μ_f equal to $\mu_f(W_i \cap U_i) + v_i^f$. Indeed,

$$\mu_f(W_i \cap U_i) \geq \frac{1}{1+\varepsilon} \lambda_\zeta(W_i \cap U_i) = \frac{a^2}{2(1+\varepsilon)} \geq \frac{a^2}{4} > |v_i^f|$$

and analogously

$$\mu_f(W_i \cap U_{k(i)}) > |v_i^f|.$$

Thus

$$0 < \mu_f(W_i \cap U_i) + v_i^f < \mu_f(W_i),$$

which, together with the absolute continuity of μ_f , yields the existence of h_f^i for $i = 1, \dots, m$.

We now estimate $|h_f^i|$. Since $|v_i^f|$ is the μ_f measure of the triangle with vertices $(0, 0)$, $(a, 0)$ and y_f^i in W_i and $a|h_f^i|$ is its height, we have

$$|v_i^f| > \frac{1}{1+\varepsilon} \frac{a^2 |h_f^i|}{2}.$$

Hence, by (68),

$$(71) \quad |h_f^i| < \frac{2(1+\varepsilon)|v_i^f|}{a^2} < \frac{(1+\varepsilon)\hat{\varepsilon}}{6} < \frac{\hat{\varepsilon}}{3}.$$

Let $G_f^i : W_i \rightarrow W_i$ be given by $G_f^i := H(h_f^i, \sqrt{\varepsilon})$ for $i = 1, \dots, m$ as in (30). Since

$$(G_f^i)^{-1}(W_i \cap U_i) = Q_f^i,$$

we have

$$(\mathcal{G}_f)_*\mu_f(W_i \cap U_i) = \mu_f(Q_f^i) = \mu_f(W_i \cap U_i) + v_i.$$

Analogously

$$(\mathcal{G}_f)_*\mu_f(W_i \cap U_{k(i)}) = \mu_f(W_i \setminus Q_f^i) = \mu_f(W_i) - (\mu_f(W_i \cap U_i) + v_i) = \mu_f(W_i \cap U_{k(i)}) - v_i.$$

By the definition of v_i^f and by the fact that W_i and W_j are disjoint for $i \neq j$, we have for $i = 1, \dots, m$

$$\begin{aligned} (\mathcal{G}_f)_*(\mu_f)(U_i) &= (\mathcal{G}_f)_*(\mu_f) \left(U_i \setminus (W_i \cup \bigcup_{\{j:k(j)=i\}} W_j) \right) \\ &\quad + (\mathcal{G}_f)_*(\mu_f)(W_i \cap U_i) + \sum_{\{j:k(j)=i\}} (\mathcal{G}_f)_*(\mu_f)(W_j \cap U_i) \\ &= \mu_f \left(U_i \setminus (W_i \cup \bigcup_{\{j:k(j)=i\}} W_j) \right) + (\mu_f(W_i \cap U_i) + v_i^f) + \sum_{\{j:k(j)=i\}} (\mu_f(W_j \cap U_i) - v_j^f) \\ &= \mu_f(U_i) + v_i^f - \sum_{\{j:k(j)=i\}} v_j^f = \mu_f(U_i) + (Bv^f)_i = \lambda_\zeta(U_i). \end{aligned}$$

Since $(G_f^i)_*(\mu_f)(W_i) = \mu_f(W_i)$, we also have

$$(\mathcal{G}_f)_*(\mu_f)(M) = \mu_f(M) = \lambda_\zeta(M),$$

and thus

$$(\mathcal{G}_f)_*(\mu_f)(U_0) = \lambda_\zeta(U_0).$$

Hence (70) is satisfied. Moreover (71) together with (32) and (33) yield (69). What is left to prove is that G_f^i depends continuously on f . By Lemma 5.2, we only need to prove that h_f^i depends continuously on f .

Take $f \in \mathcal{W}(M, \frac{1}{1+\varepsilon}, \frac{1}{1-\varepsilon})$, and let $\delta > 0$. Since v^g depends continuously on g , there exists $0 < \delta' \leq \delta$ such that, for any $g \in \mathcal{W}(M, \frac{1}{1+\varepsilon}, \frac{1}{1-\varepsilon})$,

$$\|f - g\|_{L^1} < \delta' \implies \max_{i=1, \dots, m} |v_i^f - v_i^g| < \delta.$$

We now evaluate the difference between the respective Lebesgue measures of the quadrilaterals given by $(0, 0)$, $(0, -a)$, $(a, 0)$, y_f^i and by $(0, 0)$, $(0, -a)$, $(a, 0)$, y_g^i :

$$\begin{aligned} \frac{a^2|h_f^i - h_g^i|}{2} &< (1 + \varepsilon)|\mu_f(W_i \cap U_i) + v_i^f - \mu_g(W_i \cap U_i) - v_i^g| + \int_{W_i} |f - g| d\lambda_\zeta \\ &\leq (1 + \varepsilon)(|\mu_f(W_i \cap U_i) - \mu_g(W_i \cap U_i)| + |v_i^f - v_i^g|) + \int_{W_i} |f - g| d\lambda_\zeta \\ &< (1 + \varepsilon) \left(\int_{W_i \cap U_i} |f - g| d\lambda_\zeta + \delta + \int_{W_i} |f - g| d\lambda_\zeta \right) \leq (1 + \varepsilon)(2\|f - g\|_{L^1} + \delta). \end{aligned}$$

Hence, for every $g \in \mathcal{W}(M, \frac{1}{1+\varepsilon}, \frac{1}{1-\varepsilon})$ such that $\|f - g\|_{L^1} < \delta'$, we have

$$|h_f^i - h_g^i| < \frac{6(1 + \varepsilon)\delta}{a^2},$$

which implies that h_f^i depends continuously on f .

Since G_f^i depends continuously on f , by the definition \mathcal{G}_f also depends continuously on f . Since $f \in \mathcal{W}(M, \frac{1}{1+\varepsilon}, \frac{1}{1-\varepsilon})$, and $\varepsilon < \frac{\hat{\varepsilon}}{3}$, by (69), $(\mathcal{G}_f)_*\mu_f$ is an absolutely continuous measure with density $\hat{f} = (f \circ \mathcal{G}_f^{-1}) \cdot \det D(\mathcal{G}_f^{-1})$ satisfying

$$\frac{1}{1 + \hat{\varepsilon}} < \frac{1 - \frac{\hat{\varepsilon}}{3}}{1 + \frac{\hat{\varepsilon}}{3}} < \hat{f} < \frac{1 + \frac{\hat{\varepsilon}}{3}}{1 - \frac{\hat{\varepsilon}}{3}} < \frac{1}{1 - \hat{\varepsilon}}$$

and

$$\int_{U_i} \hat{f} d\lambda_\zeta = (\mathcal{G}_f)_* \mu_f(U_i) = \lambda_\zeta(U_i) \text{ for every } i = 0, \dots, m.$$

Therefore, on each U_i the density \hat{f} satisfies the assumptions of Lemma 5.6. Hence, for each $i = 0, \dots, m$, there exists a homeomorphism $H_{\hat{f}}^i : U_i \rightarrow U_i$ which transports the measure $(\mathcal{G}_f)_* \mu_f|_{U_i}$ to $\lambda_\zeta|_{U_i}$, and such that $H_{\hat{f}}^i|_{\partial U_i} = Id$. Therefore, we can define a homeomorphism $H_{\hat{f}} : M \rightarrow M$ such that

$$H_{\hat{f}}(x) := H_{\hat{f}}^i(x) \text{ whenever } x \in U_i.$$

Then $(H_{\hat{f}} \circ \mathcal{G}_f)_* \mu_f = (H_{\hat{f}})_*((\mathcal{G}_f)_* \mu_f) = \lambda_\zeta$. Let $\mathcal{H}_f := H_{\hat{f}} \circ \mathcal{G}_f$. What is left to prove is that $f \mapsto \mathcal{H}_f$ is continuous.

By Lemma 5.6 $H_{\hat{f}}^i$ depend continuously on \hat{f} and hence $\hat{f} \mapsto H_{\hat{f}}$ is continuous. Moreover, (v) in Lemma 5.3 implies that $f \mapsto \det(D\mathcal{G}_f^{-1}) \in L^\infty(M)$ is continuous. Furthermore, by (iv) in Lemma 5.3, the homeomorphism $\mathcal{G}_f^{-1} : M \rightarrow M$ is Lipschitz with constant $\frac{5}{4}$. Thus, by Lemma 5.4 and Remark 5.5, $f \mapsto f \circ \mathcal{G}_f^{-1}$ is continuous. Hence

$$\mathcal{W}(M, \frac{1}{1+\varepsilon}, \frac{1}{1-\varepsilon}) \ni f \mapsto \hat{f} = (f \circ \mathcal{G}_f^{-1}) \cdot \det D(\mathcal{G}_f^{-1}) \in L^1(M)$$

is continuous and this implies the continuity of $f \mapsto H_{\hat{f}}$.

Now consider any $f \in \mathcal{W}(M, \frac{1}{1+\varepsilon}, \frac{1}{1-\varepsilon})$. Since $H_{\hat{f}} : M \rightarrow M$ is uniformly continuous, for any $\eta > 0$ we can find $0 < \delta$ such that

$$d_M(x, y) < \delta \Rightarrow d_M(H_{\hat{f}}(x), H_{\hat{f}}(y)) < \eta.$$

Then, for every $x \in M$ and any $g \in \mathcal{W}(M, \frac{1}{1+\varepsilon}, \frac{1}{1-\varepsilon})$ such that $d_{\text{Hom}}(H_{\hat{f}}, H_{\hat{g}}) < \eta$ and $d_{\text{Hom}}(\mathcal{G}_f, \mathcal{G}_g) < \delta$, we have

$$\begin{aligned} d_M(H_{\hat{f}} \circ \mathcal{G}_f(x), H_{\hat{g}} \circ \mathcal{G}_g(x)) &\leq d_M(H_{\hat{f}} \circ \mathcal{G}_f(x), H_{\hat{f}} \circ \mathcal{G}_g(x)) \\ &\quad + d_M(H_{\hat{f}} \circ \mathcal{G}_g(x), H_{\hat{g}} \circ \mathcal{G}_g(x)) < 2\eta. \end{aligned}$$

Analogously,

$$d_M((H_{\hat{f}} \circ \mathcal{G}_f)^{-1}(x), (H_{\hat{g}} \circ \mathcal{G}_g)^{-1}(x)) < 2\eta.$$

This concludes the proof of the continuity of $f \mapsto \mathcal{H}_f$ and the proof of the whole theorem. \square

6. LOCAL CONTINUOUS EMBEDDING OF THE MODULI SPACE

In this section, we finalize the construction of a continuous mappings on open subsets of a connected component of any stratum, which is needed to prove the main result of this paper. We do it in two steps.

Firstly, for each $\zeta \in \mathcal{M}(M, \Sigma, \kappa)$ we construct a neighborhood $\mathcal{U}_\zeta \subset \mathcal{M}(M, \Sigma, \kappa)$ of ζ , so that for every $\omega \in \mathcal{U}_\zeta$ there exists a piecewise affine homeomorphism $\mathfrak{h}_\omega : M \rightarrow M$ such that $(\mathfrak{h}_\omega)_* \lambda_\omega = f_\omega \lambda_\zeta$ with $\frac{1}{1+\varepsilon_\zeta} < f_\omega < \frac{1}{1-\varepsilon_\zeta}$, where $\varepsilon_\zeta > 0$ is given by Theorem 5.8. We will also require that $\omega \mapsto f_\omega \in L^1(M)$ is continuous.

Secondly, we use the results of the previous sections to show the existence of a homeomorphism $\mathcal{H}_\omega : M \rightarrow M$ such that $(\mathcal{H}_\omega \circ \mathfrak{h}_\omega)_* \lambda_\omega = \lambda_\zeta$. Moreover, we show that these homeomorphisms yield the existence of a continuous mapping $\mathfrak{S} : \mathcal{U}_\zeta \rightarrow \text{Flow}(M, \zeta)$ such that $\mathfrak{S}(\omega)$ is isomorphic by a homeomorphism to \mathcal{T}^ω - the vertical translation flow on (M, ω) .

Lemma 6.1. *Let $\zeta \in \mathcal{M}(M, \Sigma, \kappa)$. There exists a neighbourhood $\mathcal{U}_\zeta \subset \mathcal{M}(M, \Sigma, \kappa)$ such that, for every $\omega \in \mathcal{U}_\zeta$, the following holds:*

- (i) *there exists a triangulation $\mathcal{Y}(\omega)$ and a piecewise affine homeomorphism $\mathfrak{h}_\omega : (M, \omega) \rightarrow (M, \zeta)$ which is affine on elements of $\mathcal{Y}(\omega)$, fixes Σ and is Lipschitz with constant $\frac{11}{10}$,*
- (ii) *$(\mathfrak{h}_\omega)_* \lambda_\omega$ is an absolutely continuous measure with respect to λ_ζ with piecewise constant density f_ω satisfying $\frac{1}{1+\varepsilon_\zeta} < f_\omega < \frac{1}{1-\varepsilon_\zeta}$,*

(iii) the mapping $U_\zeta \ni \omega \mapsto f_\omega \in L^1(M, \lambda_\zeta)$ is continuous.

Moreover, for given $\epsilon > 0$, there exists $\delta > 0$ such that

(iv) for any $\bar{\omega} \in \mathcal{U}_\zeta$, if $d_{Mod}(\omega, \bar{\omega}) < \delta$ then $\mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}} : (M, \bar{\omega}) \rightarrow (M, \omega)$ is affine on elements of $\mathcal{Y}(\bar{\omega})$, $\mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega : (M, \omega) \rightarrow (M, \bar{\omega})$ is affine on elements of $\mathcal{Y}(\omega)$, they are both Lipschitz with constant $1 + \epsilon$,

$$\|Id - D(\mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}})|_A\| < \epsilon \text{ for every } A \in \mathcal{Y}(\bar{\omega})$$

and

$$\|Id - D(\mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega)|_B\| < \epsilon \text{ for every } B \in \mathcal{Y}(\omega),$$

(v) for any $\bar{\omega} \in \mathcal{U}_\zeta$ such that $d_{Mod}(\omega, \bar{\omega}) < \delta$ and for the set

$$\tilde{M}(\omega) := \{x \in M; \inf_{\sigma \in \Sigma} d_\omega(\mathcal{T}_t^\omega(x), \sigma) > 4\epsilon \text{ for all } t \in [-1, 1]\},$$

we have $\lambda_\omega(\tilde{M}(\omega)) > 1 - K\epsilon$, where $K > 0$ depends only on stratum, and for $x \in \tilde{M}(\omega)$ we have

$$d_\omega(\mathcal{T}_t^\omega(x), \mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}} \circ \mathcal{T}_t^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x)) < \epsilon \text{ for any } t \in [-1, 1].$$

and for every $\sigma \in \Sigma$

$$d_\omega(\sigma, \mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}} \circ \mathcal{T}_t^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x)) > 3\epsilon \text{ for any } t \in [-1, 1].$$

Proof. Let π be a permutation of the alphabet \mathcal{A} with d elements which belongs to a Rauzy class corresponding to $\mathcal{M}(M, \Sigma, \kappa)$. We can assume that ζ has no vertical saddle-connections and thus there is a polygonal representation $(\pi, \lambda^\zeta, \tau^\zeta)$ of ζ . Otherwise we can rotate ζ to obtain a form ζ' which does not admit vertical saddle connections, construct a triangulation $\mathcal{Y}(\zeta')$ and rotate it back together with this triangulation to obtain a triangulation $\mathcal{Y}(\zeta)$ (note that a rotation is an isometry and that it acts continuously on $\mathcal{M}(M, \Sigma, \kappa)$, see [23]).

Let $\omega \in \mathcal{M}(M, \Sigma, \kappa)$, and assume that $\omega = M(\pi, \lambda^\omega, \tau^\omega)$ for some $\lambda^\omega, \tau^\omega$. Let $\mathcal{P}(\omega) \subset \mathbb{C}$ be the polygon corresponding to ω , whose vertices $R_0(\omega), R_1(\omega), \dots, R_d(\omega), R'_1(\omega), \dots, R'_{d-1}(\omega)$ are given by

$$R_i(\omega) := \sum_{\{\alpha; \pi_0(\alpha) \leq i\}} (\lambda_\alpha^\omega + i\tau_\alpha^\omega) \quad \text{and} \quad R'_i(\omega) := \sum_{\{\alpha; \pi_1(\alpha) \leq i\}} (\lambda_\alpha^\omega + i\tau_\alpha^\omega) \quad \text{for } i = 0, \dots, d.$$

Note that $R_0(\omega) = R'_0(\omega) = 0$, $R_d(\omega) = R'_d(\omega)$. For $i = 1, \dots, d-1$ consider the vertical segments connecting $R_i(\omega)$ and $R'_i(\omega)$ with the opposite side of $\mathcal{P}(\omega)$. Denote the other endpoints of those segments by $Q_i(\omega)$ and $Q'_i(\omega)$ respectively (see Fig. 3). Since each side on the upper half of the polygon is identified with one of the sides on the lower half of the polygon, there exist representations of $Q_i(\omega)$ and $Q'_i(\omega)$ on the opposite half of the polygon which we denote by $S_i(\omega)$ and $S'_i(\omega)$ respectively. Note that

$$\text{Re}(S_i(\omega)) = T_{\pi, \lambda^\omega}^{-1}(\text{Re}(R_i(\omega))) \quad \text{and} \quad \text{Re}(S'_i(\omega)) = T_{\pi, \lambda^\omega}(\text{Re}(R'_i(\omega))),$$

where T_{π, λ^ω} is the IET given by (π, λ) . Let

$$\mathcal{V}(\omega) := \{R_0(\omega), \dots, R_d(\omega), R'_1(\omega), \dots, R'_{d-1}(\omega), S_1(\omega), \dots, S_{d-1}(\omega), S'_1(\omega), \dots, S'_{d-1}(\omega)\}.$$

$\mathcal{V}(\omega)$ is fully determined by $(\lambda^\omega, \tau^\omega)$. If ω has no vertical saddle connections, then for all distinct $x, y \in \mathcal{V}(\omega)$ we have $\text{Re}(x) \neq \text{Re}(y)$. Consider the sequence $\{V_j(\omega)\}_{j=0}^{4d-2}$, which is an ordering of $\mathcal{V}(\omega)$, such that $\{\text{Re}(V_j)\}_{j=0}^{4d-2}$ be an increasing sequence.

Let $\varepsilon_1 > 0$ be such that for every $\omega \in \mathcal{M}(M, \Sigma, \kappa)$ satisfying $d_{Mod}(\zeta, \omega) < \varepsilon_1$, the orderings of the sets $\mathcal{V}(\zeta)$ and $\mathcal{V}(\omega)$ are the same, that is

$$V_j(\omega) = R_i(\omega) \text{ iff } V_j(\zeta) = R_i(\zeta); \quad V_j(\omega) = R'_i(\omega) \text{ iff } V_j(\zeta) = R'_i(\zeta),$$

$$V_j(\omega) = S_i(\omega) \text{ iff } V_j(\zeta) = S_i(\zeta) \quad \text{and} \quad V_j(\omega) = S'_i(\omega) \text{ iff } V_j(\zeta) = S'_i(\zeta),$$

for every $j = 0, \dots, 2d$, and $\{\text{Re}(V_i(\omega))\}_{i=0}^d$ is strictly increasing.

For every $\omega \in \mathcal{M}(M, \Sigma, \kappa)$ with $d_{Mod}(\zeta, \omega) < \varepsilon_1$, we now construct a triangulation $\mathcal{Y}(\omega)$ of $\mathcal{P}(\omega)$. (We abuse the word ‘‘triangulation’’ since the edges may connect vertices of triangulation

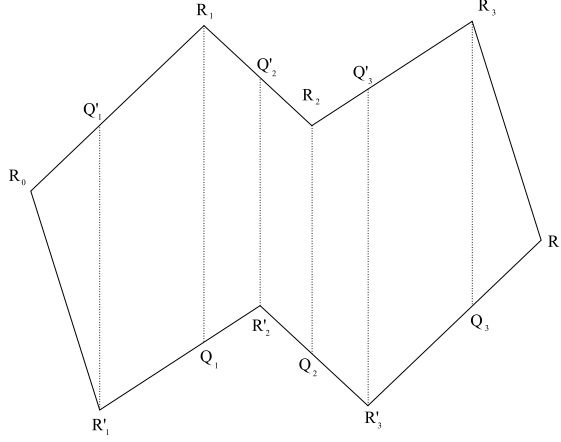


FIGURE 3. The vertices of $\mathcal{P}(\omega)$ and their projections on opposite sides.

which are actually the same points.) Let $\{r(k)\}_{0 \leq k \leq 2d}$ be the strictly increasing sequence in $\{0, \dots, 4d - 2\}$ such that $V_{r(0)}, \dots, V_{r(2d)}$ be the vertices of $\mathcal{P}(\omega)$. Let $\tilde{V}_{r(k)}(\omega) := Q_i(\omega)$ whenever $V_{r(k)}(\omega) = R_i(\omega)$ and analogously let $\tilde{V}_{r(k)}(\omega) := Q'_i(\omega)$ whenever $V_{r(k)}(\omega) = R'_i(\omega)$.

Consider the triangle given by the points $V_{r(0)} = 0, V_{r(1)}, \tilde{V}_{r(1)}$. If $r(1) = 1$, then this triangle belongs to $\mathcal{Y}(\omega)$. If $r(1) \neq 1$ and $\text{Im}(V_1(\omega)) > 0$, then we connect by segments all $V_i(\omega)$ such that $i \leq r(1)$ and $\text{Im}(V_i(\omega)) < 0$ with $V_1(\omega)$, and for all $i \leq r(1)$ such that $\text{Im}(V_i(\omega)) > 0$, we connect $V_i(\omega)$ with the point $V_{r(1)}(\omega)$ or $\tilde{V}_{r(1)}(\omega)$, whichever has the negative imaginary part. If $r(1) \neq 1$ and $\text{Im}(V_1(\omega)) < 0$, then we proceed symmetrically. The triangles obtained by using the above segments are elements of $\mathcal{Y}(\omega)$. By applying vertical reflection, we use the same construction for the triangle given by points $V_{r(2d-1)}(\omega), \tilde{V}_{r(2d-1)}(\omega), V_{r(2d)}(\omega)$.

For every $k = 1, \dots, 2d - 2$, consider the trapezoid given by $V_{r(k)}(\omega), V_{r(k+1)}(\omega), \tilde{V}_{r(k)}(\omega)$ and $\tilde{V}_{r(k+1)}(\omega)$. In each of those trapezoids, we take the diagonal connecting the top-left vertex with the bottom-right vertex. If $r(k+1) = r(k) + 1$, then the two resulting triangles belong to $\mathcal{Y}(\omega)$. If $r(k+1) \neq r(k) + 1$, then for every $r(k) < i < r(k+1)$, we connect $V_i(\omega)$ with the bottom-right vertex if $\text{Im}(V_i(\omega)) > 0$, and with the top-left vertex if $\text{Im}(V_i(\omega)) < 0$. We include the resulting triangles into $\mathcal{Y}(\omega)$. In this way we get a triangulation $\mathcal{Y}(\omega)$ of $\mathcal{P}(\omega)$ into triangles which have vertices in $\mathcal{V}(\omega)$ (see Fig. 4).

Define $\mathfrak{h}_\omega : (M, \omega) \rightarrow (M, \zeta)$ as the piecewise affine transformation, such that

$$\mathfrak{h}_\omega(V_i(\omega)) = V_i(\zeta) \text{ for every } i = 0, \dots, 4d - 2$$

and sending affinely each triangle from $\mathcal{Y}(\omega)$ with vertices $V_j(\omega), V_k(\omega), V_\ell(\omega)$ onto the triangle with vertices $V_j(\zeta), V_k(\zeta), V_\ell(\zeta)$. Note that the map \mathfrak{h}_ω is uniquely determined by the points in $\mathcal{V}(\omega)$. Moreover, since $\Sigma \subset \mathcal{V}(\omega)$, \mathfrak{h}_ω fixes Σ .

Let $0 < \varepsilon_2 < \varepsilon_1$ be such that, defining $\tilde{\mathcal{U}}_\zeta := \{\omega \in \mathcal{M}(M, \Sigma, \kappa), d_{\text{Mod}}(\zeta, \omega) < \varepsilon_2\}$, we have

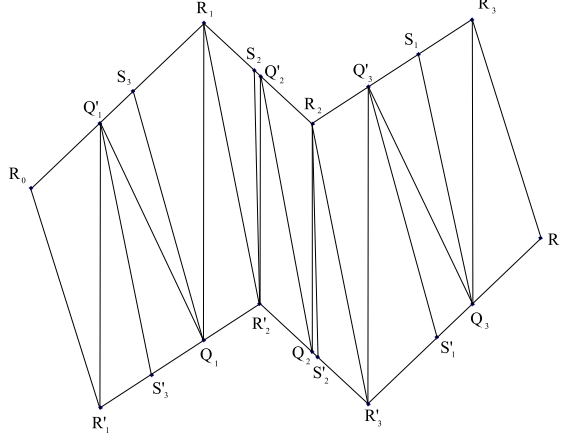
$$(72) \quad \omega \in \tilde{\mathcal{U}}_\zeta \Rightarrow 1 - \varepsilon_\zeta < \frac{\lambda_\zeta(A)}{\lambda_\omega(\mathfrak{h}_\omega^{-1}(A))} < 1 + \varepsilon_\zeta \text{ for every } A \in \mathcal{Y}(\zeta).$$

This implies that $(\mathfrak{h}_\omega)_* \lambda_\omega$ is absolutely continuous with respect to λ_ζ and has a piecewise constant density f_ω given by

$$(73) \quad f_\omega(x) = \frac{\lambda_\omega(\mathfrak{h}_\omega^{-1}(A))}{\lambda_\zeta(A)} \text{ for every } x \in A \text{ where } A \in \mathcal{Y}(\zeta).$$

Hence

$$\frac{1}{1 + \varepsilon_\zeta} < f_\omega < \frac{1}{1 - \varepsilon_\zeta}.$$

FIGURE 4. The triangulation $\mathcal{Y}(\omega)$ for the polygon in Fig. 3.

Moreover, note that for every $A \in \mathcal{Y}(\zeta)$, the mapping $\omega \rightarrow \lambda_\omega(\mathfrak{h}_\omega^{-1}(A))$ is continuous. Hence, the formula (73) implies the continuity of $\omega \mapsto f_\omega \in L^1(M, \lambda_\zeta)$.

Let $\omega \in \tilde{\mathcal{U}}_\zeta$. Then for any $\bar{\omega} \in \tilde{\mathcal{U}}_\zeta$, $\mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega$ is a continuous piecewise affine homoeomorphism which is affine on the elements of $\mathcal{Y}(\omega)$ and

$$\mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(V_i(\omega)) = V_i(\bar{\omega}).$$

Take any $A \in \mathcal{Y}(\zeta)$ and let $V_j(\zeta), V_k(\zeta), V_\ell(\zeta) \in \mathcal{V}(\zeta)$ be its vertices. Then $\text{lin}(\mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega|_{\mathfrak{h}_{\bar{\omega}}^{-1}A})$ is given by a matrix $B_A(\omega, \bar{\omega}) = [b_{ij}(\omega, \bar{\omega})]_{i,j=1,2}$, where

$$b_{1,1}(\omega, \bar{\omega}) = \frac{\text{Re}(V_k(\bar{\omega}) - V_j(\bar{\omega})) \text{Im}(V_\ell(\omega) - V_j(\omega)) - \text{Re}(V_\ell(\bar{\omega}) - V_j(\bar{\omega})) \text{Im}(V_k(\omega) - V_j(\omega))}{\text{Re}(V_k(\omega) - V_j(\omega)) \text{Im}(V_\ell(\omega) - V_j(\omega)) - \text{Re}(V_\ell(\omega) - V_j(\omega)) \text{Im}(V_k(\omega) - V_j(\omega))},$$

$$b_{1,2}(\omega, \bar{\omega}) = \frac{\text{Re}(V_\ell(\bar{\omega}) - V_j(\bar{\omega})) \text{Re}(V_k(\omega) - V_j(\omega)) - \text{Re}(V_k(\bar{\omega}) - V_j(\bar{\omega})) \text{Re}(V_\ell(\omega) - V_j(\omega))}{\text{Re}(V_k(\omega) - V_j(\omega)) \text{Im}(V_\ell(\omega) - V_j(\omega)) - \text{Re}(V_\ell(\omega) - V_j(\omega)) \text{Im}(V_k(\omega) - V_j(\omega))},$$

$$b_{2,1}(\omega, \bar{\omega}) = \frac{\text{Im}(V_k(\bar{\omega}) - V_j(\bar{\omega})) \text{Im}(V_\ell(\omega) - V_j(\omega)) - \text{Im}(V_\ell(\bar{\omega}) - V_j(\bar{\omega})) \text{Im}(V_k(\omega) - V_j(\omega))}{\text{Re}(V_k(\omega) - V_j(\omega)) \text{Im}(V_\ell(\omega) - V_j(\omega)) - \text{Re}(V_\ell(\omega) - V_j(\omega)) \text{Im}(V_k(\omega) - V_j(\omega))},$$

and

$$b_{2,2}(\omega, \bar{\omega}) = \frac{\text{Re}(V_k(\omega) - V_j(\omega)) \text{Im}(V_\ell(\bar{\omega}) - V_j(\bar{\omega})) - \text{Im} \text{Re}(V_\ell(\omega) - V_j(\omega))(V_k(\bar{\omega}) - V_j(\bar{\omega}))}{\text{Re}(V_k(\omega) - V_j(\omega)) \text{Im}(V_\ell(\omega) - V_j(\omega)) - \text{Re}(V_\ell(\omega) - V_j(\omega)) \text{Im}(V_k(\omega) - V_j(\omega))}.$$

Note that, to obtain a formula for $B_A(\bar{\omega}, \omega) = \text{lin}(\mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}}|_{\mathfrak{h}_{\bar{\omega}}^{-1}A})$, it is enough to switch ω with $\bar{\omega}$. Observe that $B_A(\omega, \omega) = Id$. Since all coefficients depend continuously on the elements of $\mathcal{V}(\bar{\omega})$ and $\mathfrak{h}_\zeta = Id$, by taking $\bar{\omega} = \zeta$ we can find $0 < \varepsilon_3 \leq \varepsilon_2$ such that for all $\omega \in \mathcal{U}_\zeta := \{\omega \in \mathcal{M}(M, \Sigma, \kappa), d_{Mod}(\zeta, \omega) < \varepsilon_3\}$, \mathfrak{h}_ω and \mathfrak{h}_ω^{-1} are Lipschitz with constant $\frac{11}{10}$. Moreover, for any $\epsilon > 0$ and any $\omega \in \mathcal{U}_\zeta$ we can find $\delta > 0$ such that for all $A \in \mathcal{Y}(\zeta)$ and for all $\bar{\omega}$ satisfying $d_{Mod}(\omega, \bar{\omega}) < \delta$, we have that $\mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega$ and $\mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}}$ are Lipschitz with constant $1 + \epsilon$ and $\|Id - B_A(\omega, \bar{\omega})\| < \epsilon$ and $\|Id - B_A(\bar{\omega}, \omega)\| < \epsilon$.

To prove (v), note that the set $\tilde{M}(\omega)$ does not contain points which are in the 4ϵ -neighbourhood of ingoing and outgoing vertical separatrix segments of length 1 starting from singular points. That is the complement $\tilde{M}(\omega)^c$ is of measure λ_ω at most $(1 + 4\epsilon)8\epsilon$ times the number of ingoing and outgoing separatrices, which determines the value of K .

For every $\bar{\omega} \in \mathcal{U}_\zeta$ denote by $X^{\bar{\omega}} : M \rightarrow \mathbb{R}^2$ the unit constant vertical vector field on $(M, \bar{\omega})$ defined on $M \setminus \Sigma$ which generates $\mathcal{T}^{\bar{\omega}}$. Then by (iv) we have that

$$(74) \quad \|X^\omega(x) - D(\mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}})|_{\mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x)} X^{\bar{\omega}}(\mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_\omega(x))\| < \epsilon.$$

Note that the vector field $D(\mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}})_{\mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x)} X^{\bar{\omega}}(\mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x))$ is well defined everywhere except on the edges of the triangulation $\mathcal{Y}(\omega)$. Since this vector field is constant on the interiors of the elements of the triangulation $\mathcal{Y}(\omega)$, we can define it on $M \setminus \Sigma$ by choosing on each edge of $\mathcal{Y}(\omega)$ one of the vectors derived from one of the two triangles forming this edge. It is worth to mention that, if the direction of an edge of $\mathcal{Y}(\bar{\omega})$ given by some triangles $\mathfrak{h}_{\bar{\omega}}^{-1}(A)$ and $\mathfrak{h}_{\bar{\omega}}^{-1}(B) \in \mathcal{Y}(\bar{\omega})$ coincides with the direction of the flow $\mathcal{T}^{\bar{\omega}}$, then the vector field $D(\mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}})_{\mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x)} X^{\bar{\omega}}(\mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x))$ takes the same values on the triangles $\mathfrak{h}_\omega^{-1}(A), \mathfrak{h}_\omega^{-1}(B) \in \mathcal{Y}(\omega)$. Hence the flow induced by this vector field is well defined and (74) holds everywhere on $M \setminus \Sigma$. Thus, in view of (74), for any $t \in [-1, 1]$ we have

$$\begin{aligned} & \left| \int_0^t \|X^\omega(\mathcal{T}_s^\omega(x)) - D(\mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}})_{\mathcal{T}_s^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x)} X^{\bar{\omega}}(\mathcal{T}_s^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x))\| ds \right| \\ &= \left| \int_0^t \|X^\omega(\mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}} \circ \mathcal{T}_s^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x)) - D(\mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}})_{\mathcal{T}_s^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x)} X^{\bar{\omega}}(\mathcal{T}_s^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x))\| ds \right| \\ &< |t| \epsilon \leq \epsilon. \end{aligned}$$

Since for every $x \in \tilde{M}(\omega)$ we have

$$\mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}} \circ \mathcal{T}_t^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x) = x + \int_0^t D(\mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}})_{\mathcal{T}_s^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x)} X^{\bar{\omega}}(\mathcal{T}_s^{\bar{\omega}}(\mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x))) ds$$

in local coordinates, we deduce that

$$\begin{aligned} & d_\omega(\mathcal{T}_t^\omega(x), \mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}} \circ \mathcal{T}_t^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x)) \\ & \leq \left| \int_0^t \|X^\omega(\mathcal{T}_s^\omega(x)) - D(\mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}})_{\mathcal{T}_s^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x)} X^{\bar{\omega}}(\mathcal{T}_s^{\bar{\omega}}(\mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x)))\| ds \right| \leq \epsilon. \end{aligned}$$

Since $x \in \tilde{M}(\omega)$, this also implies that for every $\sigma \in \Sigma$ we have

$$d_\omega(\sigma, \mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}} \circ \mathcal{T}_t^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x)) > 3\epsilon.$$

This concludes the proof of (v) and thus the proof of the whole lemma. \square

Lemma 6.2. *Let $\omega \in \mathcal{M}(M, \Sigma, \kappa)$ and let D be a rectangle in (M, ω) . For every $\epsilon > 0$ there exists $\delta > 0$ such that, for every λ_ω -measure preserving $F : D \rightarrow (M, \omega)$ satisfying*

$$(75) \quad \sup_{x \in D} d_\omega(x, F(x)) < \delta,$$

we have

$$(76) \quad \lambda_\omega(D) - \lambda_\omega(D \cap F(D)) < \epsilon.$$

Proof. Let $\epsilon > 0$. We assume that $\epsilon < \lambda_\omega(D)$, otherwise the result is obvious. Choose $\delta > 0$ such that the set

$$\hat{D} := \{x \in D; \forall y \in M, d_\omega(x, y) < \delta \Rightarrow y \in D\}$$

has measure $\lambda_\omega(\hat{D}) > \lambda_\omega(D) - \epsilon$. Then, for any λ_ω -measure preserving $F : D \rightarrow (M, \omega)$ satisfying (75), we have $F(\hat{D}) \subset D$. Hence

$$\lambda_\omega(D \cap F(D)) \geq \lambda_\omega(F(\hat{D})) = \lambda_\omega(\hat{D}) > \lambda_\omega(D) - \epsilon.$$

\square

Theorem 6.3. *Let $\zeta \in \mathcal{M}(M, \Sigma, \kappa)$ and let \mathcal{U}_ζ be the neighbourhood given by Lemma 6.1. There exists a continuous mapping $\mathfrak{S} : \mathcal{U}_\zeta \rightarrow \text{Flow}(M, \lambda_\zeta)$ such that for every $\omega \in \mathcal{U}_\zeta$ the vertical flow on (M, ω) is measure-theoretically isomorphic by a homeomorphism to the measure-preserving flow $\mathfrak{S}(\omega)$ on (M, λ_ζ) .*

Proof. By (i) and (ii) in Lemma 6.1, for every $\omega \in \mathcal{U}_\zeta$ there exists a homeomorphism $\mathfrak{h}_\omega : M \rightarrow M$ fixing Σ and such that $(\mathfrak{h}_\omega)_*\lambda_\omega = f_\omega\lambda_\zeta$, where f_ω satisfies $\frac{1}{1+\varepsilon_\zeta} < f_\omega < \frac{1}{1-\varepsilon_\zeta}$. Hence we can apply Theorem 5.8 to obtain a homeomorphism $\mathcal{H}_\omega := \mathcal{H}_{f_\omega} : M \rightarrow M$, which depends continuously on f_ω and $(\mathcal{H}_{f_\omega})_*(f_\omega\lambda_\zeta) = \lambda_\zeta$. By (iii) in Lemma 6.1, it follows that the map $\omega \mapsto f_\omega$ is continuous. Hence $\omega \mapsto \mathcal{H}_\omega$, as a composition of two continuous mappings, is also continuous. Now define a homeomorphism of M

$$\mathcal{S}_\omega := \mathcal{H}_\omega \circ \mathfrak{h}_\omega.$$

Note that $(\mathcal{S}_\omega)_*\lambda_\omega = \lambda_\zeta$ and the flow $\mathcal{S}_\omega \circ \mathcal{T}^\omega \circ \mathcal{S}_\omega^{-1}$ is λ_ζ -measure preserving. To conclude the proof we show now that the mapping $\mathcal{U}_\zeta \ni \omega \mapsto \mathcal{S}_\omega \circ \mathcal{T}^\omega \circ \mathcal{S}_\omega^{-1} =: \mathfrak{S}(\omega) \in \text{Flow}(M, \lambda_\zeta)$ is continuous.

Fix $\omega \in \mathcal{U}_\zeta$. We now prove the continuity of \mathfrak{S} in ω . On (M, ω) choose a family \mathcal{Q} of open rectangles, with vertical and horizontal sides, that generates the Borel σ -algebra on M . We may assume that for every $Q \in \mathcal{Q}$ we have $\lambda_\omega(Q) \leq \frac{1}{4}$. Note that $\mathcal{S}_\omega^{-1} : (M, \zeta) \rightarrow (M, \omega)$ is a measure-theoretic isomorphism. Hence, in view of Remark 2.6, to prove that \mathfrak{S} is continuous, it is sufficient to prove that the map $\mathcal{U}_\zeta \ni \bar{\omega} \mapsto \mathcal{S}_\omega^{-1} \circ \mathfrak{S}(\bar{\omega})_t \circ \mathcal{S}_\omega \in \text{Flow}(X, \lambda_\omega)$ is continuous. That is for every $\varepsilon > 0$ and $Q \in \mathcal{Q}$ there exists $\delta > 0$ such that

$$(77) \quad d_{Mod}(\omega, \bar{\omega}) < \delta \Rightarrow \sup_{t \in [-1, 1]} \lambda_\omega(\mathcal{T}_t^\omega Q \triangle \mathcal{S}_\omega^{-1} \circ \mathfrak{S}(\bar{\omega})_t \circ \mathcal{S}_\omega Q) < \varepsilon.$$

Fix $Q \in \mathcal{Q}$ and $\varepsilon > 0$. We now prove (77) for Q . Denote by k the number of times the ingoing and outgoing vertical separatrix segments of length 2 starting from the singular points $\sigma \in \Sigma$ intersect with Q . By extending those segments if necessary, we obtain segments $v_j \subset Q$, for $j = 1, \dots, k$, such that the endpoints of v_j lie on a horizontal sides of Q . Let $0 < \varepsilon_Q < \varepsilon$, and consider the subset $\tilde{Q} \subseteq Q$ obtained by cutting out from Q all rectangles of which the segments v_j are vertical sides and whose width is $4\varepsilon_Q$. Assume that ε_Q is small enough so that

$$(78) \quad \lambda_\omega(\tilde{Q}) > (1 - \varepsilon)\lambda_\omega(Q).$$

Note that \tilde{Q} is a union of $l \leq k + 1$ rectangles D_j for $j = 1, \dots, l$. By Lemma 6.2, there exists $\gamma > 0$ such that for every $j = 1, \dots, l$ and every λ_ω -preserving transformation $F : D_j \rightarrow M$ satisfying $\sup_{x \in D_j} d_\omega(x, F(x)) < 4\gamma$ we have

$$(79) \quad \lambda_\omega(D_j \cap F(D_j)) > (1 - \varepsilon)\lambda_\omega(D_j).$$

Take $0 < \varepsilon < \min\{\gamma, \varepsilon_Q\}$. Since $\bar{\omega} \mapsto \mathcal{H}_{\bar{\omega}}$ is continuous, we can choose $\delta > 0$ such that for every $\bar{\omega} \in \mathcal{U}_\zeta$

$$(80) \quad d_{Mod}(\omega, \bar{\omega}) < \delta \implies \left(\sup_{x \in M} d_\zeta(\mathcal{H}_\omega^{-1} \circ \mathcal{H}_{\bar{\omega}}(x), x) < \varepsilon \quad \wedge \quad \sup_{x \in M} d_\zeta(\mathcal{H}_{\bar{\omega}}^{-1} \circ \mathcal{H}_\omega(x), x) < \varepsilon \right).$$

Moreover, by applying (iv) from Lemma 6.1 for ε and taking smaller δ if necessary, we get that $\mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega : (M, \bar{\omega}) \rightarrow (M, \omega)$ and $\mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}} : (M, \omega) \rightarrow (M, \bar{\omega})$ are Lipschitz piecewise affine homeomorphisms with constant $1 + \varepsilon$. Furthermore, (v) in Lemma 6.1 gives us that the set $\tilde{M}(\omega)$ satisfies $\lambda_\omega(\tilde{M}(\omega)) > 1 - K\varepsilon$ and for $x \in \tilde{M}(\omega)$ we have

$$(81) \quad d_\omega(\mathcal{T}_t^\omega(x), \mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}} \circ \mathcal{T}_t^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x)) < \varepsilon \text{ for any } t \in [-1, 1].$$

It also implies that, for every $\sigma \in \Sigma$,

$$(82) \quad d_\omega(\sigma, \mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}} \circ \mathcal{T}_t^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega(x)) > 3\varepsilon \text{ for any } t \in [-1, 1].$$

Since $\varepsilon < \varepsilon_Q$, we have $\tilde{Q} \subset \tilde{M}(\omega)$.

We now estimate the distance between the orbits of the flows $\mathfrak{h}_\omega^{-1} \circ \mathfrak{h}_{\bar{\omega}} \circ \mathcal{T}_t^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_\omega$ and

$$\mathcal{S}_\omega^{-1} \circ \mathcal{S}_{\bar{\omega}} \circ \mathcal{T}_t^{\bar{\omega}} \circ \mathcal{S}_{\bar{\omega}}^{-1} \circ \mathcal{S}_\omega = \mathfrak{h}_\omega^{-1} \circ \mathcal{H}_\omega^{-1} \circ \mathcal{H}_{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}} \circ \mathcal{T}_t^{\bar{\omega}} \circ \mathcal{H}_{\bar{\omega}}^{-1} \circ \mathcal{H}_\omega \circ \mathfrak{h}_\omega.$$

By (80) we have that

$$d_\zeta(\mathfrak{h}_\omega(x), \mathcal{H}_{\bar{\omega}}^{-1} \circ \mathcal{H}_\omega \circ \mathfrak{h}_\omega(x)) < \varepsilon$$

for every $x \in M$. By (i) in Lemma 6.1, $\mathfrak{h}_{\bar{\omega}}^{-1} : (M, \zeta) \rightarrow (M, \omega)$ is Lipschitz with constant $\frac{11}{10}$. Thus we have

$$d_{\bar{\omega}}(\mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_{\omega}(x), \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathcal{H}_{\bar{\omega}}^{-1} \circ \mathcal{H}_{\omega} \circ \mathfrak{h}_{\omega}(x)) < \frac{11}{10}\epsilon.$$

Since $\mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_{\omega}$ is Lipschitz with constant $1 + \epsilon$ and fixes Σ , (82) implies that

$$\min_{\sigma \in \Sigma} \inf_{t \in [-1, 1]} d_{\bar{\omega}}(\mathcal{T}_t^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_{\omega}(x), \sigma) > \frac{3\epsilon}{1 + \epsilon} > 2\epsilon$$

for every $x \in \tilde{M}(\omega)$. Hence on the 2ϵ -neighbourhood of $\mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_{\omega}(x)$, $\{\mathcal{T}_t^{\bar{\omega}}\}_{t \in [-1, 1]}$ acts isometrically. Thus

$$d_{\bar{\omega}}(\mathcal{T}_t^{\bar{\omega}}(\mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_{\omega}(x)), \mathcal{T}_t^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathcal{H}_{\bar{\omega}}^{-1} \circ \mathcal{H}_{\omega} \circ \mathfrak{h}_{\omega}(x)) < \frac{11}{10}\epsilon,$$

for $t \in [-1, 1]$ and for every $x \in \tilde{M}(\omega)$. Since $\mathfrak{h}_{\bar{\omega}} : (M, \bar{\omega}) \rightarrow (M, \zeta)$ is Lipschitz with constant $\frac{11}{10}$, this implies that

$$d_{\zeta}(\mathfrak{h}_{\bar{\omega}} \circ \mathcal{T}_t^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_{\omega}(x), \mathfrak{h}_{\bar{\omega}} \circ \mathcal{T}_t^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathcal{H}_{\bar{\omega}}^{-1} \circ \mathcal{H}_{\omega} \circ \mathfrak{h}_{\omega}(x)) < \frac{121}{100}\epsilon.$$

Again by using (80) we obtain that

$$d_{\zeta}(\mathfrak{h}_{\bar{\omega}} \circ \mathcal{T}_t^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_{\omega}(x), \mathcal{H}_{\omega}^{-1} \circ \mathcal{H}_{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}} \circ \mathcal{T}_t^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathcal{H}_{\bar{\omega}}^{-1} \circ \mathcal{H}_{\omega} \circ \mathfrak{h}_{\omega}(x)) < \frac{221}{100}\epsilon.$$

Finally, since $\mathfrak{h}_{\omega}^{-1}$ is also Lipschitz with constant $\frac{11}{10}$, we obtain that

$$d_{\omega}(\mathfrak{h}_{\omega}^{-1} \circ \mathfrak{h}_{\bar{\omega}} \circ \mathcal{T}_t^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathfrak{h}_{\omega}(x), \mathfrak{h}_{\omega}^{-1} \circ \mathcal{H}_{\omega}^{-1} \circ \mathcal{H}_{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}} \circ \mathcal{T}_t^{\bar{\omega}} \circ \mathfrak{h}_{\bar{\omega}}^{-1} \circ \mathcal{H}_{\bar{\omega}}^{-1} \circ \mathcal{H}_{\omega} \circ \mathfrak{h}_{\omega}(x)) < \frac{2431}{1000}\epsilon.$$

By combining this with (81) we obtain that for every $x \in \tilde{M}(\omega)$ we have

$$(83) \quad d_{\omega}(\mathcal{T}_t^{\omega}(x), \mathcal{S}_{\omega}^{-1} \circ \mathfrak{S}(\bar{\omega})_t \circ \mathcal{S}_{\omega}(x)) < \frac{3431}{1000}\epsilon < 4\epsilon.$$

By the definition of $\tilde{M}(\omega)$, $\{\mathcal{T}_t^{\omega}\}_{t \in [-1, 1]}$ acts isometrically on the 4ϵ -neighbourhood of $x \in \tilde{M}(\omega)$. Hence

$$(84) \quad d_{\omega}(x, \mathcal{T}_{-t}^{\omega} \circ \mathcal{S}_{\omega}^{-1} \circ \mathfrak{S}(\bar{\omega})_t \circ \mathcal{S}_{\omega}(x)) < 4\epsilon.$$

Since $D_j \subseteq \tilde{Q} \subset \tilde{M}(\omega)$, (84) is satisfied for all $x \in D_j$. Consider

$$F := \mathcal{T}_{-t}^{\omega} \circ \mathcal{S}_{\omega}^{-1} \circ \mathfrak{S}(\bar{\omega})_t \circ \mathcal{S}_{\omega}.$$

Note that F is λ_{ω} -measure preserving. Thus, by (79), we get

$$\lambda_{\omega}(D_j \cap \mathcal{T}_{-t}^{\omega} \circ \mathcal{S}_{\omega}^{-1} \circ \mathfrak{S}(\bar{\omega})_t \circ \mathcal{S}_{\omega}(D_j)) > (1 - \epsilon)\lambda_{\omega}(D_j).$$

Together with \mathcal{T}^{ω} -invariance of λ_{ω} , this yields

$$\lambda_{\omega}(\mathcal{T}_t^{\omega}(D_j) \cap \mathcal{S}_{\omega}^{-1} \circ \mathfrak{S}(\bar{\omega})_t \circ \mathcal{S}_{\omega}(D_j)) > (1 - \epsilon)\lambda_{\omega}(\mathcal{T}_t^{\omega}(D_j)),$$

for every $t \in [-1, 1]$. By summing up over $j = 1, \dots, l$ we get

$$\lambda_{\omega}(\mathcal{T}_t^{\omega}(\tilde{Q}) \cap \mathcal{S}_{\omega}^{-1} \circ \mathfrak{S}(\bar{\omega})_t \circ \mathcal{S}_{\omega}(\tilde{Q})) > (1 - \epsilon)\lambda_{\omega}(\mathcal{T}_t^{\omega}(\tilde{Q}))$$

and by (78) this yields

$$\lambda_{\omega}(\mathcal{T}_t^{\omega}(Q) \cap \mathcal{S}_{\omega}^{-1} \circ \mathfrak{S}(\bar{\omega})_t \circ \mathcal{S}_{\omega}(Q)) > (1 - 2\epsilon)\lambda_{\omega}(\mathcal{T}_t^{\omega}(Q)).$$

Since $\lambda_{\omega}(Q) < \frac{1}{4}$, we have

$$\lambda_{\omega}(\mathcal{T}_t^{\omega}(Q) \Delta \mathcal{S}_{\omega}^{-1} \circ \mathfrak{S}(\bar{\omega})_t \circ \mathcal{S}_{\omega}(Q)) < 4\epsilon\lambda_{\omega}(\mathcal{T}_t^{\omega}(Q)) \leq \epsilon.$$

Thus we get (77), and this concludes the proof of the theorem. \square

Since the construction above is local, we need to show that this suffices to transport the G_{δ} -condition from the space of flows to the moduli space.

Lemma 6.4. *Let X be a metric topological space. Let $\{U_i\}_{i \in \mathbb{N}}$ be a sequence of open subsets such that $\bigcup_{i \in \mathbb{N}} U_i = X$. If $V \subseteq X$ is such that*

$$V \cap U_i \text{ is a } G_\delta\text{-set for each } i \in \mathbb{N},$$

then V is a G_δ -set.

Proof. Note that $V = \bigcap_{i \in \mathbb{N}} (V \cap U_i) \cup U_i^c$. Since X is metrizable, every closed set is a G_δ -set. To finish the proof it is enough to observe that the union of two G_δ -sets is a G_δ -set. \square

7. TRANSLATION FLOWS DISJOINT WITH THEIR INVERSES ARE G_δ -DENSE

We have the following result which follows from the proof of Corollary 3.3 in [5].

Theorem 7.1. *Let $(X, \mathcal{B}(X), \mu)$ be a nonatomic standard Borel probability space, and let $\text{Flow}(X)$ be the space of μ -invariant flows on X . The set of flows which are weakly mixing and disjoint with their inverse is G_δ -dense in $\text{Flow}(X)$.*

The following result allows us to transfer the G_δ condition onto any connected component of the moduli space.

Proposition 7.2. *Let $\mathfrak{P}\text{rop}$ be a property of a measure-preserving flow such that the set of elements having this property is a G_δ subset of the space $\text{Flow}(X)$. Then in every connected component C of \mathcal{M} , the set of translation structures for which the vertical flow has the property $\mathfrak{P}\text{rop}$ is a G_δ set in C .*

Proof. Let C be a connected component of \mathcal{M} . In view of Theorem 6.3 for every $\zeta \in C$ there exists an open neighbourhood \mathcal{U}_ζ of ζ and a continuous mapping $\mathfrak{S}_\zeta : \mathcal{U}_\zeta \rightarrow \text{Flow}(M, \lambda_\zeta)$ such that for every $\omega \in \mathcal{U}_\zeta$ the vertical flow \mathcal{T}^ω is measure-theoretically isomorphic to $\mathfrak{S}_\zeta(\omega)$. Since C is a topological manifold, it is σ -compact. Thus there exists a sequence $\{\zeta_n\}_{n \in \mathbb{N}}$ of translation structures such that $\bigcup_{n \in \mathbb{N}} \mathcal{U}_{\zeta_n} = C$. For each $n \in \mathbb{N}$ we have that

$$\begin{aligned} \mathcal{Y}_{\zeta_n} &:= \{\omega \in \mathcal{U}_{\zeta_n}; \mathcal{T}^\omega \text{ satisfies } \mathfrak{P}\text{rop}\} \\ &= \{\omega \in \mathcal{U}_{\zeta_n}; \mathfrak{S}_{\zeta_n}(\omega) \text{ satisfies } \mathfrak{P}\text{rop}\} \\ &= \mathfrak{S}_{\zeta_n}^{-1}\{\mathcal{T} \in \text{Flow}(M, \lambda_{\zeta_n}); \mathcal{T} \text{ satisfies } \mathfrak{P}\text{rop}\} \end{aligned}$$

is a G_δ set in \mathcal{U}_{ζ_n} . By Lemma 6.4, this gives that the set of $\omega \in C$ such that \mathcal{T}^ω satisfies $\mathfrak{P}\text{rop}$ is a G_δ set in C . \square

By combining Theorem 7.1 and Proposition 7.2 we get the following result.

Corollary 7.3. *The set of translation structures ζ such that the vertical flow on (M, ζ) is weakly mixing and disjoint with its inverse is a G_δ set in every connected component C of the moduli space.*

Throughout this section we use the following notation. Let $C \subset \mathcal{M}$ be a non-hyperelliptic connected component of the moduli space, i.e. C is not of the form $\mathcal{M}^{\text{hyp}}(2g-2)$ or $\mathcal{M}^{\text{hyp}}(g-1, g-1)$ for any $g \geq 2$. Let $\pi = (\pi_0, \pi_1)$ be a permutation of the alphabet \mathcal{A} of d elements from the corresponding extended Rauzy class satisfying

$$\pi_0^{-1}(1) = \pi_1^{-1}(d) \text{ and } \pi_0^{-1}(d) = \pi_1^{-1}(1).$$

This permutation exists due to Theorem 2.7 and by the choice of C , it is not symmetric. Let Ω_π be the translation matrix corresponding to π . By Corollary 4.2 there exist symbols $a_1, a_2 \in \mathcal{A}$ such that $(\Omega_\pi)_{a_1 a_2} = (\Omega_\pi)_{a_2 a_1} = 0$ and for any rationally independent vector $\tau \in \mathbb{R}^{\mathcal{A}}$ the numbers

$$(\Omega_\pi \tau)_{a_2} - (\Omega_\pi \tau)_{a_1} \text{ and } (\Omega_\pi \tau)_{a_1} - ((\Omega_\pi \tau)_{\pi_0^{-1}(1)} + (\Omega_\pi \tau)_{\pi_0^{-1}(d)})$$

are rationally independent. The proof of the following lemma goes along the same lines as the proof of Lemma 14 in [8]. It is mainly based on the recurrence of polygonal Rauzy-Veech induction.

Lemma 7.4. *The set*

$$C_* := \{M(\pi, \lambda, \tau) \in C; (\pi, \lambda, \tau) \in \Theta_\pi, \lambda_a = 0 \text{ for } a \in \mathcal{A} \setminus \{\pi_0^{-1}(1), \pi_0^{-1}(d), a_1, a_2\}\}$$

is dense in C .

Before heading to the proof of the main result we give the proof of Proposition 1.2 which treats the density of translation structures on which the vertical flow is reversible.

Proof of Proposition 1.2. By following again the proof of Lemma 14 in [8] we prove that the set

$$C_{**} := \{M(\pi, \lambda, \tau) \in C; (\pi, \lambda, \tau) \in \Theta_\pi, \lambda_a = 0 \text{ for } a \in \mathcal{A} \setminus \{\pi_0^{-1}(1), \pi_0^{-1}(d)\}\}$$

is dense in C . The vertical flow on $M(\pi, \lambda, \tau) \in C_{**}$ is measure-theoretically isomorphic to the vertical flow on a torus given by $(\lambda_{\pi_0^{-1}(1)}, \lambda_{\pi_0^{-1}(d)})$ and π - the non-trivial permutation of two elements. Since the translation flows on tori are reversible, this concludes the proof. \square

The special representations of the vertical flows associated to the translation structures from C_* as given in Lemma 7.4 are special flows over IETs of 3 intervals and under a roof functions which are piecewise constant and have discontinuity points which coincide with the discontinuity points of the IET and one additional discontinuity point inside the middle interval. After one step of either left- or right-hand side Rauzy-Veech induction we get a special representation over rotation and under a piecewise constant roof function with 4 discontinuity points. We now prove some properties of such flows.

Let $\alpha \in [0, 1)$ be an irrational number, and let $T_\alpha : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ be the rotation by α . For any $\beta \in [0, 1)$ let

$$\|\beta\| := \min \{ \{\beta\}, \{1 - \beta\} \}.$$

Let $\{q_n\}_{n \in \mathbb{N}}$ be the sequence of partial denominators associated to α . Recall that for every odd $n \in \mathbb{N}$, we have a pair of Rokhlin towers

$$\{T_\alpha^i[-\|q_n \alpha\|, 0)\}_{i=0, \dots, q_n-1} \quad \text{and} \quad \{T_\alpha^i[0, \|q_n \alpha\|)\}_{i=0, \dots, q_n-1}$$

for $n \geq 1$, which covers \mathbb{R}/\mathbb{Z} . As a corollary from Theorem 3.9 in [4] we get the following result (recall the definitions of $f^{(r)}$ and Leb^f from subsection 2.2, as well as the definition of $\mu_{t,s}$ given in (1)).

Proposition 7.5. *Let $\{(T_\alpha^f)_t\}_{t \in \mathbb{R}}$ be the special flow over the rotation by $\alpha \in [0, 1)$ under a positive roof function $f \in L^2([0, 1), \text{Leb})$. Suppose that there exists a rigidity sequence $\{r_n\}_{n \in \mathbb{N}}$ for T_α (which is a subsequence of $\{q_n\}_{n \in \mathbb{N}}$) such that, setting $b_n := r_n \int_0^1 f(y) dy$, the sequence*

$$(85) \quad \left\{ \int_0^1 \left| f^{(r_n)}(x) - b_n \right|^2 dx \right\}, \quad n \in \mathbb{N}$$

is bounded. Then there exists a probability measure $P \in \mathcal{P}(\mathbb{R}^2)$ such that, up to taking a subsequence,

$$\left(f^{(2r_n)} - 2b_n, f^{(r_n)} - b_n \right)_* \text{Leb} \rightarrow P \text{ weakly.}$$

Moreover, along the same subsequence, we have

$$\text{Leb}_{2b_n, b_n}^f \rightarrow \int_{\mathbb{R}^2} \text{Leb}_{-t, -u}^f dP(t, u).$$

To prove the next result we need the following remark.

Remark 7.6. Let $f : [0, 1) \rightarrow \mathbb{R}$ be a piecewise constant function. Let β_1, \dots, β_k be the jumps of f and let d_1, \dots, d_k be their respective values. Then for every $x \in [0, 1)$ and every odd $n \in \mathbb{N}$,

$$\sum_{i=0}^{q_n-1} (f(T_\alpha^{q_n+i}(x)) - f(T_\alpha^i(x))) = \sum_{i=0}^{q_n-1} \sum_{j=1}^k -d_j \chi_{T_\alpha^{-i}[\beta_j, \beta_j + \|q_n \alpha\|)}(x).$$

Indeed, the expression $f(T_\alpha^{q_n+i}(x)) - f(T_\alpha^i(x))$ takes non-zero value if and only if there is a discontinuity point β_j in the interval $(T_\alpha^{q_n+i}(x), T_\alpha^i(x)]$. However, we have $T_\alpha^{q_n+i}(x) = T_\alpha^i(x) - \|q_n\alpha\|$. Hence $\beta_j \in (T_\alpha^{q_n+i}(x), T_\alpha^i(x)]$ if and only if $x \in T_\alpha^{-i}[\beta_j, \beta_j + \|q_n\alpha\|]$. In other words, if we consider the (not necessarily disjoint) towers $U_j := \bigcup_{i=0}^{q_n-1} T_\alpha^{-i}[\beta_j, \beta_j + \|q_n\alpha\|]$ for $j = 1, \dots, k$, then

$$\sum_{i=0}^{q_n-1} (f(T_\alpha^{q_n+i}(x)) - f(T_\alpha^i(x))) = \sum_{j=1}^k -d_j \chi_{U_j}(x).$$

In particular, if $x \notin U_j$ for all $j = 1, \dots, k$, then $\sum_{i=0}^{q_n-1} (f(T_\alpha^{q_n+i}(x)) - f(T_\alpha^i(x))) = 0$. Otherwise, if $x \in U_j$ for some $j = 1, \dots, k$, then d_j contributes to the value of the considered expression.

We need the following theorem which is a version of Theorem 7.3 in [4]. Recall that $\xi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by $\xi(t, u) := t - 2u$.

Lemma 7.7. *There exists a set $\Lambda \subset \mathbb{R}/\mathbb{Z}$ of full Lebesgue measure such that, for every $\alpha \in \Lambda$, there exists a set $D_\alpha \in (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ of full Lebesgue measure with the property that, if $(\beta_1, \beta_2) \in D_\alpha$, then*

- the numbers $0, 1 - \alpha, \beta_1$ and β_2 are distinct;
- for every piecewise constant positive function $f : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$ with discontinuity points $0, 1 - \alpha, \beta_1, \beta_2$ and jumps d_{β_1} and d_{β_2} at β_1 and β_2 respectively, for the special flow $\{(T_\alpha^t)_t\}_{t \in \mathbb{R}}$ there exist probability measures $P, Q \in \mathcal{P}(\mathbb{R}^2)$ such that, up to a subsequence, we have

$$(86) \quad \text{Leb}_{2b_n, b_n}^f \rightarrow \int_{\mathbb{R}^2} \text{Leb}_{-t, -u}^f dP(t, u) \text{ and } \text{Leb}_{-2b_n, -b_n}^f \rightarrow \int_{\mathbb{R}^2} \text{Leb}_{-t, -u}^f dQ(t, u) \text{ weakly,}$$

where $b_n := q_n \int_0^1 f(x) dx$. Moreover, $\xi_* P$ is atomic with exactly 4 atoms in points $0, -d_{\beta_1}, -d_{\beta_2}, d_{\beta_1} + d_{\beta_2}$, while $\xi_* Q = (-\xi)_* P$ has exactly 4 atoms in points $0, d_{\beta_1}, d_{\beta_2}, -(d_{\beta_1} + d_{\beta_2})$.

Proof. Let $\Lambda \subset \mathbb{R}/\mathbb{Z}$ be the set of irrational $\alpha \in \mathbb{R}/\mathbb{Z}$ such that there exists a sequence $\{k_n\}_{n \in \mathbb{N}}$ of odd numbers such that, for some $\frac{1}{52} \leq \varepsilon \leq \frac{1}{25}$, we have $\lim_{n \rightarrow \infty} q_{k_n} \|q_{k_n} \alpha\| = \varepsilon$. The set Λ is of full Lebesgue measure. Indeed, the Gauss map $G(x) = \{\frac{1}{x}\}$ is mixing for the absolutely continuous measure with density $\frac{1}{\ln 2} \frac{1}{1+x}$, hence in particular G^2 is ergodic. For any irrational $\alpha \in \mathbb{R}/\mathbb{Z}$, let $\{a_n\}_{n \in \mathbb{N}}$ be the sequence of partial quotients of α . Then we have (see [16])

$$(87) \quad \frac{1}{2} \frac{1}{a_{n+1} + 1} < q_n \|q_n \alpha\| < \frac{1}{a_{n+1}}.$$

Recall that for any $m \in \mathbb{N}$, $G^n(\alpha) \in (\frac{1}{m+1}, \frac{1}{m}]$ iff $a_n = m$. Hence by ergodicity of G^2 , for almost every $\alpha \in [0, 1)$, $a_{n+1} = 25$ for infinitely many odd numbers n . Thus we obtain the claim.

Fix $\alpha \in \Lambda$. Recall that for every $n \in \mathbb{N}$, \mathbb{R}/\mathbb{Z} is covered by a pair of towers

$$\{T_\alpha^i[-\|q_{k_n} \alpha\|, 0]\}_{i=0, \dots, q_{k_n}-1} \quad \text{and} \quad \{T_\alpha^i[0, \|q_{k_n-1} \alpha\|]\}_{i=0, \dots, q_{k_n}-1}.$$

Note that $1 - \alpha \in T_\alpha^{q_{k_n}-1}[0, \|q_{k_n-1} \alpha\|]$. More precisely, $1 - \alpha = T_\alpha^{q_{k_n}-1}(0) + \|q_{k_n} \alpha\|$. Since $\{q_{k_n}\}_{n \in \mathbb{N}}$ is a rigidity sequence and f , regardless of the choice of discontinuity points, is of bounded variation, (86) follows from Proposition 7.5 and Koksma-Denjoy inequality:

$$|f^{(q_{k_n})}(x) - b_n| \leq \text{Var}(f) \quad \text{for every } x \in [0, 1).$$

In view of Proposition 7.5, taking a subsequence if necessary, we get

$$P = \lim_{n \rightarrow \infty} (f^{(2q_{k_n})} - 2b_n, f^{(q_{k_n})} - b_n)_* \text{Leb}$$

and

$$Q = \lim_{n \rightarrow \infty} (f^{(-2q_{k_n})} + 2b_n, f^{(-q_{k_n})} + b_n)_* \text{Leb}.$$

By applying ξ_* to both expressions we obtain

$$\xi_*P = \lim_{n \rightarrow \infty} \left(\sum_{i=0}^{q_{k_n}-1} (f \circ T_\alpha^{q_{k_n}+i} - f \circ T_\alpha^i) \right)_* \text{Leb}$$

and

$$\xi_*Q = \lim_{n \rightarrow \infty} \left(\sum_{i=1}^{q_{k_n}} (f \circ T_\alpha^{-q_{k_n}-i} - f \circ T_\alpha^{-i}) \right)_* \text{Leb}.$$

By using the invariance of the Lebesgue measure under $T_\alpha^{2q_{k_n}}$, we get $\xi_*Q = (-\xi)_*P$.

Consider the sequence of pairs of disjoint Rokhlin towers

$$V_n := \bigcup_{i=0}^{q_{k_n}-1} T_\alpha^i [3\|q_{k_n}\alpha\|, \frac{1}{3}\|q_{k_n-1}\alpha\|) \quad \text{and} \quad W_n := \bigcup_{i=0}^{q_{k_n}-1} T_\alpha^i [\frac{2}{3}\|q_{k_n-1}\alpha\|, \|q_{k_n-1}\alpha\| - 3\|q_{k_n}\alpha\|).$$

We know (see e.g. [16]) that

$$\frac{1}{2q_{k_n}} < \|q_{k_n-1}\alpha\| < \frac{1}{q_{k_n}}.$$

Hence

$$\frac{\|q_{k_n-1}\alpha\|}{\|q_{k_n}\alpha\|} < \frac{1}{q_{k_n}\|q_{k_n}\alpha\|} \rightarrow \frac{1}{\varepsilon} \quad \text{and} \quad \frac{\|q_{k_n-1}\alpha\|}{\|q_{k_n}\alpha\|} > \frac{1}{2q_{k_n}\|q_{k_n}\alpha\|} \rightarrow \frac{1}{2\varepsilon}.$$

By using the fact that $\varepsilon < \frac{1}{25}$, for sufficiently large n we obtain that

$$(88) \quad 12 < \frac{\|q_{k_n-1}\alpha\|}{\|q_{k_n}\alpha\|} < 53.$$

In view of (88) we get

$$\text{Leb}(W_n) = \text{Leb}(V_n) = q_{k_n} \left(\frac{1}{3}\|q_{k_n-1}\alpha\| - 3\|q_{k_n}\alpha\| \right) > q_{k_n}\|q_{k_n}\alpha\| > \frac{1}{53},$$

for sufficiently large $n \in \mathbb{N}$, that is the measures of V_n, W_n are bounded away from 0.

By the remark to Lemma 3.4 in [15], this implies that, for almost every $\beta_1 \in [0, 1)$, there exists an infinite set N_1 of natural numbers such that $\beta_1 \in V_n$ for each $n \in N_1$. Once such β_1 and N_1 are fixed, the same argument yields that for almost every $\beta_2 \in [0, 1)$ there exists an infinite subset $N_2 \subset N_1$ such that $\beta_2 \in W_n$ for each $n \in N_2$. Using Fubini's theorem, we get that for almost every $(\beta_1, \beta_2) \in [0, 1) \times [0, 1)$ there exist infinitely many integers n such that $\beta_1 \in V_n$ and $\beta_2 \in W_n$. Let D_α be the set of such pairs (β_1, β_2) .

Now we fix (β_1, β_2) in D_α . Extracting a subsequence if necessary, we may assume that $\beta_1 \in V_n$ and $\beta_2 \in W_n$ for all n . Since V_n and W_n are disjoint and $0, 1 - \alpha \notin W_n \cup V_n$ for all $n \in \mathbb{N}$, the points $0, 1 - \alpha, \beta_1$ and β_2 are distinct. Note that, by the choice of W_n and V_n the towers $\bigcup_{i=0}^{q_{k_n}-1} T^{-i}[\beta_1, \beta_1 + \|q_{k_n}\alpha\|)$, $\bigcup_{i=0}^{q_{k_n}-1} T^{-i}[\beta_2, \beta_2 + \|q_{k_n}\alpha\|)$ and $\bigcup_{i=0}^{q_{k_n}} T^{-n}[0, \|q_{k_n}\alpha\|)$ are pairwise disjoint. Indeed, we have the following inclusions

$$\begin{aligned} \bigcup_{i=0}^{q_{k_n}-1} T^{-i}[\beta_1, \beta_1 + \|q_{k_n}\alpha\|) &\subset \bigcup_{i=0}^{q_{k_n}-1} T_\alpha^i [3\|q_{k_n}\alpha\|, \frac{1}{3}\|q_{k_n-1}\alpha\| + \|q_{k_n}\alpha\|), \\ \bigcup_{i=0}^{q_{k_n}-1} T^{-i}[\beta_2, \beta_2 + \|q_{k_n}\alpha\|) &\subset \bigcup_{i=0}^{q_{k_n}-1} T_\alpha^i [\frac{2}{3}\|q_{k_n-1}\alpha\|, \|q_{k_n-1}\alpha\| - 2\|q_{k_n}\alpha\|), \end{aligned}$$

and

$$\bigcup_{i=0}^{q_{k_n}} T^{-i}[0, \|q_{k_n}\alpha\|) \subset \bigcup_{i=0}^{q_{k_n}-1} T_\alpha^i [0, 2\|q_{k_n}\alpha\|).$$

In view of (88), we get $\frac{1}{3}\|q_{k_n-1}\alpha\| > 4\|q_{k_n}\alpha\|$ for sufficiently large n . In particular

$$\frac{1}{3}\|q_{k_n-1}\alpha\| + \|q_{k_n}\alpha\| < \frac{2}{3}\|q_{k_n-1}\alpha\|,$$

which shows that the intervals

$$\left[0, 2\|q_{k_n}\alpha\|\right), \left[3\|q_{k_n}\alpha\|, \frac{1}{3}\|q_{k_n-1}\alpha\| + \|q_{k_n}\alpha\|\right) \text{ and } \left[\frac{2}{3}\|q_{k_n-1}\alpha\|, \|q_{k_n-1}\alpha\| - 2\|q_{k_n}\alpha\|\right)$$

are pairwise disjoint. This implies the desired disjointness of the aforementioned towers. This allow us to control the atoms of limit measures and their respective masses.

Suppose now that f has discontinuity points at $0, 1-\alpha, \beta_1$ and β_2 , where $(\beta_1, \beta_2) \in D_\alpha$ with jumps $d_0, d_{1-\alpha}, d_{\beta_1}$ and d_{β_2} respectively. Since k_n is odd, in view of Remark 7.6 the expression $\sum_{i=0}^{q_{k_n}-1} (f \circ T_\alpha^{q_{k_n}+i}(x) - f \circ T_\alpha^i(x))$ may only take non-zero values for x on towers $U_1 := \bigcup_{i=0}^{q_{k_n}-1} T^{-i}[0, \|q_{k_n}\alpha\|)$, $U_2 := \bigcup_{i=0}^{q_{k_n}-1} T^{-i}[1-\alpha, 1-\alpha + \|q_{k_n}\alpha\|)$, $U_3 := \bigcup_{i=0}^{q_{k_n}-1} T^{-i}[\beta_1, \beta_1 + \|q_{k_n}\alpha\|)$ and $U_4 := \bigcup_{i=0}^{q_{k_n}-1} T^{-i}[\beta_2, \beta_2 + \|q_{k_n}\alpha\|)$. We have proved though, that U_3 is disjoint with other towers and the same is true for U_4 . Thus we get that

$$\sum_{i=0}^{q_{k_n}-1} (f \circ T_\alpha^{q_{k_n}+i}(x) - f \circ T_\alpha^i(x)) = -d_{\beta_1} \text{ for } x \in U_3,$$

and

$$\sum_{i=0}^{q_{k_n}-1} (f \circ T_\alpha^{q_{k_n}+i}(x) - f \circ T_\alpha^i(x)) = -d_{\beta_1} \text{ for } x \in U_4.$$

On the other hand,

$$U_1 \cap U_2 = \bigcup_{i=1}^{q_{k_n}-1} T^{-i}[0, \|q_{k_n}\alpha\|), \quad U_1 \setminus U_2 = [0, \|q_{k_n}\alpha\|), \text{ and } U_2 \setminus U_1 = [\|q_{k_n}\alpha\|, 2\|q_{k_n}\alpha\|).$$

Hence

$$\begin{aligned} \sum_{i=0}^{q_{k_n}-1} (f \circ T_\alpha^{q_{k_n}+i}(x) - f \circ T_\alpha^i(x)) &= -d_0 - d_{1-\alpha} \text{ for } x \in U_1 \cap U_2, \\ \sum_{i=0}^{q_{k_n}-1} (f \circ T_\alpha^{q_{k_n}+i}(x) - f \circ T_\alpha^i(x)) &= -d_0 \text{ for } x \in [0, \|q_{k_n}\alpha\|), \end{aligned}$$

and

$$\sum_{i=0}^{q_{k_n}-1} (f \circ T_\alpha^{q_{k_n}+i}(x) - f \circ T_\alpha^i(x)) = -d_{1-\alpha} \text{ for } x \in [\|q_{k_n}\alpha\|, 2\|q_{k_n}\alpha\|).$$

Finally

$$\sum_{i=0}^{q_{k_n}-1} (f \circ T_\alpha^{q_{k_n}+i}(x) - f \circ T_\alpha^i(x)) = 0 \text{ for } x \notin U_1 \cup U_2 \cup U_3 \cup U_4.$$

Note that $-d_0 - d_{1-\alpha} = d_{\beta_1} + d_{\beta_2}$, since the sum of jumps of f is 0. Moreover

$$\text{Leb}([0, \|q_{k_n}\alpha\|) = \text{Leb}([\|q_{k_n}\alpha\|, 2\|q_{k_n}\alpha\|) = \|q_{k_n}\alpha\| \rightarrow 0,$$

$$\text{Leb}(U_3) = \text{Leb}(U_4) = q_{k_n}\|q_{k_n}\alpha\| \rightarrow \varepsilon$$

and

$$\text{Leb}(U_1 \cap U_2) = q_{k_n}\|q_{k_n}\alpha\| - \|q_{k_n}\alpha\| \rightarrow \varepsilon.$$

Hence we get that $\xi_*P = \lim_{n \rightarrow \infty} (\sum_{i=0}^{q_{k_n}-1} f \circ T_\alpha^{q_{k_n}+i} - f \circ T_\alpha^i)_* \text{Leb}$ is a measure such that $\xi_*P(\{-d_{\beta_1}\}) = \xi_*P(\{-d_{\beta_2}\}) = \xi_*P(\{d_{\beta_1} + d_{\beta_2}\}) = \varepsilon$ and $\xi_*P(\{0\}) = 1 - 3\varepsilon$. Since $\xi_*Q = (-\xi)_*P$, we obtain the final claim. \square

We now state a reformulation of the above lemma for rotations on arbitrary large circles. If needed, throughout the proof of this lemma, we identify $\mathbb{R}/x\mathbb{Z}$ with $[0, x)$ for every $x \in \mathbb{R}_{>0}$.

Lemma 7.8. *There exists a subset $\Delta_0 \subset \Delta := \{(x, y) \in \mathbb{R}_{>0}^2; 0 < y < x\}$ of full Lebesgue measure in Δ with the property that for every $(l, \alpha) \in \Delta_0$ there exists a set $D_{l, \alpha} \subset (\mathbb{R}/l\mathbb{Z}) \times (\mathbb{R}/l\mathbb{Z})$ of full Lebesgue measure such that for every $(\beta_1, \beta_2) \in D_{l, \alpha}$ we have*

- the numbers $0, l - \alpha, \beta_1$ and β_2 are distinct in $\mathbb{R}/l\mathbb{Z}$ and
- if T_α is the rotation on $\mathbb{R}/l\mathbb{Z}$ by $\alpha \in \mathbb{R}/l\mathbb{Z}$ and $h : \mathbb{R}/l\mathbb{Z} \rightarrow \mathbb{R}_{>0}$ is a piecewise constant function with exactly 4 discontinuity points at $0, l - \alpha, \beta_1, \beta_2$ and rationally independent jumps at β_1 and β_2 , then the special flow T_α^h is weakly mixing and disjoint with its inverse.

Proof. Let $\Lambda \subset [0, 1)$ and $D_\alpha \subset [0, 1) \times [0, 1)$ for $\alpha \in \Lambda$ be the sets given by Lemma 7.7. For any $l \in \mathbb{R}_{>0}$, let us also denote by $l : [0, 1) \rightarrow [0, l)$ the map given by $l(x) := lx$. We also consider l as a map between \mathbb{R}/\mathbb{Z} and $\mathbb{R}/l\mathbb{Z}$. For any $l \in \mathbb{R}_{>0}$ let $\Lambda_l := l(\Lambda) \subset [0, l)$ and set $D_{l,\alpha} := (l \times l)(D_{l^{-1}\alpha}) \subset [0, l) \times [0, 1)$ for any $\alpha \in [0, l)$. Define $\Delta_0 := \{(x, y); x \in \mathbb{R}_{>0}, y \in \Lambda_x\}$. Note that Δ_0 is of full Lebesgue measure in Δ and for every $l \in \mathbb{R}_{>0}$ and $\alpha \in \Lambda_l$ the set $D_{l,\alpha}$ is of full Lebesgue measure in $(\mathbb{R}/l\mathbb{Z}) \times (\mathbb{R}/l\mathbb{Z})$.

Take $(l, \alpha) \in \Delta_0$ and $(\beta_1, \beta_2) \in D_{l,\alpha}$. By the definition of Λ and $D_{l^{-1}\alpha}$, the points $0, l - \alpha, \beta_1$ and β_2 are distinct. Let $h : \mathbb{R}/l\mathbb{Z} \rightarrow \mathbb{R}_{>0}$ be a piecewise constant function which has exactly 4 discontinuity points at $0, l - \alpha, \beta_1$ and β_2 . Assume that the jumps d_{β_1} and d_{β_2} at β_1 and β_2 are rationally independent. Consider the special flow T_α^h on $[0, l)^h$. The map $(l^{-1} \times Id) : [0, l)^h \rightarrow [0, 1)^{hol}$ establishes an isomorphism of flows T_α^h and $T_{l^{-1}\alpha}^{hol}$. The roof function $h \circ l$ has discontinuities at $0, 1 - l^{-1}\alpha, l^{-1}\beta_1$ and $l^{-1}\beta_2$ and has jumps d_{β_1} and d_{β_2} at $l^{-1}\beta_1$ and $l^{-1}\beta_2$ respectively. Moreover, $l^{-1}\alpha \in \Lambda$ and $(l^{-1}\beta_1, l^{-1}\beta_2) \in D_\alpha$. In view of Lemma 7.7, this gives

$$\text{Leb}_{2b_n, b_n}^{hol} \rightarrow \int_{\mathbb{R}^2} \text{Leb}_{-t, -u}^{hol} dP(t, u), \quad \text{and} \quad \text{Leb}_{-2b_n, -b_n}^{hol} \rightarrow \int_{\mathbb{R}^2} \text{Leb}_{-t, -u}^{hol} dQ(t, u) \quad \text{weakly,}$$

for some increasing to infinity real sequence $\{b_n\}_{n \in \mathbb{N}}$ and measures $P, Q \in \mathcal{P}(\mathbb{R}^2)$. Furthermore, $\xi_* P$ is atomic and has atoms at $0, -d_{\beta_1}, -d_{\beta_2}$ and $d_{\beta_1} + d_{\beta_2}$, while $\xi_* Q$ is also atomic and has atoms at $0, d_{\beta_1}, d_{\beta_2}$ and $-(d_{\beta_1} + d_{\beta_2})$. Since d_{β_1} and d_{β_2} are rationally independent, Proposition 3.5 implies that $T_{l^{-1}\alpha}^{hol}$ is weakly mixing. Moreover, the rational independence of d_{β_1} and d_{β_2} also gives $\xi_* P \neq \xi_* Q$ which yields $P \neq Q$. In view of Corollary 3.4 this gives that $T_{l^{-1}\alpha}^{hol}$ is disjoint with its inverse. Since $T_{l^{-1}\alpha}^{hol}$ and T_α^h are isomorphic, T_α^h is weakly mixing and disjoint with its inverse. \square

We are now ready to give the proof of the main result of this paper.

Proof of Theorem 1.1. In view of Corollary 7.3, the set of translation structures whose associated vertical flow is weakly mixing and disjoint with its inverse is a G_δ set in every connected component of the moduli space. We now show that there is a dense subset of translation structures in each non-hyperelliptic connected component C so that the associated vertical flows are weakly mixing and disjoint with their inverses.

Fix a non-hyperelliptic connected component C of the moduli space. Recall that for some $d \geq 2$ and an alphabet \mathcal{A} of d elements, there is a permutation $\pi = (\pi_0, \pi_1) \in S_0^{\mathcal{A}}$ in the extended Rauzy class associated with C , such that

$$\pi_1(\pi_0^{-1}(1)) = d \quad \text{and} \quad \pi_1(\pi_0^{-1}(d)) = 1.$$

Let $\Omega := \Omega_\pi$ be the translation matrix corresponding to π . Then, in view of Corollary 4.2, there exist symbols $a_1, a_2 \in \mathcal{A}$ such that

$$\Omega_{a_1 a_2} = \Omega_{a_2 a_1} = 0$$

and the numbers

$$(\Omega\tau)_{a_2} - (\Omega\tau)_{a_1} \quad \text{and} \quad (\Omega\tau)_{a_i} - ((\Omega\tau)_{\pi_0^{-1}(1)} + (\Omega\tau)_{\pi_0^{-1}(d)})$$

are rationally independent for $i = 1, 2$ whenever τ is rationally independent.

Let

$$\Xi_* := \{(\pi, \lambda, \tau) \in \Theta_\pi; \lambda_a = 0 \text{ for } a \in \mathcal{A} \setminus \{\pi_0^{-1}(1), \pi_0^{-1}(d), a_1, a_2\}\}$$

Let $C_* := \{M(\pi, \lambda, \tau) \in C; (\pi, \lambda, \tau) \in \Xi_*\}$. In view of Lemma 7.4, this is a dense subset of C . Hence to prove the density of the desired property in C , it is enough to prove that this property

holds for a dense set in C_* . We prove this by finding a dense subset of parameters in Ξ_* such that the associated translation structures have the sought properties.

Note that the set $\Xi \subset \Xi_*$ given by

$$\Xi := \{(\pi, \lambda, \tau) \in \Xi_*; T_{\pi, \lambda} \text{ is ergodic}; \lambda_{\pi_0^{-1}(1)} \neq \lambda_{\pi_0^{-1}(d)}; \tau \text{ is rationally independent}\}$$

is dense in Ξ_* . Let $\zeta = M(\pi, \lambda, \tau) \in C_*$ with $(\pi, \lambda, \tau) \in \Xi$. Let \mathcal{T}^ζ be the corresponding vertical flow. Recall that it has a special representation $T_{\pi, \lambda}^h$ over the IET $T_{\pi, \lambda} : [0, |\lambda|) \rightarrow [0, |\lambda|)$ and under a piecewise constant roof function $h : [0, |\lambda|) \rightarrow \mathbb{R}_{>0}$ which is constant over exchanged intervals. Moreover, if we consider $h = \{h_a\}_{a \in \mathcal{A}}$ as a vector of values, where h_a is the value of h over the interval corresponding to a , then $h_a = -(\Omega\tau)_a$. However, since $(\pi, \lambda, \tau) \in \Xi_*$, we have that $\lambda_a = 0$ for $a \in \mathcal{A} \setminus \{\pi_0^{-1}(1), \pi_0^{-1}(d), a_1, a_2\}$. Thus we can reduce the data describing the above special representation.

Let $\hat{\pi} = (\hat{\pi}_0, \hat{\pi}_1)$ be the permutation on the alphabet $\hat{\mathcal{A}} := \{\pi_0^{-1}(1), \pi_0^{-1}(d), a_1, a_2\}$ given by

$$\hat{\pi}_0(\pi_0^{-1}(1)) = 1, \quad \hat{\pi}_0(\pi_0^{-1}(d)) = 4, \quad \hat{\pi}_0(a_1) = 2, \quad \hat{\pi}_0(a_2) = 3$$

and

$$\hat{\pi}_1(\pi_0^{-1}(1)) = 4, \quad \hat{\pi}_1(\pi_0^{-1}(d)) = 1, \quad \hat{\pi}_1(a_1) = 2, \quad \hat{\pi}_1(a_2) = 3.$$

For $a \in \hat{\mathcal{A}}$ let $\hat{\lambda}_a := \lambda_a$. Moreover, since the intervals corresponding to $a \in \mathcal{A} \setminus \hat{\mathcal{A}}$ are empty, h can be considered as a vector $\{h_a\}_{a \in \hat{\mathcal{A}}}$. Then \mathcal{T}^ζ has a special representation $T_{\hat{\pi}, \hat{\lambda}}^h$ over the IET $T_{\hat{\pi}, \hat{\lambda}} : [0, |\lambda|) \rightarrow [0, |\lambda|)$.

Consider the sets $\Xi_0, \Xi_1 \subset \Xi$ given by

$$\Xi_0 := \{(\pi, \lambda, \tau) \in \Xi; \lambda_{\pi_0^{-1}(1)} > \lambda_{\pi_0^{-1}(d)}\} \text{ and } \Xi_1 := \{(\pi, \lambda, \tau) \in \Xi; \lambda_{\pi_0^{-1}(1)} < \lambda_{\pi_0^{-1}(d)}\}.$$

We have $\Xi_0 \cup \Xi_1 = \Xi$. Suppose first that $(\pi, \lambda, \tau) \in \Xi_0$ that is $\lambda_{\pi_0^{-1}(1)} > \lambda_{\pi_0^{-1}(d)}$. Let $\phi : \{(x, y, z, v) \in \mathbb{R}_{>0}^4; x > v\} \rightarrow \mathbb{R}_{>0}^4$ be the diffeomorphism given by

$$\phi(x, y, z, v) := (x - v, v, y, z).$$

Then after one step of the polygonal right hand side Rauzy-Veech induction on $\mathcal{T}_{\hat{\pi}, \hat{\lambda}}^h$ we get a special flow $T_\alpha^{\hat{h}}$ over the rotation $T_\alpha : [0, \hat{\lambda}_{\pi_0^{-1}(1)} + \hat{\lambda}_{a_1} + \hat{\lambda}_{a_2}) \rightarrow [0, \hat{\lambda}_{\pi_0^{-1}(1)} + \hat{\lambda}_{a_1} + \hat{\lambda}_{a_2})$ by $\alpha = \alpha(\hat{\lambda}) := \hat{\lambda}_{a_1} + \hat{\lambda}_{a_2} + \hat{\lambda}_{\pi_0^{-1}(d)}$ under a piecewise constant function $\hat{h} : [0, \hat{\lambda}_{\pi_0^{-1}(1)} + \hat{\lambda}_{a_1} + \hat{\lambda}_{a_2}) \rightarrow \mathbb{R}_{>0}$ with values $h_{\pi_0^{-1}(1)}, h_{\pi_0^{-1}(1)} + h_{\pi_0^{-1}(d)}, h_{a_1}, h_{a_2}$ over the consecutive intervals of lengths given by the vector $\phi(\hat{\lambda}_{\pi_0^{-1}(1)}, \hat{\lambda}_{a_1}, \hat{\lambda}_{a_2}, \hat{\lambda}_{\pi_0^{-1}(d)})$. Recall that the flows $\mathcal{T}_{\hat{\pi}, \hat{\lambda}}^h$ and $T_\alpha^{\hat{h}}$ are isomorphic. Let $l = l(\hat{\lambda}) := \hat{\lambda}_{\pi_0^{-1}(1)} + \hat{\lambda}_{a_1} + \hat{\lambda}_{a_2}$, $\beta_1 = \beta_1(\hat{\lambda}) := \hat{\lambda}_{\pi_0^{-1}(1)}$ and $\beta_2 = \beta_2(\hat{\lambda}) := \hat{\lambda}_{\pi_0^{-1}(1)} + \hat{\lambda}_{a_1}$. Then $\hat{h} : [0, l) \rightarrow \mathbb{R}_{>0}$ has discontinuities at points $l - \alpha$, β_1 and β_2 . The jump at the point β_1 is equal to $h_{a_1} - (h_{\pi_0^{-1}(1)} + h_{\pi_0^{-1}(d)})$, while at the point β_2 equals $h_{a_2} - h_{a_1}$. Moreover, we have

$$h_{a_1} - (h_{\pi_0^{-1}(1)} + h_{\pi_0^{-1}(d)}) = -(\Omega\tau)_{a_1} + ((\Omega\tau)_{\pi_0^{-1}(1)} + (\Omega\tau)_{\pi_0^{-1}(d)}),$$

and

$$h_{a_2} - h_{a_1} = -(\Omega\tau)_{a_2} + (\Omega\tau)_{a_2}.$$

Since τ is a rationally independent vector, Corollary 4.2 yields the rational independence of the jumps at β_1 and β_2 . From now on we treat T_α as a rotation on $\mathbb{R}/l\mathbb{Z}$. Furthermore, we also treat \hat{h} as a piecewise constant function on $\mathbb{R}/l\mathbb{Z}$. Then $\hat{h} : \mathbb{R}/l\mathbb{Z} \rightarrow \mathbb{R}_{>0}$ gets an additional discontinuity point at 0.

Let us consider the diffeomorphism $\psi : \mathbb{R}_{>0}^4 \rightarrow \{(x, y, z, v) \in \mathbb{R}_{>0}^4; 0 < x - y < z < v < x\}$ given by

$$\psi(x, y, z, v) := (x + y + z + v, y + z + v, x + y, x + y + z).$$

Then

$$\psi \circ \phi : \{(x, y, z, v) \in \mathbb{R}_{>0}^4; x > v\} \rightarrow \{(x, y, z, v) \in \mathbb{R}_{>0}^4; 0 < x - y < z < v < x\}$$

is a diffeomorphism and $\psi \circ \phi(\hat{\lambda}) = (l, \alpha, \beta_1, \beta_2)$.

Let $\Delta_0 \subset \{(x, y) \in \mathbb{R}_{>0}^2; y \in (0, x)\}$ and $D_{l,\alpha} \subset (\mathbb{R}/l\mathbb{Z}) \times (\mathbb{R}/l\mathbb{Z})$ for $(l, \alpha) \in \Delta_0$ be sets given by Lemma 7.8. Then, by Lemma 7.8, for every $(l, \alpha) \in \Delta_0$ and $(\beta_1, \beta_2) \in D_{l,\alpha}$, the special flow $T_\alpha^{\hat{h}}$ over the rotation by α on $\mathbb{R}/l\mathbb{Z}$ and under a piecewise constant roof function with discontinuity points $0, l - \alpha, \beta_1, \beta_2$ and with rationally independent jumps at β_1 and β_2 is weakly mixing and disjoint with its inverse. Consider

$$\mathcal{G} := \{(x, y, z, v) \in \mathbb{R}_{>0}^4; (x, y) \in \Delta_0, \frac{y}{x} \in \mathbb{R} \setminus \mathbb{Q}, (z, v) \in D_{x,y}, 0 < x - y < z < v < x\}.$$

In view of Lemma 7.8, Δ_0 is dense in $\{(x, y) \in \mathbb{R}_{>0}^2; y < x\}$ and $D_{x,y}$ is dense in $(0, x) \times (0, x)$. Therefore \mathcal{G} is a dense set in $\{(x, y, z, v) \in \mathbb{R}_{>0}^4; 0 < x - y < z < v < x\}$. As $\psi \circ \phi$ is a diffeomorphism, the set $(\psi \circ \phi)^{-1}(\mathcal{G})$ is dense in $\{(x, y, z, v) \in \mathbb{R}_{>0}^4; x > v\}$. Hence the set

$$\Gamma_0 := \{(\pi, \lambda, \tau) \in \Xi_0; \hat{\lambda} \in (\psi \circ \phi)^{-1}(\mathcal{G}) \text{ and } \tau \text{ is a rationally independent vector}\}$$

is dense in Ξ_0 . By going along the same lines and by using the left-hand side polygonal Rauzy-Veech induction, we find a dense set $\Gamma_1 \subset \Xi_1$ which has analogous properties.

If $\gamma = (\pi, \lambda, \tau) \in \Gamma_0 \cup \Gamma_1$ then $(l(\hat{\lambda}), \alpha(\hat{\lambda})) \in \Delta_0$, $(\beta_1(\hat{\lambda}), \beta_2(\hat{\lambda})) \in D_{l(\hat{\lambda}), \alpha(\hat{\lambda})}$ and the vertical flow on $M(\gamma)$ is isomorphic to a special flow $T_{\alpha(\hat{\lambda})}^{\hat{h}}$ on $(\mathbb{R}/l(\hat{\lambda})\mathbb{Z})^{\hat{h}}$, where $\hat{h} : \mathbb{R}/l(\hat{\lambda})\mathbb{Z} \rightarrow \mathbb{R}_{>0}$ is a piecewise constant roof function with discontinuities at $0, l(\hat{\lambda}) - \alpha(\hat{\lambda}), \beta_1(\hat{\lambda}), \beta_2(\hat{\lambda})$ and the jumps at $\beta_1(\hat{\lambda})$ and $\beta_2(\hat{\lambda})$ are rationally independent. In view of Lemma 7.8, those flows are weakly mixing and disjoint with their inverses. As $\Gamma_0 \cup \Gamma_1$ is dense in Ξ , it is also dense in Ξ_* . Since $M : \Theta_\pi \rightarrow C$ given by $(\pi, \lambda, \tau) \mapsto M(\pi, \lambda, \tau)$ is continuous and $M(\Xi_*) = C_*$, we have that $M(\Gamma_0 \cup \Gamma_1)$ is dense in C_* . Moreover, by Lemma 7.4, C_* is dense in C which yields the result. □

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