SPECTRAL PROPERTIES OF COCYCLES OVER ROTATIONS

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Abstract

Let \( T: \mathbb{T}^d \to \mathbb{T}^d \) be an ergodic rotation. Given \( \varphi: \mathbb{T}^d \to \mathbb{T} \) a smooth cocycle we show that the set

\[
\{ f \in L^2(\mathbb{T}^d+1, \lambda_{d+1}) : \hat{\sigma}_f(n) = (U^n_{\varphi} f, f) = O\left( \frac{1}{|n|^{rw(\varphi)}} \right) \},
\]

where \( rw(\varphi) \) is the rank of the winding vector of \( \varphi \) is dense in the orthocomplement of the eigenfunctions of \( T \). In particular the skew product diffeomorphism \( T_{\varphi}: \mathbb{T}^{d+1} \to \mathbb{T}^{d+1} \) given by

\[
T_{\varphi}(z, \omega) = (Tz, \varphi(z)\omega)
\]

has countable Lebesgue spectrum in that orthocomplement. We construct an ergodic rotation \( T \) of \( \mathbb{T}^2 \) and a real analytic cocycle on \( \tilde{\varphi}: \mathbb{T}^2 \to \mathbb{R} \) such that an extension \( T \exp(2\pi i \tilde{\varphi}) \) is mixing in the orthocomplement of the eigenfunctions of \( T \).

Introduction

Let \( \mathbb{T}^d \) be a \( d \)-dimensional torus. We will consider an ergodic rotation of the \( d \)-dimensional torus given by

\[
T(z_1, \ldots, z_d) = (z_1 e^{2\pi i \alpha_1}, \ldots, z_d e^{2\pi i \alpha_d})
\]

where \( \alpha_1, \ldots, \alpha_d, 1 \) are independent over \( \mathbb{Q} \).

By a cocycle we mean a smooth map \( \varphi: \mathbb{T}^d \to \mathbb{T} \). Then, by Fubini Theorem a transformation \( T_{\varphi}: (\mathbb{T}^{d+1}, \lambda_{d+1}) \to (\mathbb{T}^{d+1}, \lambda_{d+1}) \) given by

\[
T_{\varphi}(z, \omega) = (Tz, \varphi(z)\omega)
\]

preserves Lebesgue measure \( \lambda_{d+1} \). The automorphism \( T_{\varphi} \) is called an extension of \( T \).

Such a cocycle \( \varphi \) can be represented as

\[
\varphi(e^{2\pi i x_1}, \ldots, e^{2\pi i x_d}) = e^{2\pi i (\tilde{\varphi}(x_1, \ldots, x_d) + m_1 x_1 + m_d x_d)}
\]

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where $m_1, ..., m_d \in \mathbb{Z}$ and $\tilde{\varphi} : \mathbb{R}^d \to \mathbb{R}$ is smooth, periodic of period 1 in each coordinate. In this representation of $\varphi$, the vector $(m_1, ..., m_d) \in \mathbb{Z}^d$ is unique, while $\tilde{\varphi}$ is unique up to an additive integer constant.

The vector $w(\varphi) = (m_1, ..., m_d)$ we call the \textit{winding vector} of a cocycle $\varphi$. The number $rw(\varphi) = \text{card}\{i : i = 1, ..., d, m_i \neq 0\}$ we call the \textit{rank} of the winding vector of a cocycle $\varphi$. For $d = 1$ the winding vector is equal to the degree $d(\varphi)$ of $\varphi$.


\textbf{Proposition 1.} If $d(\varphi) = 0$ and $\tilde{\varphi}$ is absolutely continuous, then the maximal spectral type of $T_\varphi$ is singular and is not mixing in the orthocomplement of the eigenfunctions of $T$.

In 1993, A. Iwanik, M. Lemańczyk and D. Rudolph [8] proved that

\textbf{Proposition 2.} If $d(\varphi) \neq 0$ and $\tilde{\varphi}$ is absolutely continuous and $\tilde{\varphi}'$ is of bounded variation, then $T_\varphi$ has countable Lebesgue spectrum in the orthocomplement of the eigenfunctions of $T$ and the set

$$\{ f \in L^2(T^2, \lambda_2) : \hat{\sigma}_f(n) = (U_{T_\varphi}^n f, f) = O\left(\frac{1}{|n|}\right)\}$$

is dense in that orthocomplement.

This result is a strengthening of an earlier result by Kushnirenko [11] (see also [2] pp.344).

We can interpret Proposition 1 and 2 as certain facts giving rise to a spectral stability of $T_\varphi$ where $\varphi$ is a character of $T$: indeed if we multiply $\varphi$ by a smooth cocycle $\psi$ of degree zero spectral properties of $T_\varphi$ and $T_{\varphi \psi}$ remain the same.

In this paper we will generalize these facts to multidimensional rotations for non zero winding vector smooth cocycles. In Section 3 we show that for $\varphi \in C^2(T)$, $T_\varphi$ has countable Lebesgue spectrum in the orthocomplement of the eigenfunctions of $T$ and for $\varphi \in C^2(T)$, the set

$$\{ f \in L^2(T^{d+1}, \lambda_{d+1}) : \hat{\sigma}_f(n) = (U_{T_\varphi}^n f, f) = O\left(\frac{1}{|n|^{rw(\varphi)}}\right)\}$$

is dense in that orthocomplement.

For zero winding vector smooth cocycles and $d \geq 2$ our result are rather to suggest that no spectral stability property holds. In Section 4 we construct an ergodic rotation $T$ of $\mathbb{T}^2$ and a real analytic cocycle on $\varphi : T^2 \to T$ such that an extension $T_\varphi$ is mixing in the orthocomplement of the eigenfunctions of $T$.

1 \hspace{1cm} \textbf{Notation and facts from spectral theory}

The substance of this section is classical (e.g. for an irrational rotation of the circle see [3], [8] and [13]).
Let $U$ be a unitary operator on a separable Hilbert space $H$. For any $f \in H$ we define the cyclic space $Z(f) = \text{span}\{U^n f : n \in \mathbb{Z}\}$. By the spectral measure $\sigma_f$ of $f$ we mean a Borel measure on $\mathbb{T}$ determined by the equalities

$$\hat{\sigma}_f(n) = \int_{\mathbb{T}} z^n d\sigma_f = (U^n f, f)$$

for $n \in \mathbb{Z}$.

**Theorem 1.1 (spectral theorem).** There exists a sequence $f_1, f_2, ...$ in $H$ such that

$$H = \bigoplus_{n=1}^{\infty} Z(f_n)$$

and $\sigma_{f_1} \gg \sigma_{f_2} \gg \ldots$.

Moreover, for any sequence $f'_1, f'_2, ...$ in $H$ satisfying (1) we have $\sigma_{f_1} \equiv \sigma_{f'_1}, \sigma_{f_2} \equiv \sigma_{f'_2}, \ldots$.

The spectral type of $\sigma_{f_1}$ (the equivalence class of measures) will be called the maximal spectral type of $U$. $U$ is said to have Lebesgue spectrum if $\sigma_{f_1} \equiv \lambda$ where $\lambda$ is Lebesgue measure on the circle. It is said that $U$ has Lebesgue spectrum of uniform multiplicity if $\sigma_{f_n} \equiv \lambda$ for $n = 1, 2, ..., k$ and $\sigma_{f_n} \equiv 0$ for $n > k$ where $k \in \mathbb{N} \cup \{\infty\}$.

Let $X$ be an infinite abelian group which is metric, compact and monothetic. Let $B$ be a $\sigma$-algebra of Borel sets on $X$ and $\mu$ be Haar measure on $X$. We will denote $H$ the space $L^2(X, B, \mu)$. We will consider an ergodic rotation of the group $X$ given by $T x = a \cdot x$, where $a$ is a cyclic generator of $X$.

For a cocycle (here by a cocycle we mean any Borel map) $F : X \to T$ we will consider a unitary operator $U : H \to H$ given by

$$(U f)(x) = F(x) f(Tx).$$

**Lemma 1.2.** The maximal spectral type of the operator $U$ is either discrete or continuous singular or Lebesgue.

**Lemma 1.3.** If the maximal spectral type of the operator $U$ is Lebesgue then the multiplicity function of $U$ is uniform.

**Lemma 1.4.** Suppose that $f \in H$ and $\sum_{n=1}^{\infty} |(U^n f, f)|^2 < +\infty$. Then $\sigma_f \ll \lambda$.

Denote

$$F^{(n)}(x) = \begin{cases} F(x) F(Tx) ... F(T^{n-1}x) & \text{if } n > 0 \\ 1 & \text{if } n = 0 \\ F(T^n x) F(T^{n+1}x) ... F(T^{-1}x)^{-1} & \text{if } n < 0 \end{cases}$$

**Corollary 1.1.** Suppose,

$$\sum_{n=1}^{\infty} \int_X |F^{(n)}(x) d\mu(x)|^2 < +\infty.$$ 

Then $U$ has Lebesgue spectrum of uniform multiplicity.
Let $G$ be a compact abelian group, $m$ its Haar measure and $\varphi : X \to G$ a cocycle. We will consider the extension $T_{\varphi} : (X \times G, \mu \times m) \to (X \times G, \mu \times m)$ given by

$$T_{\varphi}(x, g) = (Tx, \varphi(x)g).$$

Let us decompose

$$L^2(X \times G, \mu \times m) = \bigoplus_{\chi \in \hat{G}} H_{\chi}$$

where

$$H_{\chi} = \{ f : f(x, g) = h(x)\chi(g), h \in L^2(X, \mu) \}.$$

Observe that $H_{\chi}$ is closed $U_{T_{\varphi}}$-invariant subspace of $L^2(X \times G, \mu \times m)$, where $U_{T_{\varphi}} = f \circ T_{\varphi}$.

**Lemma 1.5.** The operator $U_{T_{\varphi}} : H_{\chi} \to H_{\chi}$ is unitarily equivalent to $U_{\chi} : H \to H$, where

$$(U_{\chi}h)(x) = \chi(\varphi(x))h(Tx).$$

## 2 Functions of bounded variation and absolutely continuous functions

Let $I^d$ denote the closed $d$-dimensional unit cube. By a partition $P$ of $I^d$, we mean a partition into cubes given by sequences

$$\{(\eta_0^{(j)}, \eta_1^{(j)}, \ldots, \eta_{m_j}^{(j)}) : 0 = \eta_0^{(j)} \leq \ldots \leq \eta_{m_j}^{(j)} = 1 , \ j = 1, \ldots, d\}.$$

Given such a partition, we define, for $j = 1, \ldots, d$ and $i = 1, \ldots, m_j - 1$ the operator $\Delta_{j,i} : C(I^d) \to C(I^d)$ by

$$\Delta_{j,i}f(x^{(1)}, \ldots, x^{(d)}) = f(x^{(1)}, \ldots, x^{(j-1)}, \eta_{i+1}^{(j)}, x^{(j+1)}, \ldots, x^{(d)}) - f(x^{(1)}, \ldots, x^{(j-1)}, \eta_{i}^{(j)}, x^{(j+1)}, \ldots, x^{(d)})$$

However, if it does not rise to a confusion, we will rather write

$$\Delta_j f(x^{(1)}, \ldots, x^{(j-1)}, \eta_i^{(j)}, x^{(j+1)}, \ldots, x^{(d)})$$

instead of $\Delta_{j,i}f(x^{(1)}, \ldots, x^{(j-1)}, \eta_i^{(j)}, x^{(j+1)}, \ldots, x^{(d)})$.

For $j \neq j'$ and $0 \leq i \leq m_j - 1$, $0 \leq i' \leq m_{j'} - 1$ we have

$$\Delta_{j,i}\Delta_{j',i'}f = \Delta_{j',i'}\Delta_{j,i}f$$

and for $j_1, \ldots, j_p$ such that $j_s \neq j_{s'}$ for $s \neq s'$ we will write

$$\Delta_{j_1,\ldots,j_p} = \Delta_{j_1,i_1} \ldots \Delta_{j_p,i_p}$$

where by the domain of $\Delta_{j_1,\ldots,j_p}$ we mean only points $(x^{(1)}, \ldots, x^{(d)})$, $x^{(j_s)} = \eta_{i_s}^{(j_s)}$ for some $i_s$. 

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Let $Q$ be a closed $d$-dimensional cube $\prod_{i=1}^{d} [a^{(i)}, b^{(i)}] \subset I^d$. Given $Q$ define for $j = 1, \ldots, d$ the operator $\Delta_j^*|Q : C^1(I^d) \to C^0(I^d)$ by

$$\Delta_j^*|Q f(x^{(1)}, \ldots, x^{(d)}) =
\begin{multline*}
f(x^{(1)}, \ldots, x^{(j-1)}, b^{(j)}, x^{(j+1)}, \ldots, x^{(d)}) - f(x^{(1)}, \ldots, x^{(j-1)}, a^{(j)}, x^{(j+1)}, \ldots, x^{(d)})
\end{multline*}$$

and let $\Delta_j^* \cdot \ldots \cdot \Delta_p^*|Q$ stand for $\Delta_1^* \cdot \ldots \cdot \Delta_p^*|Q$.

**Definition 2.1.** For a function $f : I^d \to \mathbb{C}$ we set

$$Var_d f = \sup_{P \in \mathcal{P}} \sum_{i_1, \ldots, i_d = 0}^{m_1-1} \sum_{i_2, \ldots, i_d = 0}^{m_2-1} |\Delta_1 \ldots \Delta_d f(\eta_{i_1}, \ldots, \eta_{i_d})|,$$

where $\mathcal{P}$ is the family of all partitions $P$ of $I^d$. If $Var_d f$ is finite, then $f$ is said to be of bounded variation on $I^d$ in the sense of Vitali.

**Definition 2.2.** Let $f : I^d \to \mathbb{C}$ be a function of bounded variation in the sense of Vitali. Suppose that the restriction of $f$ to each face $F = \{(x^{(1)}, \ldots, x^{(d)}) : x^{(i_s)} = 0, s = 1, \ldots, k\}$ where $1 \leq i_1 < \ldots < i_k \leq d$ ($k = 1, \ldots, d$) is of bounded variation on $F$ in the sense of Vitali. Then $f$ is said to be of bounded variation on $I^d$ in the sense of Hardy and Krause.

In what follows functions of bounded variation are those of bounded variation in the sense of Hardy and Krause.

**Remark.** If a function is of bounded variation, then it is integrable in sense of Riemann (for $d = 2$, see [7] §448).

Given $0 \leq p \leq n$ on the set $S_n$ all permutation of $\{1, \ldots, n\}$ consider the following equivalence relation

$$\sigma \equiv \sigma' \iff \sigma(\{1, \ldots, p\}) = \sigma'(\{1, \ldots, p\})$$

We will consider an expression $F(i_1, \ldots, i_n)$, $(i_k \in \mathbb{N})$ such that

$$F(i_{\sigma(1)}, \ldots, i_{\sigma(n)}) = F(i_{\sigma'(1)}, \ldots, i_{\sigma'(n)})$$

whenever $\sigma \equiv \sigma'$. By

$$\sum_{i \in \mathbb{N}} F(i)$$

we denote the sum $\sum_{[\sigma] \in S_n / \equiv} F(i_{\sigma(1)}, \ldots, i_{\sigma(n)})$.

Let $f : I^d \to \mathbb{C}$ be a function of bounded variation. Given $0 \leq k \leq d$ and $(a^{(k+1)}, \ldots, a^{(d)}) \in I^{d-k}$ consider the function $g : I^k \to \mathbb{C}$ given by

$$g(x^{(1)}, \ldots, x^{(k)}) = f(x^{(1)}, \ldots, x^{(k)}, a^{(k+1)}, \ldots, a^{(d)}).$$
For each $0 \leq p \leq d - k$ consider

$$F_p(k+1, \ldots, d) = Var^{(k+p)} f(\ldots, \ldots, 0, \ldots, 0)$$

and notice that expressions of this kind satisfy (2).

**Lemma 2.1.**

$$Var^{(k)} g \leq \sum_{p=0}^{d-k} \sum_{k+p \text{ coordinates}} Var^{(k+p)} f(\ldots, 0, \ldots, 0).$$

**Proof.** We first prove (by induction on $l$) that for a function $h : I^l \to \mathbb{C}$ and $(y^{(1)}, \ldots, y^{(l)}) \in I^l$ and a partition given by $\{(0, y^{(j)}), j = 1, \ldots, l\}$ we have

$$h(y^{(1)}, \ldots, y^{(l)}) - h(0, \ldots, 0) = \sum_{p=1}^{l} \sum_{j} \Delta_{1\ldots p} f(0, \ldots, 0).$$

1. Obviously, (3) holds for $l = 1$.
2. Assuming (3) to hold for $l$, we will prove it for $l + 1$.

$$h(y^{(1)}, \ldots, y^{(l+1)}) - h(0, \ldots, 0) =$$

$$h(y^{(1)}, \ldots, y^{(l)}) - h(0, \ldots, 0, y^{(l+1)}) + \Delta_{l+1} h(0, \ldots, 0) =$$

$$\sum_{p=1}^{l} \sum_{j} \Delta_{1\ldots p} (l+1) h(0, \ldots, 0) + \sum_{p=1}^{l} \sum_{j} \Delta_{1\ldots p} h(0, \ldots, 0) + \Delta_{l+1} h(0, \ldots, 0) =$$

$$\sum_{p=1}^{l+1} \sum_{j} \Delta_{1\ldots p} h(0, \ldots, 0).$$

Let $P$ be a partition of $I^k$ given by $\{(\eta_0^{(j)}, \eta_1^{(j)}, \ldots, \eta_{m_j}^{(j)}): 0 = \eta_0^{(j)} \leq \ldots \leq \eta_{m_j}^{(j)} = 1, \ j = 1, \ldots, k\}$. Consider a partition $P'$ of $I^d$ given by $\{(\eta_0^{(j)}, \eta_1^{(j)}, \ldots, \eta_{m_j}^{(j)}): 0 = \eta_0^{(j)} \leq \ldots \leq \eta_{m_j}^{(j)} = 1, \ j = 1, \ldots, k\} \cup \{(0, a^{(j)}, 1): j = k+1, \ldots, d\}$. Then

$$\sum_{t_1=0}^{m_1-1} \ldots \sum_{t_k=0}^{m_k-1} |\Delta_{1\ldots k} g(\eta_1^{(1)}, \ldots, \eta_k^{(k)})| =$$

$$\sum_{t_1=0}^{m_1-1} \ldots \sum_{t_k=0}^{m_k-1} |\Delta_{1\ldots k} f(\eta_1^{(1)}, \ldots, \eta_k^{(k)}, a^{(k+1)}, \ldots, a^{(d)})| \leq$$

$$\sum_{p=0}^{d-k} \sum_{t_1=0}^{m_1-1} \ldots \sum_{t_k=0}^{m_k-1} |\Delta_{1\ldots k+p} f(\eta_1^{(1)}, \ldots, \eta_k^{(k)}, 0, \ldots, 0)| \leq$$
\[ \sum_{p=0}^{d-k} \sum_{k+1,...,d,p} \text{Var}^{k+p}(f) \int_{I_p}^{k+p} \]

and consequently

\[ \text{Var}^{(k)} \leq \sum_{p=0}^{d-k} \sum_{k+1,...,d,p} \text{Var}^{(k+p)}(f) \int_{I_p}^{k+p} \]

Remark. If \( f \) is continuous then

\[ I = \lim_{\delta \to 0} \sum_{i=0}^{m_i-1} \Delta_i f \]

Let \( P \) be a partition of \( I^d \) given by \( \{(\eta_0^{(j)}, \eta_1^{(j)}, ..., \eta_n^{(j)}) : 0 = \eta_0^{(j)} \leq ... \leq \eta_n^{(j)} = 1, j = 1, ..., d\} \). Then

\[ \delta(P) = \max_{\{(i_1, ..., i_d) : 0 \leq i_s \leq m_s-1\}} \prod_{j=1}^{d} |\eta_{j+1}^{(j)} - \eta_{j}^{(j)}| \]

we will be called the diameter of the partition \( P \).

**Definition 2.3.** Let \( f, g : I^d \to \mathbb{C} \) and let \( f \) be bounded. If for each sequence of partitions \( P_k \) given by \( \{(\eta_0^{(j,k)}, \eta_1^{(j,k)}, ..., \eta_n^{(j,k)} : j = 1, ..., d\} \) such that \( \lim_{k \to \infty} \delta(P_k) = 0 \) and for any sequence \( \{\xi_i^{(k)}_{s,i} : s = 1, ..., d, k \in \mathbb{N}\} \) where \( \xi_i^{(k)}_{s,i} \in [\eta_{s,i}^{(j,k)}, \eta_{s,i+1}^{(j,k)}] \) we have

\[ \lim_{k \to \infty} \sum_{i_1=0}^{m_{1,k}-1} ... \sum_{i_d=0}^{m_{d,k}-1} f(\xi_i^{(k)}_{s,i}) \Delta_i g(\xi_i^{(1,k)}, ..., \xi_i^{(d,k)}) = I, \]

then \( I \) is called the Riemann-Stieltjes integral of \( f \) and is denoted \( \int_I f \, dg \).

**Remark.** If \( f, g \) both are functions of bounded variation and if one of the functions is continuous then \( \int_I f \, dg \) exists (for \( d = 2 \), see [7] §448).

**Remark.** If \( \int_I f \, dg \) exists and \( g \) is of bounded variation in the sense of Vitali, then

\[ |\int_I f \, dg| \leq \sup_{x \in I^d} |f(x)| \text{Var}(d)g. \]

Let \( f, g : I^d \to \mathbb{C} \) both be functions of bounded variation and let one of them is continuous. For \( 0 \leq p \leq d \) consider

\[ F_p(1, ..., d) = \Delta_{p+1,...,d}^* \int_{I_p} g(\cdot, ..., \cdot, 0, ..., 0) df(\cdot, ..., \cdot, 0, ..., 0) \]

and notice that expressions of this kind satisfy (2).

**Theorem 2.2 (Integration by parts).** We have

\[ \int_I f dg = \sum_{p=0}^{d} (-1)^p \sum_{1,...,d,p} \Delta_{p+1,...,d}^* \int_{I_p} g(\cdot, ..., \cdot, 0, ..., 0) df(\cdot, ..., \cdot, 0, ..., 0). \]

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\textbf{Proof.} For \( d = 2 \), see [7] §448. We can prove this theorem using Lemma 5.2 from [10] ch.2 §5. □

\textbf{Corollary 2.1.} If \( f \) and \( g \) be periodic of period 1 in each coordinate, then

\[
\int_{I^d} f \, dg = (-1)^d \int_{I^d} g \, df. \quad \blacksquare
\]

Given \( 0 = s_0 \leq s_1 \leq \ldots \leq s_{k-1} \leq s_k = n \) on the set \( S_n \) all permutations of \( \{1, \ldots, n\} \) consider the following equivalence relation

\[
\sigma \equiv \sigma' \iff \sigma(s_{l-1} + 1, \ldots, s_l) = \sigma'(s_{l-1} + 1, \ldots, s_l) \quad \text{for} \quad l = 1, \ldots, k.
\]

We will consider an expression \( F(i_1, \ldots, i_n), (i_k \in \mathbb{N}) \) such that

\[
(4) \quad F(i_{\sigma(1)}, \ldots, i_{\sigma(n)}) = F(i_{\sigma'(1)}, \ldots, i_{\sigma'(n)}) \quad \text{whenever} \quad \sigma \equiv \sigma'.
\]

By

\[
\sum_{i_1, \ldots, i_n | s_1, \ldots, s_{k-1}} F(i_1, \ldots, i_n) \quad \text{we denote the sum} \quad \sum_{[\sigma] \in S_n/\equiv} F(i_{\sigma(1)}, \ldots, i_{\sigma(n)}).
\]

Let \( f_1, \ldots, f_k : I^d \to \mathbb{C} \) be functions of bounded variation. For \( 0 = s_0 \leq s_1 \leq \ldots \leq s_{k-1} \leq s_k = n \) consider

\[
F_{s_1 \ldots s_k}(1, \ldots, d) = \prod_{r=1}^{k} \sum_{\alpha_r = 0}^{d-s_r+s_r-1} \sum_{1, \ldots, s_r-1, s_r+1, \ldots, d} Var^{(\alpha_r+s_r-s_r-1)} f_r(i_{\sigma(1)}, \ldots, 0, \ldots, \sigma, \ldots, 0, \ldots, 0)
\]

and notice that expressions of this kind satisfy (4).

\textbf{Lemma 2.3.} The product \( f_1 \cdot \ldots \cdot f_k \) is of bounded variation and we have

\[
Var^{(d)} f_1 \cdot \ldots \cdot f_k \leq \sum_{0=s_0 \leq s_1 \leq \ldots \leq s_{k-1} \leq s_k = n} \prod_{r=1}^{k} \sum_{\alpha_r = 0}^{d-s_r+s_r-1} \sum_{1, \ldots, s_r-1, s_r+1, \ldots, d} Var^{(\alpha_r+s_r-s_r-1)} f_r(i_{\sigma(1)}, \ldots, 0, \ldots, \sigma, \ldots, 0, \ldots, 0). \quad \blacksquare
\]
Lemma 2.4. Assume that there exists a real number a such that 0 < a ≤ |f(x)| for every x ∈ I^d. Then \( \frac{1}{x} : I^d \to \mathbb{C} \) is a function of bounded variation and we have:

\[
\text{Var}(d) \frac{1}{x} \leq \sum_{k=1}^{d} \frac{1}{a^{k+1}} \sum_{0<s_0<\ldots<s_{k-1}<s_k=d} \sum_{r=0}^{s_k} \prod_{r=1}^{k} \sum_{r=1}^{s_r} \text{Var}^{(\alpha_r+s_r-s_{r-1})} f(s_0, \ldots, 0, 0, \ldots, 0, 0).
\]

Definition 2.4. We say that a function \( f : I^d \to \mathbb{C} \) has the derivative in the sense of Vitali at \((x^{(1)}, \ldots, x^{(d)}) \in I^d \) if there exists limit

\[
\lim_{(h^{(1)}, \ldots, h^{(d)}) \to 0, h^{(i)} \neq 0, 0 \leq |h^{(i)}| < 1} \frac{\Delta f_{x^{(1)}, \ldots, x^{(d)}}}{h^{(1)} \ldots h^{(d)}}(x).
\]

This limit is called the derivative of \( f \) and is denoted \( Df(x^{(1)}, \ldots, x^{(d)}) \).

Remark. If \( f \in C^d(I^d) \) then \( Df(x) = \frac{\partial^2 f}{\partial x^{(1)}}(x) \) (see [12] ch.7 §1).

Remark. If a function \( f : I^d \to \mathbb{C} \) is of bounded variation in the sense of Vitali, then \( f \) has the derivative in the sense of Vitali almost everywhere (see [12] ch.7 §2).

Definition 2.5. (inductive) A function \( f : I^d \to \mathbb{C} \) is said to be differentiable in the sense of Hardy and Krause-

- for \( d = 1 \) if it is differentiable in the ordinary sense,
- for \( d > 1 \) if it has the derivative in the sense of Vitali in every point and for any \( j = 1, \ldots, d \) and \( a \in I \) the function \( f_j : I^d \to \mathbb{C} \)

\[
f_j(x^{(1)}, \ldots, x^{(d-1)}) = f(x^{(1)}, \ldots, x^{(j-1)}, a, x^{(j)}, \ldots, x^{(d-1)})
\]

is differentiable in the sense of Hardy and Krause.
In what follows by differentiable functions we mean those which are differentiable in the sense of Hardy and Krause. The derivative of \( f(\hat{x}^{(1)}, ..., \hat{x}^{(i_1)}, ..., \hat{x}^{(i_k)}, ..., \hat{x}^{(d)}) \) is denoted \( D_{\hat{x}^{(i_1)}, ..., \hat{x}^{(i_k)}, ..., \hat{x}^{(d)}} f(x) \).

Let \( f : I^d \to \mathbb{C} \) be a differentiable function. For \( 0 = s_0 < s_1 < ... < s_{k-1} < s_k = d \) consider

\[
F_{s_1, ..., s_k}(1, ..., d) = \prod_{r=1}^{k} D_{x^{(r-1)+1}, ..., x^{(r)}} f(x)
\]

and notice that expressions of this kind satisfy (4).

**Lemma 2.5.** The function \( \exp f : I^d \to \mathbb{C} \) is differentiable and we have

\[
D \exp f(x) = \exp f(x) \sum_{k=1}^{d} \sum_{s_0 < s_1 < ... < s_{k-1} < s_k = d} \sum_{r=1}^{k} D_{x^{(r-1)+1}, ..., x^{(r)}} f(x). \]

The number \( |P| = \prod_{i=1}^{d} (b^{(i)} - a^{(i)}) \) is called the *substance* of the cube \( P = \prod_{i=1}^{d} [a^{(i)}, b^{(i)}] \).

**Definition 2.6.** A function \( f : I^d \to \mathbb{C} \) is said to be *absolutely continuous in the sense of Vitali* if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for every system of cubes \( Q_1, ..., Q_n \) such that \( |Q_1 \cap Q_j| = 0 \) for any \( 1 \leq i \neq j \leq n \) if \( |Q_1| + ... + |Q_n| < \delta \) then

\[
|\Delta^*_1, d(Q_1, f) + ... + \Delta^*_1, d(Q_n, f)| < \varepsilon.
\]

**Remark.** If a function is absolutely continuous in the sense of Vitali then it is of bounded variation in the sense of Vitali (see [12] ch. 7 §3).

**Definition 2.7.** Let \( f : I^d \to \mathbb{C} \) be an absolutely continuous function in the sense of Vitali. Suppose the restriction \( f \) of each face \( F = \{(x^{(1)}, ..., x^{(d)}) : x^{(s)} = 0, s = 1, ..., k \} \) where \( 1 \leq i_1 < ... < i_k \leq d \) \( (k = 1, ..., d) \) is absolutely continuous function in the sense of Vitali. Then \( f \) is said to be *absolutely continuous function in the sense of Hardy and Krause*.

In what follows functions absolutely continuous are those absolutely continuous in the sense of Hardy and Krause.

**Remark.** If a function \( f \) is of bounded variation and \( g \) is absolutely continuous then

\[
\int_{I^d} f dg = \int_{I^d} f Dg d\lambda_d
\]


**Lemma 2.6.** Let \( f : I^d \to \mathbb{C} \) be an absolutely continuous function. Then for every \( g(a^{(k+1)}, ..., a^{(d)}) \in I^{d-k} \) the function \( g : I^k \to \mathbb{C} \) given by

\[
g(x^{(1)}, ..., x^{(k)}) = f(x^{(1)}, ..., x^{(k)}, a^{(k+1)}, ..., a^{(d)})
\]

is absolutely continuous.

**Proof.** Similarly as the proof of Lemma 2.1. ■

**Remark.** If a function \( f : I^d \to \mathbb{R} \) is absolutely continuous then the function \( \exp if : I^d \to \mathbb{C} \) is absolutely continuous.
3 Spectral properties in the case where the winding vector is not equal to zero

Lemma 3.1. Let $f : I^d \to \mathbb{R}$ be an absolutely continuous function such that for any $j = 1, \ldots, d$ and $x \in I^d$ we have $\Delta_j^f(x) \in \mathbb{Z}$. Suppose, $D_{\beta_{(t_1, \ldots, t_r)}} f$ is the function of bounded variation for $1 \leq i_1 < \ldots < i_k \leq d$ and there exists real a number $\alpha > 0$ such that for any $x \in I^d$ we have

$$| \sum_{k=1}^{d} (2\pi i)^k \sum_{0=s_0<s_1<\ldots<s_{k-1}<s_k=d}^{*} \sum_{r=1}^{k} D_{(\alpha_{(r-1)+1}) \ldots (\alpha_r)} f(x) | \geq \alpha > 0.$$ 

Then

$$| \int_{I^d} \exp 2\pi i f(x) dx | \leq$$

$$\sum_{k=1}^{d} \left( \frac{1}{a^{k+1}} \sum_{0=t_0<t_1<\ldots<t_{k-1}<t_k=d}^{*} \prod_{p=1}^{d-t_0+t_{p+1}} \sum_{\alpha_p=0}^{a} \sum_{1=\alpha_{r.p}, t_{p+1}=1, \ldots, t_p, u_1, \ldots, u_{k-r} \in a}^{*} \prod_{r=1}^{k} \left( D_{(\beta_{(r-1)+1}) \ldots (\beta_r)} f_{(x_{(r-1)+1}) \ldots (x_r)} \right) \right) \left( Var_{(\beta_{(r-1)+1}) \ldots (\beta_r)} D_{(\beta_{(r-1)+1}) \ldots (\beta_r)} f(x_{(r-1)+1}) \ldots (x_r) \right).$$

Proof. An application of Lemma 2.3 and Lemma 2.4 and integration by parts gives that

$$| \int_{I^d} \exp 2\pi i f(x) dx | \leq$$

$$| \int_{I^d} 1/ \left( \sum_{k=1}^{d} (2\pi i)^k \sum_{0=s_0<s_1<\ldots<s_{k-1}<s_k=d}^{*} \sum_{r=1}^{k} D_{(\alpha_{(r-1)+1}) \ldots (\alpha_r)} f | d \exp 2\pi i f(x) \right) =$$

$$| \int_{I^d} \exp 2\pi i f(x) dx (1/ \sum_{k=1}^{d} (2\pi i)^k \sum_{0=s_0<s_1<\ldots<s_{k-1}<s_k=d}^{*} \sum_{r=1}^{k} D_{(\alpha_{(r-1)+1}) \ldots (\alpha_r)} f) | \leq$$

$$Var^{(d)}(1/ \sum_{k=1}^{d} (2\pi i)^k \sum_{0=s_0<s_1<\ldots<s_{k-1}<s_k=d}^{*} \sum_{r=1}^{k} D_{(\alpha_{(r-1)+1}) \ldots (\alpha_r)} f) \leq$$
Lemma 3.2. Let \( \alpha_1, ..., \alpha_d \) be independent over \( \mathbb{Q} \) real numbers. Assume that \( \varphi : I^d \to \mathbb{R} \) is an absolutely continuous function, which is periodic of period 1 in each coordinate. Suppose, \( D_{x^{(i_1)}}...x^{(i_k)} \varphi \) is the function of bounded variation for each \( 1 \leq i_1 < ... < i_k \leq d \). Then for any \( (m_1, ..., m_d) \in \mathbb{Z}^d \) where \( m_i \neq 0 \) for \( i = 1, ..., d \) and \( N \in \mathbb{Z} \setminus \{0\} \) there exists a polynomial \( F \) of \( 4^d \) variables with nonnegative coefficients such that

\[
\left| \int_{I^d} \exp 2\pi i N(\varphi^{(n)}(x) + \sum_{k=1}^{d} m_k x^{(k)})dx \right| \leq \frac{1}{|N|^d} F(Var(\varphi) D_{x^{(i_1)}}...x^{(i_k)} f(0, ..., 0, j_1, 0, ..., 0, j_r, 0, ..., 0) : 1 \leq i_1 < ... < i_k \leq d, 1 \leq j_1 < ... < j_r \leq d)
\]

where \( \alpha = (\alpha_1, ..., \alpha_d) \) and

\[
\varphi^{(n)}(x) = \begin{cases} 
\varphi(x) + ... + \varphi(x + (n - 1)\alpha) & \text{for } n > 0 \\
0 & \text{for } n = 0 \\
-(\varphi(x + n\alpha) + ... + \varphi(x - \alpha)) & \text{for } n < 0.
\end{cases}
\]
Proof. Let \( f(x^{(1)}, \ldots, x^{(d)}) = N(\tilde{\varphi}^{(n)}(x) + \sum_{k=1}^{d} m_k n x^{(k)}) \). Then

\[
D_{x^{(1)} \cdots x^{(i)} f}(x) = N(D_{x^{(1)} \cdots x^{(i)} f}(x) + m_i n) \quad \text{for} \quad i = 1, \ldots, d
\]

and

\[
D_{x^{(1)} \cdots x^{(i_1)} \cdots x^{(i_k)} f}(x) = N(D_{x^{(1)} \cdots x^{(i_1)} \cdots x^{(i_k)} f}(x)) \quad \text{for} \quad 1 \leq i_1 < \ldots < i_k \leq d \quad \text{and} \quad k > 1.
\]

We will consider a real number \( \frac{1}{2} > \varepsilon > 0 \).

Since for each \( 1 \leq i_1 < \ldots < i_k \leq d \) the function \( D_{x^{(1)} \cdots x^{(i_k)} f}(x) \) is integrable in the sense of Riemann and the rotation of \( \alpha \) is monochromatic, there exists a natural number \( n_0 \) such that for any \( |n| \geq n_0, 1 \leq i_1 < \ldots < i_k \leq d \) and \( x \in I^d \) we have

\[
\left| \frac{D_{x^{(1)} \cdots x^{(i_k)} f}(x)}{n} \right| = \int_{I^d} D_{x^{(1)} \cdots x^{(i_k)} f}(x) dx < \varepsilon.
\]

From

\[
\int_{I^d} D_{x^{(1)} \cdots x^{(i_k)} f}(x) dx = \int_{I^d-k} \left( \int_{I^d} D_{x^{(1)} \cdots x^{(i_k)} f}(x) dx^{(i_1)} \cdots dx^{(i_k)} dx^{(1)} \cdots dx^{(d)} \right)
\]

we obtain that for \( |n| \geq n_0 \)

\[
|D_{x^{(1)} \cdots x^{(i_k)} f}(x)| < \varepsilon |n|.
\]

Let \( |n| \geq \max(n_0, d!2^{d-2} M) \) where \( M = \max_{i=1, \ldots, d} |m_i| + 1 \). Then for any \( x \in I^d \) we have

\[
\left| \sum_{k=1}^{d} (2\pi i)^{k} \sum_{0 \leq s_0 < s_1 < \ldots < s_{k-1} < s_k = d} \prod_{r=1}^{k} D_{x^{(r-1+1)} \cdots x^{(r)}} f(x) \right| \geq (2\pi |N|)^d \prod_{k=1}^{d} |D_{x^{(k)}} \tilde{\varphi}^{(n)}(x) + m_k n| - \sum_{k=1}^{d-1} (2\pi |N|)^{k} \sum_{0 \leq s_0 < s_1 < \ldots < s_{k-1} < s_k = d} \prod_{r=1}^{k} |D_{x^{(r-1+1)} \cdots x^{(r)}} \tilde{\varphi}^{(n)}(x) + m_{a_{s_{r-1}+1 \ldots s_r}} n| \geq (2\pi |N|)^d \prod_{k=1}^{d} |n|(|m_k| - \varepsilon) - \sum_{k=1}^{d-1} (2\pi |N|)^{k} \sum_{0 \leq s_0 < s_1 < \ldots < s_{k-1} < s_k = d} \sum_{s_1 < \ldots < s_{k-1}} (M|n|)^k \geq
\]
\[(\pi |Nn|^d - d!(2M\pi |Nn|)^{d-1} \geq (\pi |N|^d |n|^{d-1} (|n| - d!2^{d-1}M) \geq \frac{1}{2} (\pi |N|^d) = C |n|^d.\]

By Lemma 3.1 we have

\[| \int_{\mathbb{R}^d} \exp 2\pi i N(\varphi^{(n)}(x) + \sum_{k=1}^{d} m_k x^{(k)})dx | \leq \]

\[\sum_{l=1}^{d} \frac{1}{C^{l+1}|n|^{d(l+1)}} \sum_{0=t_0 < t_1 < ... < t_{l-1}<t_l=d} \prod_{p=1}^{l} \sum_{t_p \in \mathbb{Z}^d} \sum_{\alpha_p=0}^{*} \sum_{1,...,t_p-1,t_p+1,...,d} \]

\[\sum_{k=1}^{d} (2\pi)^k \sum_{0=s_0 < s_1 < ... < s_{k-1}<s_k=d} \prod_{r=1}^{k} \sum_{\beta_r=0}^{*} \sum_{1,...,s_r-1,s_r+1,...,\alpha_p+\beta_r-1} \sum_{1,...,t_p} \sum_{u_r} \]

\[Var(\beta_r+u_r-u_r-1) (ND_{x^{(s_r-1+1)}...x^{(s_r)}} \varphi(\alpha(...0,...0,...,0,...0)+na_{r-1+...}) \leq \]

\[\sum_{l=1}^{d} \frac{|n|^d}{C^{l+1}|n|^{d(l+1)}} \sum_{0=t_0 < ... < t_{l-1}<t_l=d} \prod_{p=1}^{l} \sum_{\alpha_p=0}^{*} \sum_{1,...,t_p-1,t_p+1,...,d} \]

\[\sum_{k=1}^{d} (2\pi |N|)^k \sum_{0=s_0 < s_1 < ... < s_{k-1}<s_k=d} \prod_{r=1}^{k} \sum_{\beta_r=0}^{*} \sum_{1,...,s_r-1,s_r+1,...,\alpha_p+\beta_r-1} \sum_{1,...,t_p} \sum_{u_r} \]

\[Var(\beta_r+u_r-u_r-1) D_{x^{(s_r-1+1)}...x^{(s_r)}} \varphi(...) \leq \sum_{l=1}^{d} \frac{|n|^d}{|n|^{d(l+1)}} F_l = \frac{1}{|n|^d} F. \]

**Remark.** With the same assumption as the one in Lemma 3.2 we can prove that for any \((r_1,...,r_d) \in \mathbb{Z}^d\) there exists a polynomial \(F\) of \(4^d\) variables with nonnegative coefficients such that

\[| \int_{\mathbb{R}^d} \exp 2\pi i N(\varphi^{(n)}(x) + \sum_{k=1}^{d} (Nm_k n + r_k) x^{(k)})dx | \leq \]

\[\frac{F}{|n|^d}.\]
Theorem 3.3. Let \(\alpha_1, ..., \alpha_d, 1\) be independent over \(\mathbb{Q}\) real numbers. Let a cocycle \(\varphi : \mathbb{T}^d \to \mathbb{T}\) be represented as

\[
\varphi(e^{2\pi i x_1}, ..., e^{2\pi i x_d}) = e^{2\pi i \sum \alpha_i x_i + \sum m_i x_i + m_d x_d}
\]

where \(\tilde{\varphi} : \mathbb{I}^d \to \mathbb{R}\) satisfies the same assumption as the one in Lemma 3.2. If \(r \in \mathbb{N}\) then the set

\[
\{ f \in L^2(\mathbb{I}^{d+1}, \lambda_{d+1}) : \hat{\sigma}_f(n) = (U^n f, f) = O\left(\frac{1}{|n|}\right) \}
\]

is dense in the orthocomplement of the eigenfunctions of \(T\).

Proof. For simplicity we may assume that \(m_1 \neq 0, ..., m_k \neq 0\). By Lemma 2.1 there exists a real number \(M > 0\) such that for any \(1 \leq i_1 < \ldots, i_p \leq k, 1 \leq j_1 < \ldots, j_l \leq k\) and \((x^{(k+1)}, \ldots, x^{(d)}) \in I^{d-k}\) we have

\[
V^{r_1(n)}D_{x^{(n)}}\tilde{\varphi}(0, \ldots, 0, j_1, 0, \ldots, 0, j_l, 0, \ldots, 0, x^{(k+1)}, \ldots, x^{(d)}) \leq M
\]

Let \(P\) be a trigonometric polynomial given by

\[
P(z_1, ..., z_d, \omega) = \sum_{r_1=\sigma_{-1}}^{R_1} \ldots \sum_{r_d=\sigma_{-1}}^{R_d} \sum_{s \neq 0} a_{r_1 \ldots r_d s} z_1^{r_1} \ldots z_d^{r_d} \omega^s
\]

where \(a_{r_1 \ldots r_d s} \in \mathbb{C}\). Then

\[
|\langle U^n_{T_\omega} P, P \rangle| = |\int_{I^{d+1}} P(T^n z, \varphi^{(n)}(z) \omega) |\tilde{P}(z, \omega)|dzd\omega| =
\]

\[
\left| \int_{I^{d+1}} \sum_{r_1, \ldots, r_d, s} a_{r_1 \ldots r_d s} \exp 2\pi i \sum_{j=1}^{d} r_j x_j^{(j)} + n \alpha_j + s \varphi^{(n)}(x) + s \sum_{j=1}^{d} m_j nx_j^{(j)} + \frac{(n-1)n}{2} \alpha_j + sy \right|
\]

\[
\sum_{r_1, \ldots, r_d, s} a_{r_1 \ldots r_d s} \exp 2\pi i \left(\sum_{j=1}^{d} r_j^{(j)} x_j + sy\right) dx(1) \ldots dx(d) dy \leq
\]

\[
\sum_{r_1, \ldots, r_d, r_1', \ldots, r_d'} |a_{r_1 \ldots r_d a r_1' \ldots r_d'}| \int_{I^d} \exp 2\pi i \left[ \varphi^{(n)}(x) + [sn \sum_{j=1}^{d} m_j x_j^{(j)} + \sum_{j=1}^{d} (r_j - r_j') x_j^{(j)}] dx \right]
\]

\[
\sum_{r_1, \ldots, r_d, r_1', \ldots, r_d'} |a_{r_1 \ldots r_d a r_1' \ldots r_d'}| \int_{I^{d-k}} \exp 2\pi i \left[ \sum_{j=1}^{d} (r_j - r_j') x_j^{(j)} dx^{(k+1)} \ldots dx(d) \right]
\]

\[
|\int_{I^k} \exp 2\pi i \varphi^{(n)}(x) + [sn \sum_{j=1}^{k} m_j x_j^{(j)} + \sum_{j=1}^{k} (r_j - r_j') x_j^{(j)}] dx^{(1)} \ldots dx^{(k)} | \leq
\]

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\[ \sum_{r_1, \ldots, r_\ell, r_1', \ldots, r_\ell'} |a_{r_1 \ldots r_\ell} a_{r_1' \ldots r_\ell'}| \frac{F_{r_1 \ldots r_\ell - r_1' \ldots r_\ell'}(M)}{|n|^k} = O\left( \frac{1}{|n|^k} \right). \]

**Corollary 3.1.** If \( \varphi \in C^2 \) and \( \text{rw}(\varphi) = k > 0 \) then the set
\[
\{ f \in L^2(\mathbb{T}^{d+1}, \lambda_{d+1}) : \hat{\sigma}_f(n) = (U^n_\varphi f, f) = O\left( \frac{1}{|n|^k} \right) \}
\]
is dense in the orthocomplement of the eigenfunctions of \( T. \)

Let \( w(\varphi) \neq 0 \). For simplicity we assume that \( m_1 \neq 0 \). Suppose, there exists a real number \( R > 0 \) such that for each \( (x^{(2)}, \ldots, x^{(d)}) \in I^{d-1} \)
\[ V(\varphi^{(1)}) \frac{\partial \hat{\varphi}}{\partial x^{(1)}}(\cdot, x^{(2)}, \ldots, x^{(d)}) \leq R. \]

In the same manner as in the proof of Theorem 3.3 we can show that
\[ \hat{\sigma}_\chi_N(n) = O\left( \frac{1}{|n|^k} \right) \text{ for } N \neq 0 \]
where \( \chi_N(z_1, \ldots, z_d, \omega) = \omega^N. \) From this and by Corollary 1.1 we conclude that \( T_\varphi \) has countable Lebesgue spectrum in the orthocomplement of the eigenfunctions of \( T. \)

**Corollary 3.2.** If \( \varphi \in C^2 \) and \( w(\varphi) \neq 0 \) then \( T_\varphi \) has countable Lebesgue spectrum in the orthocomplement of the eigenfunctions of \( T. \)

### 4 Spectral properties in the case where the winding vector is equal zero

**Lemma 4.1.** If \( 0 < |x| \leq |y| \leq \frac{1}{2}, \) then
\[ 4|x| \leq |e^{2\pi ix} - 1| \leq 2\pi|x|, \]
\[ \frac{2\pi y}{x} \leq | \frac{e^{2\pi iy} - 1}{e^{2\pi ix} - 1} | \leq \frac{|y|}{x}. \]

**Lemma 4.2.** Assume \( n \in \mathbb{N} \) and take \( a \in \mathbb{R} \) such that \( 0 < a < 1. \) Then there exist \( n \) pairwise disjoint subintervals \( I_1, \ldots, I_n \) of \( I \) such that for \( x \in I \setminus \bigcup_{i=1}^n I_i \) we have \( \cos n\pi x \geq a \) moreover \( |I_i| = \frac{a}{n}. \)

**Proof.** Set \( I_i = \left[ \frac{2i - 1}{2n} - \frac{a}{2n}, \frac{2i - 1}{2n} + \frac{a}{2n} \right]. \) Then
\[ I \setminus \bigcup_{i=1}^n I_i = \bigcup_{i=1}^n \left[ \frac{2i - 2}{2n}, \frac{2i - 1}{2n} - \frac{a}{2n} \right] \cup \left( \frac{2i - 1}{2n} + \frac{a}{2n}, \frac{2i}{2n} + \frac{a}{2n} \right). \]
If \( x \in I \setminus \bigcup_{i=1}^{n} I_i \), then there exists a natural number \( i \) such that
\[
x \in \left[ \frac{2i - 2}{2n}, \frac{2i - 1}{2n} \right] \cup \left( \frac{2i - 1}{2n} + \frac{a}{2n}, \frac{2i}{2n} \right].
\]
Then \( \frac{a}{2n} < |x - \frac{2i - 1}{2n}| \leq \frac{1}{2n} \), whence \( \frac{a}{2} < |nx - 2i - 1| \leq \frac{1}{2} \) and finally
\[
a < 2|nx - \frac{2i - 1}{2}| \leq |\sin \pi(nx - i + \frac{1}{2})| \leq |\cos \pi nx|. \quad \blacksquare
\]

**Lemma 4.3.** Let \( f : I \rightarrow \mathbb{R} \) be an absolutely continuous function such that \( f' \) is of bounded variation and \( f'(0) = f'(1) \), \( f(1) - f(0) \in \mathbb{Z} \). Suppose there exists a real number \( a \) such that \( |f'(x)| \geq a > 0 \) for \( x \in I \setminus \bigcup_{i=1}^{n} (a_i, b_i) \) (where \( 0 \leq a_1 < b_1 < \ldots < a_s < b_s < 1 \) or \( 0 < a_1 < b_1 < \ldots < a_s < 1 < b_s \)). Then
\[
(7) \quad \left| \int_0^1 e^{2\pi i f(x)} dx \right| \leq \frac{1}{2\pi} \frac{\text{Var} f'}{a^2} + \frac{s}{\pi a} + \sum_{i=1}^{s} (b_i - a_i).
\]

**Proof.** Let \( D = \bigcup_{i=1}^{n} (a_i, b_i) \) and \( a_{s+1} = a_1 \). Then
\[
\left| \int_0^1 e^{2\pi i f(x)} dx \right| \leq \left| \int_{I \setminus D} e^{2\pi i f(x)} dx \right| + \sum_{i=1}^{s} (b_i - a_i) =
\]
\[
\left| \int_{I \setminus D} \frac{1}{2\pi i f'(x)} e^{2\pi i f(x)} dx \right| + \sum_{i=1}^{s} (b_i - a_i) =
\]
\[
\left| \sum_{i=1}^{s} \left( \frac{2\pi i f'(a_{i+1})}{2\pi i f'(a_i)} - \frac{2\pi i f'(b_i)}{2\pi i f'(b_{i+1})} \right) \frac{1}{2\pi i f'(b_{i+1})} \int_{b_i}^{b_{i+1}} e^{2\pi i f(x)} dx \right| + \sum_{i=1}^{s} (b_i - a_i) \leq
\]
\[
\frac{1}{2\pi i} \sum_{i=1}^{s} \frac{1}{|f'(a_i)|} + \frac{1}{|f'(b_i)|} + \frac{1}{2\pi i} \sum_{i=1}^{s} \text{Var}[b_i, a_{i+1}] \frac{1}{|f'(x)|} + \sum_{i=1}^{s} (b_i - a_i) \leq
\]
\[
\frac{1}{2\pi} \frac{\text{Var} f'}{a^2} + \frac{s}{\pi a} + \sum_{i=1}^{s} (b_i - a_i). \quad \blacksquare
\]

Given a real number \( \alpha \in [0, 1) \), let \([0; a_1, a_2, \ldots]\) be its continued fraction expansion where \( a_n \) are positive integer numbers. Put
\[
q_0 = 1, \quad q_1 = a_1, \quad q_{n+1} = a_{n+1}q_n + q_{n-1},
\]
\[
p_0 = 0, \quad p_1 = 1, \quad p_{n+1} = a_{n+1}p_n + p_{n-1}.
\]

The rationals \( p_n/q_n \) are called the *convergents* of \( \alpha \) and the inequality
\[
\frac{1}{2q_nq_{n+1}} < |\alpha - p_n/q_n| < \frac{1}{q_nq_{n+1}}
\]

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Corollary 4.1. If \( (\alpha, \beta) \) satisfies \( (A, B) \) and \( A, B \geq 2 \), we say that a pair \( (\alpha, \beta) \) belongs to \( \mathbb{Z} \) if it holds.

Lemma 4.4. We first prove

\[
B^{8s_{2m_k}} < \frac{1}{2}q_{2u_k+1}
\]

(8)

\[
A^{8q_{2u_k+1}} < \frac{1}{2}q_{2m_k+1}
\]

(9)

where \( p_n/q_n \) and \( r_n/s_n \) are convergents of \( \alpha \) and \( \beta \).

Obviously, the set \( \{(\alpha, \beta) : (\alpha, \beta) \text{ satisfies } (A, B)\} \) is uncountable.

For a pair \( (\alpha, \beta) \) satisfying \( (A, B) \) we define real analytic functions \( \psi_1, \psi_2 : \mathbb{R} \to \mathbb{R} \) periodic of period 1 given by

\[
\psi_1(x) = \sum_{k=1}^{\infty} \frac{1}{2\pi i q_{2u_k} A^{q_{2u_k}}} (e^{2\pi i q_{2u_k} x} - e^{-2\pi i q_{2u_k} x})
\]

\[
\psi_2(y) = \sum_{k=1}^{\infty} \frac{1}{2\pi i s_{2m_k} A^{s_{2m_k}}} (e^{2\pi i s_{2m_k} y} - e^{-2\pi i s_{2m_k} y}).
\]

We first prove

Lemma 4.4. For any integer numbers \( h_1, h_2, N \neq 0 \) we have

\[
\lim_{|m| \to \infty} \int_{\mathbb{T}^2} e^{2\pi i N(\psi_1^{(m)}(x) + \psi_2^{(m)}(y)) + h_1 x + h_2 y} dx dy = 0.
\]

Corollary 4.1. If \( (\alpha, \beta) \) satisfies \( (A, B) \) then \( \alpha, \beta, 1 \) are independent over \( \mathbb{Q} \).

Proof. Suppose, \( \alpha, \beta, 1 \) are dependent over \( \mathbb{Q} \). Then there exist \( m_1, m_2, m_3 \in \mathbb{Z} \) such that \( m_1 \alpha + m_2 \beta = m_3 \). Let \( t_n/u_n \) be convergents of \( m_1 \alpha \) and \( m_2 \beta \).

Then

\[
\sum_{p=0}^{u_n-1} \psi_1(\cdot + p|m_1|\alpha) + \sum_{p=0}^{u_n-1} \psi_2(\cdot + p|m_2|\beta)
\]

uniformly converges to 0 (see [6], p. 189). From

\[
\psi_1^{(u_n|m_1m_2)}(x) + \psi_2^{(u_n|m_1m_2)}(y) =
\]

\[
\sum_{k=0}^{[m_1]-1} \sum_{l=0}^{[m_2]-1} \sum_{p=0}^{u_n-1} (\psi_1(\cdot + k\alpha + l|m_1|u_n\alpha + p|m_1|\alpha) + \psi_2(\cdot + k|m_2|u_n\beta + l\beta + p|m_2|\beta))
\]

we have

\[
\sup_{(x,y) \in \mathbb{T}^2} |\psi_1^{(u_n|m_1m_2)}(x) + \psi_2^{(u_n|m_1m_2)}(y)|
\]

\[
\leq |m_1m_2| \left( \sup_{x \in I} \sum_{p=0}^{u_n-1} \psi_1(x + p|m_1|\alpha) + \sup_{y \in I} \sum_{p=0}^{u_n-1} \psi_2(y + p|m_2|\beta) \right)
\]

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hence
\[ \psi_1^{(u_n|m_1m_2)}(\cdot) + \psi_2^{(u_n|m_1m_2)}(\cdot) \]
uniformly converges to 0 in \( I^2 \). It follows that
\[ \lim_{n \to \infty} \int_{I^2} e^{2\pi i (\psi_1^{(u_n|m_1m_2)}(x) + \psi_2^{(u_n|m_1m_2)}(y))} \, dx \, dy = 1, \]
which contradicts Lemma 4.4. \( \blacksquare \)

**Proof of Lemma 4.4.** From (8) and (9) for every \( k \in \mathbb{N} \)
\[
B^{8^2k} < \frac{1}{2} q_{2m+1} < \frac{1}{2} s_{2m+1}^2
\]
\[
A^{8^2k} < \frac{1}{2} s_{2m+1} < \frac{1}{2} s_{2m+1}^2.
\]
Hence for any \( m \geq \min(A^{8^2k}, B^{8^2k}) \) there exists natural number \( k \) such that
\[
A^{8^2k} \leq m \leq \frac{1}{2} q_{2m+1}
\]
or
\[
B^{8^2k} \leq m \leq \frac{1}{2} s_{2m+1}^2.
\]
In the first case
\[
| \int_{I^2} e^{2\pi i [N(\psi_1^{(m)}(x) + \psi_2^{(m)}(y)) + h_1 x + h_2 y]} \, dx \, dy | =
\]
\[
| \int_{I} e^{2\pi i [N(\psi_1^{(m)}(x) + h_1 x)]} | \int_{I} e^{2\pi i [N(\psi_2^{(m)}(y)) + h_2 y]} \, dy | \leq | \int_{I} e^{2\pi i [N(\psi_1^{(m)}(x) + h_1 x)]} \, dx |.
\]
From
\[
\psi_1'(x) = \sum_{l=1}^{\infty} \frac{1}{A^{q_2n}} (e^{2\pi i q_2n x} + e^{-2\pi i q_2n x})
\]
it follows that for any natural number \( m \)
\[
| \psi_1^{(m)'}(x) | = | \sum_{j=0}^{m-1} \psi_1'(x + j \alpha) |
\]
\[
\leq \sum_{l=1}^{\infty} \frac{1}{A^{q_2n}} (e^{2\pi i q_2n x} e^{2\pi i q_2n x} \alpha - 1 + e^{-2\pi i q_2n x} e^{-2\pi i q_2n x} \alpha - 1) =
\]
\[
\sum_{l=1}^{\infty} \frac{1}{A^{q_2n}} (e^{2\pi i q_2n x} \alpha - 1) (e^{2\pi i q_2n x} + e^{-2\pi i q_2n x} (x + (m-1) \alpha)) \geq
\]
\[
\frac{2}{A^{q_2n}} | e^{2\pi i q_2n x} \alpha - 1 | | \cos 2\pi q_2n (x + (m-1) \alpha) | =
\]
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\[
\sum_{l=1}^{k-1} \frac{1}{A^{2q_{2n_l}}} e^{2\pi i q_{2n_l}\alpha} - 1 - \sum_{l=k+1}^{\infty} \frac{2}{A^{q_{2n_l}}} e^{2\pi i q_{2n_l}\alpha} - 1.
\]

From \(|q_{2n_l} - p_{2n_l}| > \frac{1}{2q_{2n_l+1}}\) and (5) we have
\[
|e^{2\pi i q_{2n_l}\alpha} - 1| \geq 4|q_{2n_l}\alpha - q_{2n_l}| > \frac{2}{q_{2n_l+1}}
\]
hence \(\frac{1}{|e^{2\pi i q_{2n_l}\alpha} - 1|} < \frac{2}{q_{2n_l+1}}\) for any natural \(l\). From \(m \leq \frac{1}{2} q_{2n_l+1}\) and \(|q_{2n_l}\alpha - p_{2n_l}| < \frac{1}{2q_{2n_l+1}}\) for any \(l \geq k\) it follows that
\[
0 < |q_{2n_l}\alpha - p_{2n_l}| \leq |mq_{2n_l}\alpha - mp_{2n_l}| \leq \frac{1}{2} q_{2n_l+1} |q_{2n_l}\alpha - p_{2n_l}| < \frac{1}{2}.
\]

From (6) for \(l \geq k\)
\[
\frac{m}{2} \leq |e^{2\pi i q_{2n_l}\alpha} - 1| \leq m.
\]

From Lemma 4.2 there exist subintervals \(I_1, ..., I_{2q_{2n}}\) of \(I\) such that for any \(x \in I \setminus \bigcup_{i=1}^{2q_{2n}} I_i\) we have
\[
|\cos 2\pi q_{2n_k}(x + \frac{(m-1)\alpha}{2})| \geq \frac{1}{A^{q_{2n_k}}},
\]
moreover \(|I_i| = \frac{1}{2q_{2n_k} A^{q_{2n_k}}}\) for \(i = 1, ..., 2q_{2n_k}\).

It follows that for \(x \in I \setminus \bigcup_{i=1}^{2q_{2n}} I_i\) we have
\[
|\psi_1^{(m)}(x)| \geq -2 \sum_{l=1}^{k-1} \frac{q_{2n_l+1}}{A^{2q_{2n_l}}} + \frac{m}{A^{q_{2n_k}}} - \sum_{l=k+1}^{\infty} \frac{2m}{A^{q_{2n_l}}} \geq -q_{2n_k-1} + \frac{m}{A^{q_{2n_k}}} - \frac{4m}{A^{q_{2n_k+1}}}.
\]

From \(A^{q_{2n_k}} \leq m \leq \frac{1}{2} q_{2n_k+1}\) we have
\[
4q_{2n_k} \leq A^{q_{2n_k}} = \frac{A^{q_{2n_k}}}{A^{q_{2n_k+4}}} \leq \frac{m}{A^{4q_{2n_k}}} \quad \text{and} \quad q_{2n_k} + 2 \leq A^{q_{2n_k}} \leq \frac{1}{2} q_{2n_k+1},
\]
hence
\[
16A^{q_{2n_k}} \leq A^{q_{2n_k} + 4} \leq A^{q_{2n_k+1}}.
\]

For this reason for \(x \in I \setminus \bigcup_{i=1}^{2q_{2n}} I_i\)
\[
|\psi_1^{(m)}(x)| \geq -\frac{m}{4A^{2q_{2n_k}}} + \frac{m}{A^{q_{2n_k}}} - \frac{m}{4A^{q_{2n_k}}} = \frac{m}{2A^{q_{2n_k}}},
\]
hence \(|N\psi_1^{(m)}(x) + h_1| \geq |N| \frac{m}{2A^{2q_{2n_k}}} - |h_1|\). From (7) for any natural \(m\) such that \(\frac{m}{A^{q_{2n_k}}} \geq A^{q_{2n_k}} \geq 4\frac{m}{N}\) we have
\[
\int_I e^{2\pi i |N\psi_1^{(m)}(x) + h_1|} dx \leq \frac{1}{2\pi} \Var(N\psi_1^{(m)} + h_1) + \frac{2q_{2n_k}}{N} + \frac{1}{A^{q_{2n_k}}} \leq
\]

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Lemma 4.5. It follows that the set \( \{ h \mid \text{which converges to } h \} \) is the set of all \( h \in H \) such that \( \lim_{|m| \to \infty} \int_{B^{8m_k}} e^{2\pi i[N(\psi^{(m)}_1(x) + \psi^{(m)}_2(y)) + h_1 x + h_2 y]} dxdy = 0. \)

If \( m < 0 \) then

\[
\left| \int_{B^{8m_k}} e^{2\pi i[N(\psi^{(m)}_1(x) + \psi^{(m)}_2(y)) + h_1 x + h_2 y]} dxdy \right| =
\]

\[
\left| \int_{B^{8m_k}} e^{2\pi i[N - (\psi^{(-m)}_1(x + m\alpha) + \psi^{(-m)}_2(y + m\beta)) + h_1 x + h_2 y]} dxdy \right| =
\]

\[
\left| \int_{B^{8m_k}} e^{2\pi i[N(\psi^{(-m)}_1(x) + \psi^{(-m)}_2(y)) - h_1 x - h_2 y]} dxdy \right|.
\]

It follows that

\[
\lim_{|m| \to \infty} \int_{B^{8m_k}} e^{2\pi i[N(\psi^{(m)}_1(x) + \psi^{(m)}_2(y)) + h_1 x + h_2 y]} dxdy = 0.
\]

Lemma 4.5. Let \( U : H \to H \) be a unitary operator on a Hilbert space \( H \). Then the set \( \{ h \in H : \lim_{|m| \to \infty} (U^m h, h) = 0 \} \) is closed in \( H \).

Proof. Let \( h_n \in H \) be a sequence such that \( \lim_{|m| \to \infty} (U^m h_n, h_n) = 0 \) which convergence to \( h \in H \). Let \( \varepsilon > 0 \). We take a natural number \( n \) such that \( \| h - h_n \| < \min\left\{ \frac{\varepsilon}{2(2|n| + 1)}, 1 \right\} \). Let \( m_0 \) be a natural number such that for any \( |m| \geq m_0 \) we have \( (|U^m h_n, h_n|) < \frac{\varepsilon}{2} \). Then for \( |m| \geq m_0 \)

\[
(|U^m h|) = (|U^m (h - h_n, h) + (U^m h_n, h - h_n) + (U^m h_n, h_n)| \leq
\]

\[
\| h - h_n \| \| h \| + \| h_n \| + \| h - h_n \| + (|U^m h_n, h_n|) \leq
\]

\[
\| h - h_n \| (2 \| h \| + 1) + (|U^m h_n, h_n|) < \varepsilon.
\]

Theorem 4.6. There exist real numbers \( \alpha \) and \( \beta \) such that \( \alpha, \beta \) are independent over \( \mathbb{Q} \) and a cocycle \( \varphi : T^2 \to T \) given by

\[
\varphi(e^{2\pi i x}, e^{2\pi i y}) = e^{2\pi i (\psi_1(x) + \psi_2(y))}
\]

where \( \psi_1, \psi_2 \) are real analytic function which are periodic of period 1 such that \( T_\varphi \) is mixing in the orthocomplement of the eigenfunctions of \( T \) where \( T \) is the rotation on \( T^2 \) given by \( T(z_1, z_2) = (e^{2\pi i \alpha} z_1, e^{2\pi i \beta} z_2) \).
Proof. We take $\alpha, \beta, \psi_1, \psi_2$ like in Lemma 4.4. By Lemma 4.5 is sufficient to show that $T_\varphi$ is mixing in the set of trigonometric polynomials given by

$$P(z_1, z_2, \omega) = \sum_{k_1=-K_1}^{K_1} \sum_{k_2=-K_2}^{K_2} \sum_{l=-L}^{L} a_{k_1, k_2, l} z_1^{k_1} z_2^{k_2} \omega^l,$$

where $a_{k_1, k_2, l} \in \mathbb{C}$.

$$|\langle U_m T_\varphi P, P \rangle| =$$

$$\left| \int_{\mathbb{T}^2} \sum_{k_1, k_2, l} a_{k_1, k_2, l} e^{2\pi i (\alpha k_1 + \beta k_2)} z_1^{k_1} z_2^{k_2} (\varphi^{(m)}(z_1, z_2))^{\omega^l} \sum_{k'_1, k'_2, l'} \bar{a}_{k'_1, k'_2, l'} z_1^{-k'_1} z_2^{-k'_2} \omega^{-l'} dz_1 dz_2 d\omega \right| =$$

$$\left| \sum_{k_1, k_2, l} a_{k_1, k_2, l} \bar{a}_{k'_1, k'_2, l'} e^{2\pi i (\alpha k_1 + \beta k_2)} \int_{\mathbb{T}^2} z_1^{k_1-k'_1} z_2^{k_2-k'_2} (\varphi^{(m)}(z_1, z_2))^{l} dz_1 dz_2 \right| \leq$$

$$\sum_{k_1, k_2, l} \left| a_{k_1, k_2, l} \bar{a}_{k'_1, k'_2, l'} \right| \left| \int_{\mathbb{T}^2} e^{2\pi i [(\psi_1^{(m)}(x) + \psi_2^{(m)}(y))+(k_1-k'_1)x+(k_2-k'_2)y]} dx dy \right|.$$

Consequently $\lim_{|m| \to \infty} |\langle U_m T_\varphi P, P \rangle| = 0$ and the proof is complete. ■

References


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