Spectral properties of cocycles over rotations

Krzysztof Frączek *

Abstract

Let $T:\mathbb{T}^d\to\mathbb{T}^d$ be an ergodic rotation. Given $\varphi:\mathbb{T}^d\to\mathbb{T}$ a smooth cocycle we show that the set

$$\{f\in L^2(\mathbb{T}^{d+1},\lambda_{d+1}):\hat{\sigma}_f(n)=(U^n_{T_\varphi}f,f)=O(\frac{1}{|n|^{rw(\varphi)}})\},$$

where $rw(\varphi)$ is the rank of the winding vector of φ is dense in the orthocomplement of the eigenfunctions of T. In particular the skew product diffeomorphism $T_\varphi:\mathbb{T}^{d+1}\to\mathbb{T}^{d+1}$ given by

$$T_\varphi(z,\omega)=(Tz,\varphi(z)\omega)$$

has countable Lebesgue spectrum in that orthocomplement. We construct an ergodic rotation T of $\mathbb{T}^2$ and a real analytic cocycle on $\tilde{\varphi}:\mathbb{T}^2\to\mathbb{R}$ such that an extension $T_{\exp(2\pi i\tilde{\varphi})}$ is mixing in the orthocomplement of the eigenfunctions of T.

Introduction

Let $\mathbb{T}^d$ be a d-dimensional torus. We will consider an ergodic rotation of the d-dimensional torus given by

$$T(z_1,...,z_d)=(z_1e^{2\pi i\alpha_1},...,z_de^{2\pi i\alpha_d})$$

where $\alpha_1,...,\alpha_d,1$ are independent over $\mathbb{Q}$.
By a *cocycle* we mean a smooth map $\varphi:\mathbb{T}^d\to\mathbb{T}$. Then, by Fubini Theorem a transformation $T_\varphi:(\mathbb{T}^{d+1},\lambda_{d+1})\to(\mathbb{T}^{d+1},\lambda_{d+1})$ given by

$$T_\varphi(z,\omega)=(Tz,\varphi(z)\omega)$$

preserves Lebesgue measure λ_{d+1}. The automorphism T_φ is called an *extension* of T.
Such a cocycle φ can be represented as

$$\varphi(e^{2\pi ix_1},...,e^{2\pi ix_d})=e^{2\pi i(\tilde{\varphi}(x_1,...,x_d)+m_1x_1+m_dx_d)}$$

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where $m_1, ..., m_d \in \mathbb{Z}$ and $\tilde{\varphi} : \mathbb{R}^d \to \mathbb{R}$ is smooth, periodic of period 1 in each coordinate. In this representation of φ, the vector $(m_1, ..., m_d) \in \mathbb{Z}^d$ is unique, while $\tilde{\varphi}$ is unique up to an additive integer constant.
The vector $w(\varphi) = (m_1, ..., m_d)$ we call the *winding vector* of a cocycle φ. The number $rw(\varphi) = card\{i : i = 1, ..., d,\ m_i \neq 0\}$ we call the *rank of the winding vector* of a cocycle φ. For $d = 1$ the winding vector is equal to the degree $d(\varphi)$ of φ.
In 1991, P. Gabriel, M. Lemańczyk and P. Liardet [4] proved that

Proposition 1. *If $d(\varphi) = 0$ and $\tilde{\varphi}$ is absolutely continuous, then the maximal spectral type of T_φ is singular and is not mixing in the orthocomplement of the eigenfunctions of T.*

In 1993, A. Iwanik, M. Lemańczyk and D. Rudolph [8] proved that

Proposition 2. *If $d(\varphi) \neq 0$ and $\tilde{\varphi}$ is absolutely continuous and $\tilde{\varphi}'$ is of bounded variation, then T_φ has countable Lebesgue spectrum in the orthocomplement of the eigenfunctions of T and the set*

$$\{f \in L^2(\mathbb{T}^2, \lambda_2) : \hat{\sigma}_f(n) = (U^n_{T_\varphi} f, f) = O(\frac{1}{|n|})\}$$

is dense in that orthocomplement.

This result is a strengthening of an earlier result by Kushnirenko [11] (see also [2] pp.344).
We can interpret Proposition 1 and 2 as certain facts giving rise to a spectral stability of T_φ where φ is a character of $\mathbb{T}$: indeed if we multiply φ by a smooth cocycle ψ of degree zero spectral properties of T_φ and $T_{\varphi\psi}$ remain the same.
In this paper we will generalize these facts to multidimensional rotations for non zero winding vector smooth cocycles. In Section 3 we show that for $\varphi \in C^2(\mathbb{T})$, T_φ has countable Lebesgue spectrum in the orthocomplement of the eigenfunctions of T and for $\varphi \in C^{2d}(\mathbb{T})$, the set

$$\{f \in L^2(\mathbb{T}^{d+1}, \lambda_{d+1}) : \hat{\sigma}_f(n) = (U^n_{T_\varphi} f, f) = O(\frac{1}{|n|^{rw(\varphi)}})\}$$

is dense in that orthocomplement.
For zero winding vector smooth cocycles and $d \geq 2$ our result are rather to suggest that no spectral stability property holds. In Section 4 we construct an ergodic rotation T of $\mathbb{T}^2$ and a real analytic cocycle on $\varphi : \mathbb{T}^2 \to \mathbb{T}$ such that an extension T_φ is mixing in the orthocomplement of the eigenfunctions of T.

1 Notation and facts from spectral theory

The substance of this section is classical (e.g. for an irrational rotation of the circle see [5], [8] and [13]).

Let U be a unitary operator on a separable Hilbert space H. For any $f \in H$ we define the *cyclic space* $\mathbb{Z}(f) = span\{U^n f : n \in \mathbb{Z}\}$. By the *spectral measure* σ_f of f we mean a Borel measure on $\mathbb{T}$ determined by the equalities

$$\hat{\sigma}_f(n) = \int_{\mathbb{T}} z^n d\sigma_f = (U^n f, f)$$

for $n \in \mathbb{Z}$.

Theorem 1.1 (spectral theorem). *There exists a sequence $f_1, f_2, \ldots$ in H such that*

(1) $$H = \bigoplus_{n=1}^{\infty} \mathbb{Z}(f_n) \quad and \quad \sigma_{f_1} \gg \sigma_{f_2} \ldots \ .$$

Moreover, for any sequence $f_1', f_2', \ldots$ in H satisfying (1) we have $\sigma_{f_1} \equiv \sigma_{f_1'}, \sigma_{f_2} \equiv \sigma_{f_2'}, \ldots$.

The spectral type of σ_{f_1} (the equivalence class of measures) will be called the *maximal spectral type* of U. U is said to have *Lebesgue spectrum* if $\sigma_{f_1} \equiv \lambda$ where λ is Lebesgue measure on the circle. It is said that U has *Lebesgue spectrum of uniform multiplicity* if $\sigma_{f_n} \equiv \lambda$ for $n = 1, 2, \ldots, k$ and $\sigma_{f_n} \equiv 0$ for $n > k$ where $k \in \mathbb{N} \cup \{\infty\}$.

Let X be an infinite abelian group which is metric, compact and monothetic. Let $\mathcal{B}$ be a σ-algebra of Borel sets on X and μ be Haar measure on X. We will denote H the space $L^2(X, \mathcal{B}, \mu)$. We will consider an ergodic rotation of the group X given by $Tx = a \cdot x$, where a is a cyclic generator of X.
For a cocycle (here by a cocycle we mean any Borel map) $F : X \to \mathbb{T}$ we will consider a unitary operator $U : H \to H$ given by

$$(Uf)(x) = F(x)f(Tx).$$

Lemma 1.2. *The maximal spectral type of the operator U is either discrete or continuous singular or Lebesgue.*

Lemma 1.3. *If the maximal spectral type of the operator U is Lebesgue then the multiplicity function of U is uniform.*

Lemma 1.4. *Suppose that $f \in H$ and $\sum_{n=-\infty}^{\infty} |(U^n f, f)|^2 < +\infty$. Then $\sigma_f \ll \lambda$.*

Denote

$$F^{(n)}(x) = \begin{cases} F(x)F(Tx)\ldots F(T^{n-1}x) & if \quad n > 0 \\ 1 & if \quad n = 0 \\ (F(T^n x)F(T^{n+1}x)\ldots F(T^{-1}x))^{-1} & if \quad n < 0 \end{cases}$$

Corollary 1.1. *Suppose,*

$$\sum_{n=-\infty}^{\infty} |\int_X F^{(n)}(x) d\mu(x)|^2 < +\infty.$$

Then U has Lebesgue spectrum of uniform multiplicity.

Let G be a compact abelian group, m its Haar measure and $\varphi : X \to G$ a cocycle. We will consider the extension $T_\varphi : (X \times G, \mu \times m) \to (X \times G, \mu \times m)$ given by

$$T_\varphi(x, g) = (Tx, \varphi(x)g).$$

Let us decompose

$$L^2(X \times G, \mu \times m) = \bigoplus_{\chi \in \widehat{G}} H_\chi$$

where

$$H_\chi = \{f : f(x, g) = h(x)\chi(g), h \in L^2(X, \mu)\}.$$

Observe that H_χ is closed U_{T_φ}-invariant subspace of $L^2(X \times G, \mu \times m)$, where $U_{T_\varphi} = f \circ T_\varphi$.

Lemma 1.5. *The operator $U_{T_\varphi} : H_\chi \to H_\chi$ is unitarily equivalent to $U_\chi : H \to H$, where*

$$(U_\chi h)(x) = \chi(\varphi(x))h(Tx).$$

2 Functions of bounded variation and absolutely continuous functions

Let I^d denote the closed d-dimensional unit cube. By a *partition* P of I^d, we mean a partition into cubes given by sequences

$$\{(\eta_0^{(j)}, \eta_1^{(j)}, ..., \eta_{m_j}^{(j)}) : 0 = \eta_0^{(j)} \leq ... \leq \eta_{m_j}^{(j)} = 1 \ , \ j = 1, ..., d\}.$$

Given such a partition, we define, for $j = 1, ..., d$ and $i = 1, ..., m_j - 1$ the operator $\Delta_{j,i} : \mathbb{C}^{I^d} \to \mathbb{C}^{I^d}$ by

$$\Delta_{j,i} f(x^{(1)}, ..., x^{(d)}) =$$

$$f(x^{(1)}, ..., x^{(j-1)}, \eta_{i+1}^{(j)}, x^{(j+1)}, ..., x^{(d)}) - f(x^{(1)}, ..., x^{(j-1)}, \eta_i^{(j)}, x^{(j+1)}, ..., x^{(d)})$$

However, if it does not rise to a confusion, we will rather write

$\Delta_j f(x^{(1)}, ..., x^{(j-1)}, \eta_i^{(j)}, x^{(j+1)}, ..., x^{(d)})$ instead of $\Delta_{j.i} f(x^{(1)}, ..., x^{(j-1)}, \eta_i^{(j)}, x^{(j+1)}, ..., x^{(d)})$.

For $j \neq j'$ and $0 \leq i \leq m_j - 1, \ 0 \leq i' \leq m_{j'} - 1$ we have

$$\Delta_{j,i}\Delta_{j',i'} f = \Delta_{j',i'}\Delta_{j,i} f$$

and for $j_1, ..., j_p$ such that $j_s \neq j_{s'}$ for $s \neq s'$ we will write

$$\Delta_{j_1,...,j_p} = \Delta_{j_1,i_1} ... \Delta_{j_p,i_p}$$

where by the domain of $\Delta_{j_1,...,j_p}$ we mean only points $(x^{(1)}, ..., x^{(d)})$, $x^{(j_s)} = \eta_{i_s}^{(j_s)}$ for some i_s.

Let Q be a closed d-dimensional cube $\prod_{i=1}^{d}[a^{(i)}, b^{(i)}] \subset I^d$. Given Q define for $j = 1, ..., d$ the operator $\Delta_j^*|_Q : \mathbb{C}^{I^d} \to \mathbb{C}^{I^d}$ by

$$\Delta_j^*|_Q f(x^{(1)}, ..., x^{(d)}) =$$

$$f(x^{(1)}, ..., x^{(j-1)}, b^{(j)}, x^{(j+1)}, ..., x^{(d)}) - f(x^{(1)}, ..., x^{(j-1)}, a^{(j)}, x^{(j+1)}, ..., x^{(d)})$$

and let $\Delta_{j_1,...,j_p}^*|_Q$ stand for $\Delta_{j_1}^*|_Q ... \Delta_{j_p}^*|_Q$.

Definition 2.1. For a function $f : I^d \to \mathbb{C}$ we set

$$Var^{(d)} f = \sup_{P \in \mathcal{P}} \sum_{i_1=0}^{m_1-1} ... \sum_{i_d=0}^{m_d-1} |\Delta_{1...d} f(\eta_{i_1}^{(1)}, ..., \eta_{i_d}^{(d)})|,$$

where $\mathcal{P}$ is the family of all partitions P of I^d. If $Var^{(d)} f$ is finite, then f is said to be *of bounded variation on I^d in the sense of Vitali.*

Definition 2.2. Let $f : I^d \to \mathbb{C}$ be a function of bounded variation in the sense of Vitali. Suppose that the restriction of f to each face $F = \{(x^{(1)}, ..., x^{(d)}) : x^{(i_s)} = 0, s = 1, ..., k\}$ where $1 \leq i_1 < ... < i_k \leq d$ $(k = 1, ..., d)$ is of bounded variation on F in the sense of Vitali. Then f is said to be *of bounded variation on I^d in the sense of Hardy and Krause.*

In what follows functions of bounded variation are those of bounded variation in the sense of Hardy and Krause.

Remark. If a function is of bounded variation, then it is integrable in sense of Riemann (for $d = 2$, see [7] §448).

Given $0 \leq p \leq n$ on the set $\mathcal{S}_n$ all permutations of $\{1, ..., n\}$ consider the following equivalence relation

$$\sigma \equiv \sigma' \quad \text{iff} \quad \sigma(\{1, ..., p\}) = \sigma'(\{1, ..., p\})$$

We will consider an expression $F(i_1, ..., i_n)$, $(i_k \in \mathbb{N})$ such that

$$F(i_{\sigma(1)}, ..., i_{\sigma(n)}) = F(i_{\sigma'(1)}, ..., i_{\sigma'(n)}) \text{ whenever } \sigma \equiv \sigma'. \tag{2}$$

By

$$\sum_{i_1,...,i_n;p}^{*} F(i_1, ..., i_n) \text{ we denote the sum } \sum_{[\sigma] \in S_N/\equiv} F(i_{\sigma(1)}, ..., i_{\sigma(n)}).$$

Let $f : I^d \to \mathbb{C}$ be a function of bounded variation. Given $0 \leq k \leq d$ and $(a^{(k+1)}, ..., a^{(d)}) \in I^{d-k}$ consider the function $g : I^k \to \mathbb{C}$ given by

$$g(x^{(1)}, ..., x^{(k)}) = f(x^{(1)}, ..., x^{(k)}, a^{(k+1)}, ..., a^{(d)}).$$

For each $0 \le p \le d-k$ consider

$$F_p(k+1,...,d) = Var^{(k+p)} f(\underbrace{\overbrace{\cdot,...,\cdot}^{k},\overbrace{\cdot,...,\cdot}^{p}}_{k+p\ coordinates},0,...,0)$$

and notice that expressions of this kind satisfy (2).

Lemma 2.1.

$$Var^{(k)} g \le \sum_{p=0}^{d-k} \sum_{k+1,...,d;p}^{*} Var^{(k+p)} f(\overbrace{\cdot,...,\cdot}^{k+p},0,...,0).$$

Proof. We first prove (by induction on l) that for a function $h : I^l \to \mathbb{C}$ and $(y^{(1)},...,y^{(l)}) \in I^l$ and a partition given by $\{(0,y^{(j)},1) : j = 1,...,l\}$ we have

$$h(y^{(1)},...,y^{(l)}) - h(0,...,0) = \sum_{p=1}^{l} \sum_{1,...,l;p}^{*} \Delta_{1...p} f(0,...,0). \tag{3}$$

1. Obviously, (3) holds for $l = 1$.
2. Assuming (3) to hold for l, we will prove it for $l+1$.

$$h(y^{(1)},...,y^{(l+1)}) - h(0,...,0) =$$

$$h(y^{(1)},...y^{(l)},y^{(l+1)}) - h(0,...,0,y^{(l+1)}) + \Delta_{l+1} h(0,...,0) =$$

$$\sum_{p=1}^{l} \sum_{1,...,l;p}^{*} \Delta_{1...p\ l+1} h(0,...,0) + \sum_{p=1}^{l} \sum_{1,...,l;p}^{*} \Delta_{1...p} h(0,...,0) + \Delta_{l+1} h(0,...,0) =$$

$$\sum_{p=1}^{l+1} \sum_{1,...,l+1;p}^{*} \Delta_{1...p} h(0,...,0).$$

Let P be a partition of I^k given by $\{(\eta_0^{(j)},\eta_1^{(j)},...,\eta_{m_j}^{(j)}) : 0 = \eta_0^{(j)} \le ... \le \eta_{m_j}^{(j)} = 1\ ,\ j = 1,...,k\}$. Consider a partition P' of I^d given by $\{(\eta_0^{(j)},\eta_1^{(j)},...,\eta_{m_j}^{(j)}) : 0 = \eta_0^{(j)} \le ... \le \eta_{m_j}^{(j)} = 1,\ j = 1,...,k\} \cup \{(0,a^{(j)},1) : j = k+1,...,d\}$. Then

$$\sum_{i_1=0}^{m_1-1} ... \sum_{i_k=0}^{m_k-1} |\Delta_{1...k} g(\eta_{i_1}^{(1)},...,\eta_{i_k}^{(k)})| =$$

$$\sum_{i_1=0}^{m_1-1} ... \sum_{i_k=0}^{m_k-1} |\Delta_{1...k} f(\eta_{i_1}^{(1)},...,\eta_{i_k}^{(k)},a^{(k+1)},...,a^{(d)})| \le$$

$$\sum_{p=0}^{d-k} \sum_{k+1,...,d;p}^{*} \sum_{i_1=0}^{m_1-1} ... \sum_{i_k=0}^{m_k-1} |\Delta_{1...k+p} f(\eta_{i_1}^{(1)},...,\eta_{i_k}^{(k)},0,...,0)| \le$$

$$\sum_{p=0}^{d-k} \sum_{k+1,\ldots,d;p}^{*} Var^{(k+p)} f(\overbrace{\cdot,\ldots,\cdot}^{k+p},0,\ldots,0)$$

and consequently

$$Var^{(k)} g \leq \sum_{p=0}^{d-k} \sum_{k+1,\ldots,d;p}^{*} Var^{(k+p)} f(\overbrace{\cdot,\ldots,\cdot}^{p+k},0,\ldots,0). \blacksquare$$

Let P be a partition of I^d given by $\{(\eta_0^{(j)},\eta_1^{(j)},\ldots,\eta_{m_j}^{(j)}) : 0 = \eta_0^{(j)} \leq \ldots \leq \eta_{m_j}^{(j)} = 1\ ,\ j = 1,\ldots,d\}$. Then

$$\delta(P) = \max_{\{(i_1,\ldots,i_d):0\leq i_s\leq m_s-1\}} \prod_{j=1}^{d} |\eta_{i_j+1}^{(j)} - \eta_{i_j}^{(j)}|$$

we will be called the *diameter* of the partition P.

Definition 2.3. Let $f,g : I^d \to \mathbb{C}$ and let f be bounded. If for each sequence of partitions P_k given by $\{(\eta_0^{(j,k)},\eta_1^{(j,k)},\ldots,\eta_{m_{j,k}}^{(j,k)}) : j = 1,\ldots,d\}$ such that $\lim_{k\to\infty}\delta(P_k) = 0$ and for any sequence $\{\xi_{i_1\ldots i_d}^{(k)} : i_s = 1,\ldots,m_{s,k}-1,\ s = 1,\ldots,d,\ k\in\mathbb{N}\}$ where $\xi_{i_1\ldots i_d}^{(k)} \in \prod_{j=1}^{d}\left[\eta_{i_j}^{(j,k)},\eta_{i_j+1}^{(j,k)}\right]$ we have

$$\lim_{k\to\infty} \sum_{i_1=0}^{m_{1,k}-1} \cdots \sum_{i_d=0}^{m_{d,k}-1} f(\xi_{i_1\ldots i_d}^{(k)})\Delta_{1..d} g(\eta_{i_1}^{(1,k)},\ldots,\eta_{i_d}^{(d,k)}) = I,$$

then I is called *the Riemann-Stieltjes integral of* and is denoted $\int_{I^d} fdg$.

Remark. If f,g both are functions of bounded variation and if one of the functions is continuous then $\int_{I^d} fdg$ exists (for $d = 2$, see [7] §448).

Remark. If $\int_{I^d} fdg$ exists and g is of bounded variation in the sense of Vitali, then

$$|\int_{I^d} fdg| \leq \sup_{x\in I^d} |f(x)| Var^{(d)} g.$$

Let $f,g : I^d \to \mathbb{C}$ both be functions of bounded variation and let one of them is continuous. For $0 \leq p \leq d$ consider

$$F_p(1,\ldots,d) = \Delta_{p+1..d}^{*}|_{I^d} \int_{I^p} g(\underbrace{\cdot,\ldots,\cdot}_{p\ coord.},0,\ldots,0) df(\underbrace{\cdot,\ldots,\cdot}_{p\ coord.},0,\ldots,0)$$

and notice that expressions of this kind satisfy (2).

Theorem 2.2 (integration by parts). *We have*

$$\int_{I^d} fdg = \sum_{p=0}^{d} (-1)^p \sum_{1,\ldots,d;p}^{*} \Delta_{p+1..d}^{*}|_{I^d} \int_{I^p} g(\overbrace{\cdot,\ldots,\cdot}^{p},0,\ldots,0) df(\overbrace{\cdot,\ldots,\cdot}^{p},0,\ldots,0).$$

Proof. For $d = 2$, see [7] §448. We can prove this theorem using Lemma 5.2 from [10] ch.2 §5. ■

Corollary 2.1. *If f and g be periodic of period 1 in each coordinate, then*

$$\int_{I^d} f dg = (-1)^d \int_{I^d} g df. \ ■$$

Given $0 = s_0 \leq s_1 \leq ... \leq s_{k-1} \leq s_k = n$ on the set $\mathcal{S}_n$ all permutations of $\{1, ..., n\}$ consider the following equivalence relation

$$\sigma \equiv \sigma' \quad \text{iff} \quad \sigma(\{s_{l-1}+1, ..., s_l\}) = \sigma'(\{s_{l-1}+1, ..., s_l\}) \text{ for } l = 1, ..., k.$$

We will consider an expression $F(i_1, ..., i_n)$, $(i_k \in \mathbb{N})$ such that

$$F(i_{\sigma(1)}, ..., i_{\sigma(n)}) = F(i_{\sigma'(1)}, ..., i_{\sigma'(n)}) \text{ whenever } \sigma \equiv \sigma'. \tag{4}$$

By

$$\sum^{*}_{i_1,...,i_n;s_1,...,s_{k-1}} F(i_1, ..., i_n) \text{ we denote the sum } \sum_{[\sigma]\in S_N/\equiv} F(i_{\sigma(1)}, ..., i_{\sigma(n)}).$$

Let $f_1, ..., f_k : I^d \to \mathbb{C}$ be functions of bounded variation. For $0 = s_0 \leq s_1 \leq ... \leq s_{k-1} \leq s_k = n$ consider

$$F_{s_1...s_k}(1, ..., d) =$$

$$\prod_{r=1}^{k} \sum_{\alpha_r=0}^{d-s_r+s_{r-1}} \sum^{*}_{1,...,s_{r-1},s_r+1,...,d;\alpha_r} Var^{(\alpha_r+s_r-s_{r-1})} f_r(\underbrace{\underbrace{\underbrace{\cdot, ..., \cdot}_{\alpha_r}, 0, ..., 0}_{s_{r-1}}, \cdot, ..., \cdot}_{s_r}, 0, ..., 0)$$

and notice that expressions of this kind satisfy (4).

Lemma 2.3. *The product $f_1 \cdot ... \cdot f_k$ is of bounded variation and we have*

$$Var^{(d)} f_1 \cdot ... \cdot f_k \leq$$

$$\sum_{0=s_0\leq s_1\leq ...\leq s_{k-1}\leq s_k=d} \sum^{*}_{1,...,d;s_1,...,s_{k-1}} \prod_{r=1}^{k} \sum_{\alpha_r=0}^{d-s_r+s_{r-1}} \sum^{*}_{1,...,s_{r-1},s_r+1,...,d;\alpha_r}$$

$$Var^{(\alpha_r+s_r-s_{r-1})} f_r(\underbrace{\underbrace{\underbrace{\cdot, ..., \cdot}_{\alpha_r}, 0, ..., 0}_{s_{r-1}}, \cdot, ..., \cdot}_{s_r}, 0, ..., 0). \ ■$$

Let $f : I^d \to \mathbb{C}$ be a function of bounded variation. For $0 = s_0 < s_1 < ... < s_{k-1} < s_k = d$ consider

$$F_{s_1...s_k}(1, ..., d) =$$

$$\prod_{r=1}^{k} \sum_{\alpha_r=0}^{d-s_r+s_{r-1}} \sum_{1,...,s_{r-1},s_r+1,...,d;\alpha_r}^{*} Var^{(\alpha_r+s_r-s_{r-1})} f(\underbrace{\underbrace{\underbrace{\cdot, ..., \cdot}_{\alpha_r}, 0, ..., 0}_{s_{r-1}}, \cdot, ..., \cdot}_{s_r}, 0, ..., 0)$$

and notice that expressions of this kind satisfy (4).

Lemma 2.4. *Assume that there exists a real number* a *such that* $0 < a \leq |f(x)|$ *for every* $x \in I^d$. *Then* $\frac{1}{f} : I^d \to \mathbb{C}$ *is a function of bounded variation and we have*

$$Var^{(d)} \frac{1}{f} \leq$$

$$\sum_{k=1}^{d} \frac{1}{a^{k+1}} \sum_{0=s_0<s_1<...<s_{k-1}<s_k=d} \sum_{1,...,d;s_1,...,s_{k-1}}^{*} \prod_{r=1}^{k} \sum_{\alpha_r=0}^{d-s_r+s_{r-1}} \sum_{1,...,s_{r-1},s_r+1,...,d;\alpha_r}^{*}$$

$$Var^{(\alpha_r+s_r-s_{r-1})} f(\underbrace{\underbrace{\underbrace{\cdot, ..., \cdot}_{\alpha_r}, 0, ..., 0}_{s_{r-1}}, \cdot, ..., \cdot}_{s_r}, 0, ..., 0). \blacksquare$$

Definition 2.4. We say that a function $f : I^d \to \mathbb{C}$ has the *derivative in the sense of Vitali* at $(x^{(1)}, ..., x^{(d)}) \in I^d$ if there exists limit

$$\lim_{\substack{(h^{(1)},...,h^{(d)}) \to 0 \\ h^{(i)} \neq 0, 0 \leq x^{(i)}+h^{(i)} \leq 1}} \frac{\Delta^*_{1..d}|_{\prod_{i=1}^{d}[x^{(i)}, x^{(i)}+h^{(i)}]} f(x^{(1)}, ..., x^{(d)})}{h^{(1)}...h^{(d)}}.$$

This limit is called the *derivative* of f and is denoted $Df(x^{(1)}, ..., x^{(d)})$.

Remark. If $f \in C^d(I^d)$ then $Df(x) = \frac{\partial^d f}{\partial x^{(1)}...\partial x^{(d)}}(x)$ (see [12] ch.7 §1).

Remark. If a function $f : I^d \to \mathbb{C}$ is of bounded variation in the sense of Vitali, then f has the derivative in the sense of Vitali almost everywhere (see [12] ch.7 §2).

Definition 2.5. (inductive) A function $f : I^d \to \mathbb{C}$ is said to be *differentiable in the sense of Hardy and Krause*
-for $d = 1$ if it is differentiable in the ordinary sense,
-for $d > 1$ if it has the derivative in the sense of Vitali in every point and for any $j = 1, ..., d$ and $a \in I$ the function $f_j : I^d \to \mathbb{C}$

$$f_j(x^{(1)}, ..., x^{(d-1)}) = f(x^{(1)}, ..., x^{(j-1)}, a, x^{(j)}, ..., x^{(d-1)})$$

is differentiable in the sense of Hardy and Krause.

In what follows by differentiable functions we mean those which are differentiable in the sense of Hardy and Krause. The derivative of $f(\hat{x}^{(1)},...,x^{(i_1)},...,x^{(i_k)},...,\hat{x}^{(d)})$ is denoted $D_{x^{(i_1)}...x^{(i_k)}}f(x)$.

Let $f : I^d \to \mathbb{C}$ be a differentiable function. For $0 = s_0 < s_1 < ... < s_{k-1} < s_k = d$ consider

$$F_{s_1...s_k}(1,...,d) = \prod_{r=1}^{k} D_{x^{(s_{r-1}+1)}...x^{(s_r)}}f(x)$$

and notice that expressions of this kind satisfy (4).

Lemma 2.5. *The function* $\exp f : I^d \to \mathbb{C}$ *is differentiable and we have*

$$D\exp f(x) = \exp f(x)\sum_{k=1}^{d}\sum_{0=s_0<s_1<...<s_{k-1}<s_k=d}\sum_{1,...,d;s_1,...,s_{k-1}}^{*}\prod_{r=1}^{k} D_{x^{(s_{r-1}+1)}...x^{(s_r)}}f(x). \blacksquare$$

The number $|P| = \prod_{i=1}^{d}(b^{(i)} - a^{(i)})$ is called the *substance* of the cube $P = \prod_{i=1}^{d}\left[a^{(i)}, b^{(i)}\right]$.

Definition 2.6. A function $f : I^d \to \mathbb{C}$ is said to be *absolutely continuous in the sense of Vitali* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every system of cubes $Q_1,...,Q_n$ such that $|Q_i \cap Q_j| = 0$ for any $1 \le i \ne j \le n$ if $|Q_1| + ... + |Q_n| < \delta$ then

$$|\Delta^*_{1..d}|_{Q_1}f| + ... + |\Delta^*_{1..d}|_{Q_n}f| < \varepsilon.$$

Remark. If a function is absolutely continuous in the sense of Vitali then is of bounded variation in the sense of Vitali (see [12] ch.7 §3).

Definition 2.7. Let $f : I^d \to \mathbb{C}$ be an absolutely continuous function in the sense of Vitali. Suppose the restriction f of each face $F = \{(x^{(1)},...,x^{(d)}) : x^{(i_s)} = 0, s = 1,...,k\}$ where $1 \le i_1 < ... < i_k \le d$ $(k = 1,...,d)$ is absolutely continuous function in the sense of Vitali. Then f is said to be *absolutely continuous function in the sense of Hardy and Krause.*

In what follows functions absolutely continuous are those absolutely continuous in the sense of Hardy and Krause.

Remark. If a function f is of bounded variation and g is absolutely continuous then

$$\int_{I^d} f dg = \int_{I^d} fDg d\lambda_d$$

(see [12] ch.7 §3 and [7] §448[1]).

Lemma 2.6. *Let* $f : I^d \to \mathbb{C}$ *be an absolutely continuous function. Then for every* $g(a^{(k+1)},...,a^{(d)}) \in I^{d-k}$ *the function* $g : I^k \to \mathbb{C}$ *given by*

$$g(x^{(1)},...,x^{(k)}) = f(x^{(1)},...,x^{(k)},a^{(k+1)},...,a^{(d)})$$

is absolutely continuous.

Proof. Similarly as the proof of Lemma 2.1. $\blacksquare$

Remark. If a function $f : I^d \to \mathbb{R}$ is absolutely continuous then the function $\exp if : I^d \to \mathbb{C}$ is absolutely continuous.

3 Spectral properties in the case where the winding vector is not equal to zero

Lemma 3.1. *Let $f : I^d \to \mathbb{R}$ be an absolutely continuous function such that for any $j = 1, ..., d$ and $x \in I^d$ we have $\Delta_j^*|_{I^d} f(x) \in \mathbb{Z}$. Suppose, $D_{x^{(i_1)}..x^{(i_k)}} f$ is the function of bounded variation for $1 \le i_1 < ... < i_k \le d$ and there exists real a number $a > 0$ such that for any $x \in I^d$ we have*

$$|\sum_{k=1}^{d}(2\pi i)^k \sum_{0=s_0<s_1<...<s_{k-1}<s_k=d} \sum^{*}_{1,...,d;s_1,...,s_{k-1}} \prod_{r=1}^{k} D_{x^{(s_{r-1}+1)}...x^{(s_r)}} f(x)| \ge a > 0.$$

Then

$$|\int_{I^d} \exp 2\pi i f(x)dx| \le$$

$$\sum_{l=1}^{d} \frac{1}{a^{l+1}} \sum_{0=t_0<t_1<...<t_{l-1}<t_l=d} \sum^{*}_{1,...,d;t_1,...,t_{l-1}} \prod_{p=1}^{l} \sum_{\alpha_p=0}^{d-t_p+t_{p-1}} \sum^{*}_{1,...,t_{p-1},t_p+1,...,d;\alpha_p}$$

$$\sum_{k=1}^{d}(2\pi)^k \sum_{0=s_0<s_1<...<s_{k-1}<s_k=d} \sum^{*}_{1,...,d;s_1,...,s_{k-1}} \sum_{0=u_0\le u_1\le...\le u_{k-1}\le u_k \alpha_p+t_p-t_{p-1}}$$

$$\sum^{*}_{1,..,\alpha_p,t_{p-1}+1,..,t_p;u_1,..,u_{k-1}} \prod_{r=1}^{k} \sum_{\beta_r=0}^{d-t_p+t_{p-1}+u_r-u_{r-1}} \sum^{*}_{1,...,u_{r-1},u_r+1,...,\alpha_p,t_{p-1}+1,...,t_p;\beta_r}$$

$$Var^{(\beta_r+u_r-u_{r-1})} D_{x^{(s_{r-1}+1)}...x^{(s_r)}} f(\underbrace{\underbrace{\underbrace{\cdot,...,\cdot}_{\beta_r},0,...,0}_{u_{r-1}},\cdot,...,\cdot}_{u_r},0,...,0).$$

Proof. An application of Lemma 2.3 and Lemma 2.4 and integration by parts gives that

$$|\int_{I^d} \exp 2\pi i f(x)dx| \le$$

$$|\int_{I^d} 1/(\sum_{k=1}^{d}(2\pi i)^k \sum_{0=s_0<s_1<...<s_{k-1}<s_k=d} \sum^{*}_{1,...,d;s_1,...,s_{k-1}} \prod_{r=1}^{k} D_{x^{(s_{r-1}+1)}...x^{(s_r)}} f) d \exp 2\pi i f(x)| =$$

$$|\int_{I^d} \exp 2\pi i f(x) d(1/\sum_{k=1}^{d}(2\pi i)^k \sum_{0=s_0<s_1<...<s_{k-1}<s_k=d} \sum^{*}_{1,...,d;s_1,...,s_{k-1}} \prod_{r=1}^{k} D_{x^{(s_{r-1}+1)}...x^{(s_r)}} f)| \le$$

$$Var^{(d)}(1/\sum_{k=1}^{d}(2\pi i)^k \sum_{0=s_0<s_1<...<s_{k-1}<s_k=d} \sum^{*}_{1,...,d;s_1,...,s_{k-1}} \prod_{r=1}^{k} D_{x^{(s_{r-1}+1)}...x^{(s_r)}} f) \le$$

$$\sum_{l=1}^{d}\frac{1}{a^{l+1}}\sum_{0=t_0<t_1<...<t_{l-1}<t_l=d}\sum^{*}_{1,...,d;t_1,...,t_{l-1}}\prod_{p=1}^{l}\sum_{\alpha_p=0}^{d-t_p+t_{p-1}}\sum^{*}_{1,...,t_{p-1},t_p+1,...,d;\alpha_p}$$

$$\sum_{k=1}^{d}(2\pi)^k\sum_{0=s_0<s_1<...<s_{k-1}<s_k=d}\sum^{*}_{1,...,d;s_1,...,s_{k-1}}Var^{(\alpha_p+t_p-t_{p-1})}\prod_{r=1}^{k}$$

$$D_{x^{(s_{r-1}+1)}...x^{(s_r)}}f(\underbrace{\underbrace{\underbrace{\cdot,...,\cdot}_{\alpha_p},0,...,0}_{t_{p-1}},\cdot,...,\cdot}_{t_p},0,...,0)\leq$$

$$\sum_{l=1}^{d}\frac{1}{a^{l+1}}\sum_{0=t_0<t_1<...<t_{l-1}<t_l=d}\sum^{*}_{1,...,d;t_1,...,t_{l-1}}\prod_{p=1}^{l}\sum_{\alpha_p=0}^{d-t_p+t_{p-1}}\sum^{*}_{1,...,t_{p-1},t_p+1,...,d;\alpha_p}$$

$$\sum_{k=1}^{d}(2\pi)^k\sum_{0=s_0<s_1<...<s_{k-1}<s_k=d}\sum^{*}_{1,...,d;s_1,...,s_{k-1}}\sum_{0=u_0\leq u_1\leq...\leq u_{k-1}\leq u_k\alpha_p+t_p-t_{p-1}}$$

$$\sum^{*}_{1,..,\alpha_p,t_{p-1}+1,..,t_p;u_1,..,u_{k-1}}\prod_{r=1}^{k}\sum_{\beta_r=0}^{d-t_p+t_{p-1}+u_r-u_{r-1}}\sum^{*}_{1,...,u_{r-1},u_r+1,...,\alpha_p,t_{p-1}+1,...,t_p;\beta_r}$$

$$Var^{(\beta_r+u_r-u_{r-1})}D_{x^{(s_{r-1}+1)}...x^{(s_r)}}f(\underbrace{\underbrace{\underbrace{\cdot,...,\cdot}_{\beta_r},0,...,0}_{u_{r-1}},\cdot,...,\cdot}_{u_r},0,...,0).\blacksquare$$

Lemma 3.2. *Let $\alpha_1,...,\alpha_d,1$ be independent over $\mathbb{Q}$ real numbers. Assume that $\tilde{\varphi}:I^d\to\mathbb{R}$ is an absolutely continuous function, which is periodic of period 1 in each coordinate. Suppose, $D_{x^{(i_1)}...x^{(i_k)}}\tilde{\varphi}$ is the function of bounded variation for each $1\leq i_1<...<i_k\leq d$. Then for any $(m_1,...,m_d)\in\mathbb{Z}^d$ where $m_i\neq 0$ for $i=1,...,d$ and $N\in\mathbb{Z}\setminus\{0\}$ there exists a polynomial F of 4^d variables with nonnegative coefficients such that*

$$|\int_{I^d}\exp 2\pi iN(\tilde{\varphi}^{(n)}(x)+\sum_{k=1}^{d}m_knx^{(k)})dx|\leq$$

$$\frac{1}{|n|^d}F(Var^{(r)}D_{x^{(i_1)}..x^{(i_k)}}f(0,...,0,\overset{j_1}{\cdot},0,...,0,\overset{j_r}{\cdot},0,...,0):$$

$$1\leq i_1<...<i_k\leq d,\ 1\leq j_1<...<j_r\leq d)$$

where $\alpha=(\alpha_1,...,\alpha_d)$ and

$$\tilde{\varphi}^{(n)}(x)=\begin{cases}\tilde{\varphi}(x)+...+\tilde{\varphi}(x+(n-1)\alpha) & for\quad n>0\\ 0 & for\quad n=0\\ -(\tilde{\varphi}(x+n\alpha)+...+\tilde{\varphi}(x-\alpha)) & for\quad n<0.\end{cases}$$

Proof. Let $f(x^{(1)},...,x^{(d)}) = N(\tilde{\varphi}^{(n)}(x) + \sum_{k=1}^d m_k n x^{(k)})$. Then

$$\begin{aligned} D_{x^{(i)}} f(x) &= N(D_{x^{(i)}}\tilde{\varphi}^{(n)}(x) + m_i n) \text{ for } i=1,...,d \text{ and} \\ D_{x^{(i_1)}...x^{(i_k)}} f(x) &= N D_{x^{(i_1)}...x^{(i_k)}}\tilde{\varphi}^{(n)}(x) \text{ for } 1 \le i_1 < ... < i_k \le d \text{ and } k > 1. \end{aligned}$$

We will consider a real number $\frac{1}{2} > \varepsilon > 0$.
Since for each $1 \le i_1 < ... < i_k \le d$ the function $D_{x^{(i_1)}...x^{(i_k)}}\tilde{\varphi}(x)$ is integrable in the sense of Riemann and the rotation of α is monoergodic, there exists a natural number n_0 such that for any $|n| \ge n_0$, $1 \le i_1 < ... < i_k \le d$ and $x \in I^d$ we have

$$|\frac{D_{x^{(i_1)}...x^{(i_k)}}\tilde{\varphi}^{(n)}(x)}{n} - \int_{I^d} D_{x^{(i_1)}...x^{(i_k)}}\tilde{\varphi}(x)dx| < \varepsilon.$$

From

$$\int_{I^d} D_{x^{(i_1)}...x^{(i_k)}}\tilde{\varphi}(x)dx =$$

$$\int_{I^{d-k}}(\int_{I^k} D_{x^{(i_1)}...x^{(i_k)}}\tilde{\varphi}(x)dx^{(i_1)}...dx^{(i_k)})dx^{(1)}...d\hat{x^{(i_1)}}...d\hat{x^{(i_k)}}...dx^{(d)} =$$

$$\int_{I^{d-k}} \Delta^*_{i_1..i_k}\tilde{\varphi}(x)dx^{(1)}...d\hat{x^{(i_1)}}...d\hat{x^{(i_k)}}...dx^{(d)} = 0$$

we obtain that for $|n| \ge n_0$

$$|D_{x^{(i_1)}...x^{(i_k)}}\tilde{\varphi}^{(n)}(x)| < \varepsilon|n|.$$

Let $|n| \ge \max(n_0, d!2^{d-2}M)$ where $M = \max_{i=1,..,d}|m_i| + 1$. Then for any $x \in I^d$ we have

$$|\sum_{k=1}^d (2\pi i)^k \sum_{0=s_0<s_1<...<s_{k-1}<s_k=d} \sum^*_{1,...,d;s_1,...,s_{k-1}} \prod_{r=1}^k D_{x^{(s_{r-1}+1)}...x^{(s_r)}} f(x)| \ge$$

$$(2\pi|N|)^d \prod_{k=1}^d |D_{x^{(k)}}\tilde{\varphi}^{(n)}(x) + m_k n| -$$

$$\sum_{k=1}^{d-1}(2\pi|N|)^k \sum_{0=s_0<s_1<...<s_{k-1}<s_k=d} \sum^*_{1,...,d;s_1,...,s_{k-1}} \prod_{r=1}^k |D_{x^{(s_{r-1}+1)}...x^{(s_r)}}\tilde{\varphi}^{(n)}(x) + n a_{s_{r-1}+1...s_r}| \ge$$

where

$$a_{s_{r-1}+1...s_r} = \begin{cases} m_{i_{s_r}} & for \quad s_{r-1}+1 = s_r \\ 0 & for \quad s_{r-1}+1 < s_r \end{cases}$$

$$\ge (2\pi|N|)^d \prod_{k=1}^d |\, n|(|m_k| - \varepsilon) -$$

$$\sum_{k=1}^{d-1}(2\pi|N|)^k \sum_{0=s_0<s_1<...<s_{k-1}<s_k=d} \sum^*_{1,...,d;s_1,...,s_{k-1}} (M|n|)^k \ge$$

$$(\pi|Nn|)^d - d!(2M\pi|Nn|)^{d-1} \geq$$

$$(\pi|N|)^d|n|^{d-1}(|n| - d!2^{d-1}M) \geq \frac{1}{2}(\pi|Nn|^d) = C|n|^d.$$

By Lemma 3.1 we have

$$|\int_{I^d} \exp 2\pi i N(\tilde{\varphi}^{(n)}(x) + \sum_{k=1}^{d} m_k n x^{(k)})dx| \leq$$

$$\sum_{l=1}^{d} \frac{1}{C^{l+1}|n|^{d(l+1)}} \sum_{0=t_0<t_1<...<t_{l-1}<t_l=d} \sum^{*}_{1,...,d;t_1,...,t_{l-1}} \prod_{p=1}^{l} \sum_{\alpha_p=0}^{d-t_p+t_{p-1}} \sum^{*}_{1,...,t_{p-1},t_p+1,...,d;\alpha_p}$$

$$\sum_{k=1}^{d}(2\pi)^k \sum_{0=s_0<s_1<...<s_{k-1}<s_k=d} \sum^{*}_{1,...,d;s_1,...,s_{k-1}} \sum_{0\leq u_1\leq...\leq u_{k-1}\leq\alpha_p+t_p-t_{p-1}} \sum^{*}_{1,..,\alpha_p,t_{p-1}+1,..,t_p;u_1,..,u_{k-1}}$$

$$\prod_{r=1}^{k} \sum_{\beta_r=0}^{d-t_p+t_{p-1}+u_r-u_{r-1}} \sum^{*}_{1,...,u_{r-1},u_r+1,...,\alpha_p,t_{p-1}+1,...,t_p;\beta_r}$$

$$Var^{(\beta_r+u_r-u_{r-1})}(ND_{x^{(s_{r-1}+1)}...x^{(s_r)}}\tilde{\varphi}^{(n)}(\underbrace{\underbrace{\underbrace{\cdot,...,\cdot,}_{\beta_r}0,...,0,}_{u_{r-1}}\cdot,...,\cdot,}_{u_r}0,...,0)+na_{s_{r-1}+1...s_r}) \leq$$

$$\sum_{l=1}^{d} \frac{1}{C^{l+1}|n|^{d(l+1)}} \sum_{0<t_1<...<t_{l-1}<d} \sum^{*}_{1,...,d;t_1,...,t_{l-1}} \prod_{p=1}^{l} \sum_{\alpha_p=0}^{d-t_p+t_{p-1}} \sum^{*}_{1,...,t_{p-1},t_p+1,...,d;\alpha_p}$$

$$\sum_{k=1}^{d}(2\pi|N|)^k \sum_{0=s_0<s_1<...<s_{k-1}<s_k=d} \sum^{*}_{1,...,d;s_1,...,s_{k-1}} \sum_{0\leq u_1\leq...\leq u_{k-1}\leq\alpha_p+t_p-t_{p-1}}$$

$$\sum^{*}_{1,..,\alpha_p,t_{p-1}+1,..,t_p;u_1,..,u_{k-1}} \prod_{r=1}^{k} \sum_{\beta_r=0}^{d-t_p+t_{p-1}+u_r-u_{r-1}} \sum^{*}_{1,...,u_{r-1},u_r+1,...,\alpha_p,t_{p-1}+1,...,t_p;\beta_r} |n|$$

$$Var^{(\beta_r+u_r-u_{r-1})}D_{x^{(s_{r-1}+1)}...x^{(s_r)}}\tilde{\varphi}(...) \leq \sum_{l=1}^{d} \frac{|n|^{dl}}{|n|^{d(l+1)}}F_l = \frac{1}{|n|^d}F. \blacksquare$$

Remark. With the same assumption as the one in Lemma 3.2 we can prove that for any $(r_1, ..., r_d) \in \mathbb{Z}^d$ there exists a polynomial F of 4^d variables with nonnegative coefficients such that

$$|\int_{I^d} \exp 2\pi i(N\tilde{\varphi}^{(n)}(x) + \sum_{k=1}^{d}(Nm_k n + r_k)x^{(k)})dx| \leq \frac{F}{|n|^d}.$$

Theorem 3.3. *Let $\alpha_1, ..., \alpha_d, 1$ be independent over $\mathbb{Q}$ real numbers. Let a cocycle $\varphi : \mathbb{T}^d \to \mathbb{T}$ be represented as*

$$\varphi(e^{2\pi i x_1}, ..., e^{2\pi i x_d}) = e^{2\pi i(\tilde{\varphi}(x_1,...,x_d)+m_1x_1+m_dx_d)}$$

where $\tilde{\varphi} : I^d \to \mathbb{R}$ satisfies the same assumption as the one in Lemma 3.2. If $rw(\varphi) = k > 0$ then the set

$$\{f \in L^2(\mathbb{T}^{d+1}, \lambda_{d+1}) : \hat{\sigma}_f(n) = (U^n_{T_\varphi} f, f) = O(\frac{1}{|n|^k})\}$$

is dense in the orthocomplement of the eigenfunctions of T.

Proof. For simplicity we may assume that $m_1 \neq 0, ..., m_k \neq 0$. By Lemma 2.1 there exists a real number $M > 0$ such that for any $1 \leq i_1 < ..., i_p \leq k$, $1 \leq j_1 < ..., j_l \leq k$ and $(x^{(k+1)}, ..., x^{(d)}) \in I^{d-k}$ we have

$$Var^{(l)} D_{x^{(i_1)}..x^{(i_p)}} \tilde{\varphi}(0, ..., 0, \overset{j_1}{\cdot}, 0, ..., 0, \overset{j_l}{\cdot}, 0, ..., 0, x^{(k+1)}, ..., x^{(d)}) \leq M$$

Let P be a trigonometric polynomial given by

$$P(z_1, ..., z_d, \omega) = \sum_{r_1=-R_1}^{R_1} ... \sum_{r_d=-R_d}^{R_d} \sum_{s=-S \; s\neq 0}^{S} a_{r_1...r_d s} z_1^{r_1} ... z_d^{r_d} \omega^s$$

where $a_{r_1...r_d s} \in \mathbb{C}$. Then

$$|(U^n_{T_\varphi} P, P)| = |\int_{\mathbb{T}^{d+1}} P(T^n z, \varphi^{(n)}(z)\omega)\bar{P}(z, \omega) dz d\omega| =$$

$$|\int_{I^{d+1}} \sum_{r_1,...,r_d,s} a_{r_1...r_d s} \exp 2\pi i[\sum_{j+1}^{d} r_j(x^{(j)} + n\alpha_j)+$$

$$+s\tilde{\varphi}^{(n)}(x) + s\sum_{j=1}^{d} m_j(nx^{(j)} + \frac{(n-1)n}{2}\alpha_j) + sy]$$

$$\sum_{r'_1,...,r'_d,s'} \bar{a}_{r'_1...r'_d s'} \exp 2\pi i(\sum_{j=1}^{d} r'_j x^{(j)} + s'y) dx^{(1)} ... dx^{(d)} dy| \leq$$

$$\sum_{r_1,...,r_d,r'_1,...,r'_d,s} |a_{r_1...r_d s} a_{r'_1...r'_d s}|| \int_{I^d} \exp 2\pi i[s\tilde{\varphi}^{(n)}(x)+[sn\sum_{j=1}^{d} m_j x^{(j)}+\sum_{j=1}^{d}(r_j - r'_j)x^{(j)}]dx|$$

$$\sum_{r_1,...,r_d,r'_1,...,r'_d,s} |a_{r_1...r_d s} a_{r'_1...r'_d s}|| \int_{I^{d-k}} \exp 2\pi i \sum_{j=k+1}^{d} (r_j - r'_j)x^{(j)} \; dx^{(k+1)} ... dx^{(d)}|$$

$$|\int_{I^k} \exp 2\pi i[s\tilde{\varphi}^{(n)}(x) + sn\sum_{j=1}^{k} m_j x^{(j)} + \sum_{j=1}^{k}(r_j - r'_j)x^{(j)}] dx^{(1)} ... dx^{(k)}| \leq$$

$$\sum_{r_1,\dots,r_d,r'_1,\dots,r'_d,s} |a_{r_1\dots r_d s} a_{r'_1\dots r'_d s}| \frac{F_{s,r_1-r'_1,\dots,r_k-r'_k}(M)}{|n|^k} = O(\frac{1}{|n|^k}). \blacksquare$$

Corollary 3.1. *If $\varphi \in C^{2d}$ and $rw(\varphi) = k > 0$ then the set*

$$\{f \in L^2(\mathbb{T}^{d+1}, \lambda_{d+1}) : \hat{\sigma}_f(n) = (U^n_{T_\varphi} f, f) = O(\frac{1}{|n|^k})\}$$

is dense in the orthocomplement of the eigenfunctions of T.$\blacksquare$

Let $w(\varphi) \neq 0$. For simplicity we assume that $m_1 \neq 0$. Suppose, there exists a real number $R > 0$ such that for each $(x^{(2)}, \dots, x^{(d)}) \in I^{d-1}$

$$Var^{(1)} \frac{\partial \tilde{\varphi}}{\partial x^{(1)}}(\cdot, x^{(2)}, \dots, x^{(d)}) \leq R$$

In the same manner as in the proof of Theorem 3.3 we can show that

$$\hat{\sigma}_{\chi_N}(n) = O(\frac{1}{|n|}) \text{ for } N \neq 0$$

where $\chi_N(z_1, \dots, z_d, \omega) = \omega^N$. From this and by Corollary 1.1 we conclude that T_φ has countable Lebesgue spectrum in the orthocomplement of the eigenfunctions of T.

Corollary 3.2. *If $\varphi \in C^2$ and $w(\varphi) \neq 0$ then T_φ has countable Lebesgue spectrum in the orthocomplement of the eigenfunctions of T.*$\blacksquare$

4 Spectral properties in the case where the winding vector is equal zero

Lemma 4.1. *If $0 < |x| \leq |y| \leq \frac{1}{2}$, then*

$$4|x| \leq |e^{2\pi i x} - 1| \leq 2\pi|x|, \tag{5}$$

$$\frac{2}{\pi}|\frac{y}{x}| \leq |\frac{e^{2\pi i y} - 1}{e^{2\pi i x} - 1}| \leq |\frac{y}{x}|. \tag{6}$$

Lemma 4.2. *Assume $n \in \mathbb{N}$ and take $a \in \mathbb{R}$ such that $0 < a < 1$. Then there exist n pair wise disjoint subintervals $I_1, \dots, I_n$ of I such that for $x \in I \setminus \bigcup_{i=1}^n I_i$ we have $|\cos n\pi x| \geq a$ moreover $|I_i| = \frac{a}{n}$.*

Proof. Set $I_i = [\frac{2i-1}{2n} - \frac{a}{2n}, \frac{2i-1}{2n} + \frac{a}{2n}]$. Then

$$I \setminus \bigcup_{i=1}^n I_i = \bigcup_{i=1}^n [\frac{2i-2}{2n}, \frac{2i-1}{2n} - \frac{a}{2n}) \cup (\frac{2i-1}{2n} + \frac{a}{2n}, \frac{2i}{2n}].$$

If $x \in I \setminus \bigcup_{i=1}^{n} I_i$, then there exists a natural number i such that

$$x \in [\frac{2i-2}{2n}, \frac{2i-1}{2n} - \frac{a}{2n}) \cup (\frac{2i-1}{2n} + \frac{a}{2n}, \frac{2i}{2n}].$$

Then $\frac{a}{2n} < |x - \frac{2i-1}{2n}| \leq \frac{1}{2n}$, whence $\frac{a}{2} < |nx - \frac{2i-1}{2}| \leq \frac{1}{2}$ and finally

$$a < 2|nx - \frac{2i-1}{2}| \leq |\sin \pi(nx - i + \frac{1}{2})| \leq |\cos \pi nx|. \blacksquare$$

Lemma 4.3. *Let $f : I \to \mathbb{R}$ be an absolutely continuous function such that f' is of bounded variation and $f'(0) = f'(1)$, $f(1) - f(0) \in \mathbb{Z}$. Suppose there exists a real number a such that $|f'(x)| \geq a > 0$ for $x \in I \setminus \bigcup_{i=1}^{s}(a_i, b_i)$ (where $0 \leq a_1 < b_1 < ... < a_s < b_s < 1$ or $0 < a_1 < b_1 < ... < a_s < 1 < b_s$). Then*

$$|\int_0^1 e^{2\pi i f(x)} dx| \leq \frac{1}{2\pi} \frac{Var f'}{a^2} + \frac{s}{\pi a} + \sum_{i=1}^{s} (b_i - a_i). \tag{7}$$

Proof. Let $D = \bigcup_{i=1}^{s}(a_i, b_i)$ and $a_{s+1} = a_1$. Then

$$|\int_0^1 e^{2\pi i f(x)} dx| \leq |\int_{I \setminus D} e^{2\pi i f(x)} dx| + \sum_{i=1}^{s} (b_i - a_i) =$$

$$|\int_{I \setminus D} \frac{1}{2\pi i f'(x)} de^{2\pi i f(x)}| + \sum_{i=1}^{s} (b_i - a_i) =$$

$$|\sum_{i=1}^{s} (\frac{e^{2\pi i f(a_{i+1})}}{2\pi f'(a_{i+1})} - \frac{e^{2\pi i f(b_i)}}{2\pi f'(b_i)} - \frac{1}{2\pi} \int_{b_i}^{a^{i+1}} e^{2\pi i f(x)} d\frac{1}{f'(x)})| + \sum_{i=1}^{s} (b_i - a_i) \leq$$

$$\frac{1}{2\pi} \sum_{i=1}^{s} (\frac{1}{|f'(a_i)|} + \frac{1}{|f'(b_i)|}) + \frac{1}{2\pi} \sum_{i=1}^{s} Var_{[b_i, a_{i+1}]} \frac{1}{f'(x)}) + \sum_{i=1}^{s} (b_i - a_i) \leq$$

$$\frac{1}{2\pi} \frac{Var f'}{a^2} + \frac{s}{\pi a} + \sum_{i=1}^{s} (b_i - a_i). \blacksquare$$

Given a real number $\alpha \in [0, 1)$, let $[0; a_1, a_2, ...]$ be its continued fraction expansion where a_n are positive integer numbers. Put

$$q_0 = 1,\ q_1 = a_1,\ q_{n+1} = a_{n+1} q_n + q_{n-1},$$

$$p_0 = 0,\ p_1 = 1,\ p_{n+1} = a_{n+1} p_n + p_{n-1}.$$

The rationals p_n/q_n are called the *convergents* of α and the inequality

$$\frac{1}{2 q_n q_{n+1}} < |\alpha - \frac{p_n}{q_n}| < \frac{1}{q_n q_{n+1}}$$

holds.
Given $A, B \geq 2$, we say that a pair $(\alpha,\beta) \in [0,1)^2$ satisfies (A,B) if there exists strictly increasing sequences $\{n_k\}, \{m_k\}$ of natural numbers such that

$$B^{8s_{2m_k}} < \frac{1}{2}q_{2n_k+1} \tag{8}$$

$$A^{8q_{2n_{k+1}}} < \frac{1}{2}s_{2m_k+1} \tag{9}$$

where p_n/q_n and r_n/s_n are convergents of α and β.
Obviously, the set $\{(\alpha,\beta) : (\alpha,\beta) \text{ satisfies } (A,B)\}$ is uncountable.
For a pair (α,β) satisfying (A,B) we define real analytic functions $\psi_1, \psi_2 : \mathbb{R} \to \mathbb{R}$ periodic of period 1 given by

$$\psi_1(x) = \sum_{k=1}^{\infty} \frac{1}{2\pi i q_{2n_k} A^{q_{2n_k}}}(e^{2\pi i q_{2n_k} x} - e^{-2\pi i q_{2n_k} x})$$

$$\psi_2(y) = \sum_{k=1}^{\infty} \frac{1}{2\pi i s_{2m_k} A^{s_{2m_k}}}(e^{2\pi i s_{2m_k} y} - e^{-2\pi i s_{2m_k} y}).$$

We first prove

Lemma 4.4. *For any integer numbers $h_1, h_2, N \neq 0$ we have*

$$\lim_{|m|\to\infty} \int_{I^2} e^{2\pi i[N(\psi_1^{(m)}(x)+\psi_2^{(m)}(y))+h_1x+h_2y]}dxdy = 0.$$

Corollary 4.1. *If (α,β) satisfies (A,B) then $\alpha, \beta, 1$ are independent over $\mathbb{Q}$.*

Proof. Suppose, $\alpha, \beta, 1$ are dependent over $\mathbb{Q}$. Then there exist $m_1, m_2, m_3 \in \mathbb{Z}$ such that $m_1\alpha + m_2\beta = m_3$. Let t_n/u_n are convergents of $m_1\alpha$ and $m_2\beta$. Then

$$\sum_{p=0}^{u_n-1} \psi_1(\cdot + p|m_1|\alpha), \sum_{p=0}^{u_n-1} \psi_2(\cdot + p|m_2|\beta)$$

uniformly converges to 0 (see [6], p. 189). From

$$\psi_1^{(u_n|m_1m_2|)}(x) + \psi_2^{(u_n|m_1m_2|)}(y) =$$

$$\sum_{k=0}^{|m_1|-1}\sum_{l=0}^{|m_2|-1}\sum_{p=0}^{u_n-1}(\psi_1(x+k\alpha+l|m_1|u_n\alpha+p|m_1|\alpha)+\psi_2(y+k|m_2|u_n\beta+l\beta+p|m_2|\beta))$$

we have

$$\sup_{(x,y)\in I^2} |\psi_1^{(u_n|m_1m_2|)}(x) + \psi_2^{(u_n|m_1m_2|)}(y)|$$

$$\leq |m_1m_2|(\sup_{x\in I}|\sum_{p=0}^{u_n-1}\psi_1(x+p|m_1|\alpha)| + \sup_{y\in I}|\sum_{p=0}^{u_n-1}\psi_2(y+p|m_2|\beta)|)$$

hence

$$\psi_1^{(u_n|m_1m_2|)}(\cdot)+\psi_2^{(u_n|m_1m_2|)}(\cdot)$$

uniformly converges to 0 in I^2. It follows that

$$\lim_{n\to\infty}\int_{I^2} e^{2\pi i(\psi_1^{(u_n|m_1m_2|)}(x)+\psi_2^{(u_n|m_1m_2|)}(y))}dxdy=1,$$

which contradicts Lemma 4.4. ■

Proof of Lemma 4.4. From (8) and (9) for every $k\in\mathbb{N}$

$$B^{8s_{2m_k}}<\frac{1}{2}q_{2n_k+1}<\frac{1}{2}s_{2m_k+1}$$

$$A^{8q_{2n_k}}<\frac{1}{2}s_{2m_{k-1}+1}<\frac{1}{2}q_{2n_k+1}.$$

Hence for any $m\geq\min(A^{8q_{2n_1}},B^{8s_{2m_1}})$ there exists natural number k such that

$$A^{8q_{2n_k}}\leq m\leq\frac{1}{2}q_{2n_k+1}$$

or

$$B^{8s_{2m_k}}\leq m\leq\frac{1}{2}s_{2m_k+1}.$$

In the first case

$$|\int_{I^2} e^{2\pi i[N(\psi_1^{(m)}(x)+\psi_2^{(m)}(y))+h_1x+h_2y]}dxdy|=$$

$$|\int_I e^{2\pi i[N(\psi_1^{(m)}(x)+h_1x]}dx||\int_I e^{2\pi i[N(\psi_2^{(m)}(y))+h_2y]}dy|\leq|\int_I e^{2\pi i[N(\psi_1^{(m)}(x)+h_1x]}dx|.$$

From

$$\psi_1'(x)=\sum_{l=1}^{\infty}\frac{1}{A^{q_{2n_l}}}(e^{2\pi iq_{2n_l}x}+e^{-2\pi iq_{2n_l}x})$$

it follows that for any natural number m

$$|\psi_1^{(m)'}(x)|=|\sum_{j=0}^{m-1}\psi_1'(x+j\alpha)|=$$

$$|\sum_{l=1}^{\infty}\frac{1}{A^{q_{2n_l}}}(e^{2\pi iq_{2n_l}x}\frac{e^{2\pi iq_{2n_l}m\alpha}-1}{e^{2\pi iq_{2n_l}\alpha}-1}+e^{-2\pi iq_{2n_l}x}\frac{e^{-2\pi iq_{2n_l}m\alpha}-1}{e^{-2\pi iq_{2n_l}\alpha}-1})|=$$

$$|\sum_{l=1}^{\infty}\frac{1}{A^{q_{2n_l}}}\frac{e^{2\pi iq_{2n_l}m\alpha}-1}{e^{2\pi iq_{2n_l}\alpha}-1}(e^{2\pi iq_{2n_l}x}+e^{-2\pi iq_{2n_l}(x+(m-1)\alpha)})|\geq$$

$$\frac{2}{A^{q_{2n_k}}}|\frac{e^{2\pi iq_{2n_k}m\alpha}-1}{e^{2\pi iq_{2n_k}\alpha}-1}||\cos 2\pi q_{2n_k}(x+\frac{(m-1)\alpha}{2})|-$$

$$\sum_{l=1}^{k-1}\frac{1}{A^{q_{2n_l}}}\frac{4}{|e^{2\pi i q_{2n_l}\alpha}-1|}-\sum_{l=k+1}^{\infty}\frac{2}{A^{q_{2n_l}}}|\frac{e^{2\pi i q_{2n_l}m\alpha}-1}{e^{2\pi i q_{2n_l}\alpha}-1}|.$$

From $|q_{2n_l}\alpha - p_{2n_l}| > \frac{1}{2q_{2n_l+1}}$ and (5) we have

$$|e^{2\pi i q_{2n_l}\alpha}-1| \geq 4|q_{2n_l}\alpha - q_{2n_l}| > \frac{2}{q_{2n_l+1}}$$

hence $\frac{1}{|e^{2\pi i q_{2n_l}\alpha}-1|} < \frac{q_{2n_l+1}}{2}$ for any natural l. From $m \leq \frac{1}{2}q_{2n_l+1}$ and $|q_{2n_l}\alpha - p_{2n_l}| < \frac{1}{q_{2n_l+1}}$ for any $l \geq k$ it follows that

$$0 < |q_{2n_l}\alpha - p_{2n_l}| \leq |mq_{2n_l}\alpha - mp_{2n_l}| \leq \frac{1}{2}q_{2n_l+1}|q_{2n_l}\alpha - p_{2n_l}| < \frac{1}{2}.$$

From (6) for $l \geq k$

$$\frac{m}{2} \leq |\frac{e^{2\pi i q_{2n_l}m\alpha}-1}{e^{2\pi i q_{2n_l}\alpha}-1}| \leq m.$$

From Lemma 4.2 there exist subintervals $I_1, ..., I_{2q_{2n_k}}$ of I such that for any $x \in I \setminus \bigcup_{i=1}^{2q_{2n_k}} I_i$ we have

$$|\cos 2\pi q_{2n_k}(x+\frac{(m-1)\alpha}{2})| \geq \frac{1}{A^{q_{2n_k}}};$$

moreover $|I_i| = \frac{1}{2q_{2n_k}A^{q_{2n_k}}}$ for $i = 1, ..., 2q_{2n_k}$.
It follows that for $x \in I \setminus \bigcup_{i=1}^{2q_{2n_k}} I_i$ we have

$$|\psi_1^{(m)'}(x)| \geq -2\sum_{l=1}^{k-1}\frac{q_{2n_l+1}}{A^{q_{2n_l}}}+\frac{m}{A^{2q_{2n_k}}}-\sum_{l=k+1}^{\infty}\frac{2m}{A^{q_{2n_l}}} \geq$$

$$-q_{2n_{k-1}+1}+\frac{m}{A^{2q_{2n_k}}}-\frac{2m}{A^{q_{2n_k+1}}}\frac{A}{A-1} \geq -q_{2n_k}+\frac{m}{A^{2q_{2n_k}}}-\frac{4m}{A^{q_{2n_k+1}}}.$$

From $A^{8q_{2n_k}} \leq m \leq \frac{1}{2}q_{2n_k+1}$ we have

$$4q_{2n_k} \leq A^{6q_{2n_k}} = \frac{A^{8q_{2n_k}}}{A^{2q_{2n_k}}} \leq \frac{m}{A^{2q_{2n_k}}} \quad and \quad q_{2n_k}+2 \leq A^{8q_{2n_k}} \leq \frac{1}{2}q_{2n_k+1}$$

hence

$$16A^{2q_{2n_k}} \leq A^{2q_{2n_k}+4} \leq A^{2q_{2n_k+1}}.$$

For this reason for $x \in I \setminus \bigcup_{i=1}^{2q_{2n_k}} I_i$

$$|\psi_1^{(m)'}(x)| \geq -\frac{m}{4A^{2q_{2n_k}}}+\frac{m}{A^{2q_{2n_k}}}-\frac{m}{4A^{2q_{2n_k}}}=\frac{m}{2A^{2q_{2n_k}}},$$

hence $|N\psi_1^{(m)'}(x)+h_1| \geq |N|\frac{m}{2A^{2q_{2n_k}}} - |h_1|$. From (7) for any natural m such that $\frac{m}{A^{2q_{2n_k}}} \geq A^{6q_{2n_k}} \geq 4|\frac{h_1}{N}|$ we have

$$|\int_I e^{2\pi i[N(\psi_1^{(m)}(x)+h_1x]}dx| \leq \frac{1}{2\pi}\frac{Var(N\psi_1^{(m)'}+h_1)}{(\frac{|N|m}{4A^{2q_{2n_k}}})^2}+\frac{2q_{2n_k}}{\pi\frac{|N|m}{4A^{2q_{2n_k}}}}+\frac{1}{A^{q_{2n_k}}} \leq$$

$$\frac{8}{\pi}\frac{A^{4q_{2n_k}}}{|N|^2m^2}|N|mVar\psi_1' + \frac{8A^{4q_{2n_k}}}{\pi|N|m} + \frac{1}{A^{q_{2n_k}}} \leq$$

$$\frac{8}{\pi}\frac{A^{4q_{2n_k}}}{|N|m}(Var\psi_1'+1) + \frac{1}{A^{q_{2n_k}}} \leq \frac{c_1}{A^{q_{2n_k}}}.$$

Similarly we can get that there exists a constant c_2 such that if $B^{8s_{2m_k}} \leq m \leq \frac{1}{2}s_{2m_k+1}$ then

$$|\int_I e^{2\pi i[N(\psi_2^{(m)}(y)+h_2y]}dy| \leq \frac{c_2}{B^{s_{2m_k}}}.$$

Therefore

$$\lim_{m\to\infty}\int_{I^2} e^{2\pi i[N(\psi_1^{(m)}(x)+\psi_2^{(m)}(y))+h_1x+h_2y]}dxdy = 0.$$

If $m < 0$ then

$$|\int_{I^2} e^{2\pi i[N(\psi_1^{(m)}(x)+\psi_2^{(m)}(y))+h_1x+h_2y]}dxdy| =$$

$$|\int_{I^2} e^{2\pi i[N-(\psi_1^{(-m)}(x+m\alpha)+\psi_2^{(-m)}(y+m\beta))+h_1x+h_2y]}dxdy| =$$

$$|\int_{I^2} e^{2\pi i[N(\psi_1^{(-m)}(x)+\psi_2^{(-m)}(y))-h_1x-h_2y]}dxdy|.$$

It follows that

$$\lim_{|m|\to\infty}\int_{I^2} e^{2\pi i[N(\psi_1^{(m)}(x)+\psi_2^{(m)}(y))+h_1x+h_2y]}dxdy = 0. \blacksquare$$

Lemma 4.5. *Let $U : H \to H$ be a unitary operator on a Hilbert space H. Then the set $\{h \in H : \lim_{|m|\to\infty}(U^m h, h) = 0\}$ is closed in H.*

Proof. Let $h_n \in H$ be a sequence such that $\lim_{|m|\to\infty}(U^m h_n, h_n) = 0$ which convergence to $h \in H$. Let $\varepsilon > 0$. We take a natural number n such that $\| h - h_n \| < \min\{\frac{\varepsilon}{2(2\|h\|+1)}, 1\}$. Let m_0 be a natural number such that for any $|m| \geq m_0$ we have $|(U^m h_n, h_n)| < \frac{\varepsilon}{2}$. Then for $|m| \geq m_0$

$$|(U^m h, h)| = |(U^m(h - h_n), h) + (U^m h_n, h - h_n) + (U^m h_n, h_n)| \leq$$

$$\| h - h_n \| \| h \| + \| h_n \| \| h - h_n \| + |(U^m h_n, h_n)| \leq$$

$$\| h - h_n \| (2 \| h \| + 1) + |(U^m h_n, h_n)| < \varepsilon. \blacksquare$$

Theorem 4.6. *There exist real numbers α and β such that $\alpha, \beta, 1$ are independent over $\mathbb{Q}$ and a cocycle $\varphi : \mathbb{T}^2 \to \mathbb{T}$ given by*

$$\varphi(e^{2\pi ix}, e^{2\pi iy}) = e^{2\pi i(\psi_1(x)+\psi_2(y))}$$

where ψ_1, ψ_2 are real analytic function which are periodic of period 1 such that T_φ is mixing in the orthocomplement of the eigenfunctions of T where T is the rotation on $\mathbb{T}^2$ given by $T(z_1, z_2) = (e^{2\pi i\alpha}z_1, e^{2\pi i\beta}z_2)$.

Proof. We take $\alpha, \beta, \psi_1, \psi_2$ like in Lemma 4.4. By Lemma 4.5 is sufficient to show that T_φ is mixing in the set of trigonometric polynomials given by

$$P(z_1, z_2, \omega) = \sum_{k_1=-K_1}^{K_1} \sum_{k_2=-K_2}^{K_2} \sum_{l=-L \ l\neq 0}^{L} a_{k_1,k_2,l} z_1^{k_1} z_2^{k_2} \omega^l$$

where $a_{k_1,k_2,l} \in \mathbb{C}$.

$$|(U_{T_\varphi}^m P, P)| =$$

$$|\int_{\mathbb{T}^3} \sum_{k_1,k_2,l} a_{k_1,k_2,l} e^{2\pi i(\alpha k_1+\beta k_2)} z_1^{k_1} z_2^{k_2} (\varphi^{(m)}(z_1,z_2))^l \omega^l \sum_{k_1',k_2',l'} \bar{a}_{k_1',k_2',l'} z_1^{-k_1'} z_2^{-k_2'} \omega^{-l'} dz_1 dz_2 d\omega| =$$

$$|\sum_{k_1,k_2,k_1',k_2',l} a_{k_1,k_2,l} \bar{a}_{k_1',k_2',l'} e^{2\pi i(\alpha k_1+\beta k_2)} \int_{\mathbb{T}^2} z_1^{k_1-k_1'} z_2^{k_2-k_2'} (\varphi^{(m)}(z_1,z_2))^l dz_1 dz_2| \leq$$

$$\sum_{k_1,k_2,k_1',k_2',l} |a_{k_1,k_2,l} \bar{a}_{k_1',k_2',l'}| |\int_{I^2} e^{2\pi i[l(\psi_1^{(m)}(x)+\psi_2^{(m)}(y))+(k_1-k_1')x+(k_2-k_2')y]} dxdy|.$$

Consequently $\lim_{|m|\to\infty} |(U_{T_\varphi}^m P, P)| = 0$ and the proof is complete. ■

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Krzysztof Frączek, Department of Mathematics and Computer Science,
Nicholas Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland
fraczek@mat.uni.torun.pl