ERGODIC PROPERTIES OF THE IDEAL GAS MODEL FOR INFINITE BILLIARDS

KRZYSZTOF FRĄCZEK

Abstract. In this paper we study ergodic properties of the Poisson suspension (the ideal gas model) of the billiard flow $(b_t)_{t\in\mathbb{R}}$ on the plane with a Λ-periodic pattern ($\Lambda \subset \mathbb{R}^2$ is a lattice) of polygonal scatterers. We prove that if the billiard table is additionally rational then for a.e. direction $\theta \in S^1$ the Poisson suspension of the directional billiard flow $(b_t^\theta)_{t\in\mathbb{R}}$ is weakly mixing. This gives the weak mixing of the Poisson suspension of $(b_t)_{t\in\mathbb{R}}$. We also show that for a certain class of such rational billiards (including the periodic version of the classical wind-tree model) the Poisson suspension of $(b_t^\theta)_{t\in\mathbb{R}}$ is not mixing for a.e. $\theta \in S^1$.

1. Introduction

In this paper we deal with billiard dynamical systems on the plane with a Λ-periodic pattern ($\Lambda \subset \mathbb{R}^2$ is a lattice) of polygonal scatterers. We focus only on a rational billiards, i.e. the angles between any pair of sides of the polygons (also different polygons) are rational multiplicities of π. The most celebrated example of such billiard table is the periodic version of the wind-tree model introduced by P. and T. Ehrenfest in 1912 [10], in which the scatterers are $\mathbb{Z}^2$-translates of the rectangle $[0, a] \times [0, b]$, where $0 < a, b < 1$.

The *billiard flow* $(b_t)_{t\in\mathbb{R}}$ on a polygonal table $\mathcal{T} \subset \mathbb{R}^2$ (the boundary of the table consists of intervals) describes the unit speed free motion of a billiard ball, i.e. a point mass, on the interior of $\mathcal{T}$ with elastic collision (angle of incidence equals to the angle of reflection) from the boundary of $\mathcal{T}$. The phase space $\mathcal{T}^1$ of $(b_t)_{t\in\mathbb{R}}$ consists of points $(x, \theta) \in \mathcal{T} \times S^1$ such that if x belongs to the boundary of $\mathcal{T}$ then $\theta \in S^1$ is an inward direction. The billiard flow preserves the volume measure $\mu\times\lambda$, where μ is the area measure on $\mathcal{T}$ and λ the Lebesgue measure on S^1. For more details on billiards see [24].

Suppose that $\mathcal{T}$ is the table of a Λ-periodic rational polygonal billiard. Then the volume measure of $\mathcal{T}$ is infinite. Since the table is Λ-periodic, the set $D \subset S^1$ of directions of all sides in $\mathcal{T}$ is finite. Denote by Γ the group of isometries of S^1 generated by reflections through the axes with directions from D. Since the table is rational, Γ is a finite dihedral group. Therefore the phase space $\mathcal{T}^1$ splits into the family $\mathcal{T}^1_\theta = \mathcal{T} \times \Gamma\theta$, $\theta \in S^1/\Gamma$ of invariant subsets for $(b_t)_{t\in\mathbb{R}}$. The restriction of $(b_t)_{t\in\mathbb{R}}$ to $\mathcal{T}^1_\theta$ is called the *direction billiard flow* in direction θ and is denoted by $(b_t^\theta)_{t\in\mathbb{R}}$. The flow $(b_t^\theta)_{t\in\mathbb{R}}$ preserves μ_θ the product of μ and the counting measure of $\Gamma\theta$; this measure is also infinite. Using the standard unfolding process described in [18] (see also [24]), we obtain a connected translation surface $(M_{\mathcal{T}}, \omega_{\mathcal{T}})$ such that the directional linear flow $(\varphi_t^{\mathcal{T},\theta})_{t\in\mathbb{R}}$ on $(M_{\mathcal{T}}, \omega_{\mathcal{T}})$ is isomorphic to the flow $(b_t^\theta)_{t\in\mathbb{R}}$ for every $\theta \in S^1$. Moreover, $(M_{\mathcal{T}}, \omega_{\mathcal{T}})$ is a $\mathbb{Z}^2$-cover of a compact connected translation surface.

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We are interested in ergodic properties of the directional flows $(b_t^\theta)_{t\in\mathbb{R}}$ (or equivalently $(\varphi_t^{\mathcal{T},\theta})_{t\in\mathbb{R}}$) in typical (a.e.) direction. Recently, some progress has been done in understanding this problem, especially for periodic wind-tree model. In this model, Avila and Hubert in [2] proved the recurrence of $(b_t^\theta)_{t\in\mathbb{R}}$ for a.e. direction. The non-ergodicity for a.e. direction was shown by the author and Ulcigrai in [16]. Moreover, Delecroix, Hubert and Leliévre proved in [7] that for a.e. direction the diffusion rate of a.e. orbit is 2/3. For more complicated scatterers some related results were obtained in [8, 14, 26]. Ergodic properties for non-periodic wind-tree models were also recently studied by Málaga Sabogal and Troubetzkoy in [22, 23].

Unlike the approach presented in the mentioned articles, we do not study the dynamics of a single billiard ball (a point particle), i.e. the flow $(b_t^\theta)_{t\in\mathbb{R}}$. We are interested in dynamical properties of infinite (countable and locally finite) configurations of point particles without mutual interactions. Formally, we deal with the Poisson suspension of the flow $(b_t^\theta)_{t\in\mathbb{R}}$ modelling the ideal gas behaviour in $\mathcal{T}$, see [6, Ch. 9]. Given a measure-preserving flow $(T_t)_{t\in\mathbb{R}}$ on an infinite measure space $(X,\mathcal{B},\mu)$, its Poisson suspension $(T_t^*)_{t\in\mathbb{R}}$ is a flow acting on the probability space $(X^*,\mathcal{B}^*,\mu^*)$ of infinite and locally finite configurations of particles in X. The measure μ^* is the Poisson point process with intensity measure μ, i.e. the distribution of the number of particles in any finite measure set $A\in\mathcal{B}$ is the Poisson distribution with intensity $\mu(A)$, and $(T_t^*)_{t\in\mathbb{R}}$ moves infinite configurations of particles according to the flow $(T_t)_{t\in\mathbb{R}}$.

The main result of the paper is the following:

Theorem 1.1. *Let $(b_t)_{t\in\mathbb{R}}$ be the billiard flow on a Λ-periodic rational polygonal billiard table $\mathcal{T}$. Then for a.e. $\theta\in S^1$ the Poisson suspension of the directional billiard flow $(b_t^\theta)_{t\in\mathbb{R}}$ is weakly mixing. Moreover, the Poisson suspension of $(b_t)_{t\in\mathbb{R}}$ is also weakly mixing.*

In fact, we prove much more general result (Theorem 5.4) concerning $\mathbb{Z}^d$-covers of compact translation surfaces and their directional flows. Since $(b_t^\theta)_{t\in\mathbb{R}}$ can be treated as a directional flow on the translation surface $(M_{\mathcal{T}},\omega_{\mathcal{T}})$, Theorem 1.1 is a direct consequence of Theorem 5.4. Moreover, in Section 6 we give a criterion (Theorem 6.3) for the absence of mixing for the Poisson suspension of typical directional flows on some $\mathbb{Z}^d$-covers of compact translation surfaces. Its necessary condition (the existence of "good" cylinders) for the absence of mixing coincides with the condition for recurrence provided by [2]. This allows proving the absence of mixing for the Poisson suspension of $(b_t^\theta)_{t\in\mathbb{R}}$ (for a.e. direction) for the standard periodic wind-tree model, as well as for other recurrent billiards studied in [14, Sec. 9] and [26, Sec. 8.3].

2. Poisson point process and Poisson suspension

Let $(X,\mathcal{B},\mu)$ be a standard σ-finite atomless measure space with $\mu(X)=\infty$. Denote by $(X^*,\mathcal{B}^*,\mu^*)$ the associated Poisson point process. For relevant background material concerning Poisson point processes, see [20] and [21]. Then X^* is the space of countable subsets (configurations) of X and the σ-algebra $\mathcal{B}^*$ is generated by the subsets of the form

$$C_{A,n} := \{\overline{x}\in X^* : \operatorname{card}(\overline{x}\cap A)=n\} \text{ for } A\in\mathcal{B} \text{ with } 0<\mu(A)<+\infty \text{ and } n\geq 0.$$

For every $A\in\mathcal{B}$ with $0<\mu(A)<+\infty$ denote by $C_A : X^*\to\mathbb{Z}_{\geq 0}$ the measurable map given by $C_A(\overline{x})=\operatorname{card}(\overline{x}\cap A)$. Then μ^* is a unique probability measure on $\mathcal{B}^*$ such that:

(i) for any pairwise disjoint collection of finite measure sets $A_1,\ldots,A_k$ in $\mathcal{B}$ the random variables $C_{A_1},\ldots,C_{A_k}$ on $(X^*,\mathcal{B}^*,\mu^*)$ are jointly independent;

(ii) for any $A \in \mathcal{B}$ with $0 < \mu(A) < +\infty$ the random variable C_A on $(X^*, \mathcal{B}^*, \mu^*)$ has Poisson distribution with intensity $\mu(A)$, i.e.

$$\mu^*(C_{A,n}) = e^{-\mu(A)} \frac{\mu(A)^n}{n!} \quad \text{for} \quad n \geq 0.$$

The existence and uniqueness of the measure μ^* can be found, for instance, in [20].

Poisson suspension is a classical notion introduced in statistical mechanics to model so called ideal gas. For an infinite measure-preserving dynamical system its Poisson suspension is a probability measure-preserving system describing the dynamics of infinite (countable) configurations of particles without mutual interactions. For relevant background material we refer the reader to [6]. More formally, for any $(T_t)_{t\in\mathbb{R}}$ measure preserving flow on $(X, \mathcal{B}, \mu)$ by its *Poisson suspension* we mean the flow $(T^*_t)_{t\in\mathbb{R}}$ acting on $(X^*, \mathcal{B}^*, \mu^*)$ by $T^*_t(\overline{x}) = \{T_t y : y \in \overline{x}\}$. Since $(T^*_t)_{t\in\mathbb{R}}$ preserves the measure of any set $C_{A,n}$ and these sets generate the whole σ-algebra $\mathcal{B}^*$, the flow preserves the probability measure μ^*.

A proof of the following folklore result for measure-preserving maps can be found in [27] and [9]. In the setting of group actions, the proof runs in the same way.

Proposition 2.1. *The flow $(T^*_t)_{t\in\mathbb{R}}$ is ergodic if and only if it is weak mixing and if and only if the flow $(T_t)_{t\in\mathbb{R}}$ has no invariant subset of positive and finite measure.*

The flow $(T^*_t)_{t\in\mathbb{R}}$ is mixing if and only if for all $A \in \mathcal{B}$ with $0 < \mu(A) < \infty$ we have $\mu(A \cap T_{-t}A) \to 0$ as $t \to +\infty$.

Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ be two standard σ-finite atomless measure spaces. Assume that $(T_t)_{t\in\mathbb{R}}$ is a measure-preserving flow on $(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \times \nu)$ such that $T_t(x, y) = (T^y_t x, y)$. Then $(T^y_t)_{t\in\mathbb{R}}$ is a measure-preserving flow on $(X, \mathcal{B}, \mu)$ for a.e. $y \in Y$. By a standard Fubini argument, one gets the following result.

Lemma 2.2. *Suppose that for a.e. $y \in Y$ the flow $(T^y_t)_{t\in\mathbb{R}}$ has no invariant subset of positive and finite measure. Then the flow $(T_t)_{t\in\mathbb{R}}$ enjoys the same property.*

3. $\mathbb{Z}^d$-covers of compact translation surfaces

For relevant background material concerning translation surfaces and interval exchange transformations (IETs) we refer the reader to [24], [28], [29] and [30]. Let M be a be a surface (not necessary compact) and let ω be an Abelian differential (holomorphic 1-form) on M. The pair (M, ω) is called a *translation surface*. Denote by $\Sigma \subset M$ the set of zeros of ω. For every $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ denote by $X_\theta = X^\omega_\theta$ the directional vector field in direction θ on $M \setminus \Sigma$, i.e. $\omega(X_\theta) = e^{i\theta}$ on $M \setminus \Sigma$. Then the corresponding *directional flow* $(\varphi^\theta_t)_{t\in\mathbb{R}} = (\varphi^{\omega,\theta}_t)_{t\in\mathbb{R}}$ (also known as a *translation flow*) on $M \setminus \Sigma$ preserves the area measure μ_ω ($\mu_\omega(A) = |\int_A \frac{i}{2}\omega \wedge \overline{\omega}|$).

We use the notation $(\varphi^v_t)_{t\in\mathbb{R}}$ for the *vertical flow* (corresponding to $\theta = \frac{\pi}{2}$) and $(\varphi^h_t)_{t\in\mathbb{R}}$ for the *horizontal flow* respectively ($\theta = 0$).

Assume that the surface M is compact. Suppose that $\widetilde{M}$ is a $\mathbb{Z}^d$-covering of M and $p : \widetilde{M} \to M$ is its covering map. For any holomorphic 1-form ω on M denote by $\widetilde{\omega}$ the pullback of the form ω by the map p. Then $(\widetilde{M}, \widetilde{\omega})$ is a translation surface, called a $\mathbb{Z}^d$-cover of the translation surface (M, ω).

All $\mathbb{Z}^d$-covers of M up to isomorphism are in one-to-one correspondence with $H_1(M, \mathbb{Z})^d$. For any pair ξ_1, ξ_2 in $H_1(M, \mathbb{Z})$ denote by $\langle \xi_1, \xi_2 \rangle$ the algebraic intersection number of ξ_1 with ξ_2. Then the $\mathbb{Z}^d$-cover $\widetilde{M}_\gamma$ determined by $\gamma \in H_1(M, \mathbb{Z})^d$ has the following properties: if $\sigma : [t_0, t_1] \to M$ is a close curve in M and

$$n := \langle \gamma, [\sigma] \rangle = (\langle \gamma_1, [\sigma] \rangle, \ldots, \langle \gamma_d, [\sigma] \rangle) \in \mathbb{Z}^d$$

($[\sigma] \in H_1(M, \mathbb{Z})$), then σ lifts to a path $\widetilde{\sigma} : [t_0, t_1] \to \widetilde{M}_\gamma$ such that $\sigma(t_1) = n \cdot \sigma(t_0)$, where $\cdot$ denotes the action of $\mathbb{Z}^d$ by deck transformations on $\widetilde{M}_\gamma$.

Let (M,ω) be a compact translation surface and let $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ be its $\mathbb{Z}^d$-cover. Let us consider the vertical flow $(\widetilde{\varphi}^v_t)_{t\in\mathbb{R}}$ on $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ for which the flow $(\varphi^v_t)_{t\in\mathbb{R}}$ on (M,ω) is uniquely ergodic. Let $I\subset M\setminus\Sigma$ be a horizontal interval in (M,ω) with no self-intersections. Then the Poincaré (first return) map $T:I\to I$ for the flow $(\varphi^v_t)_{t\in\mathbb{R}}$ is a uniquely ergodic interval exchange transformation (IET). Denote by $(I_\alpha)_{\alpha\in\mathcal{A}}$ the family of exchanged intervals. Let $\tau:I\to\mathbb{R}_{>0}$ be the corresponding first return time map. Then τ is constant over each interval I_α, $\alpha\in\mathcal{A}$.

For every $\alpha\in\mathcal{A}$ we denote by $\xi_\alpha=\xi_\alpha(\omega,I)\in H_1(M,\mathbb{Z})$ the homology class of any loop formed by the orbit segment of $(\varphi^v_t)_{t\in\mathbb{R}}$ starting at any $x\in\operatorname{Int}I_\alpha$ and ending at Tx together with the segment of I that joins Tx and x.

Proposition 3.1 (see Lemma 2.1 in [16] for $d=1$)**.** *Let $I\subset M\setminus\Sigma$ be a horizontal interval in (M,ω) with no self-intersections. Then for every $\gamma\in H_1(M,\mathbb{Z})^d$ the vertical flow $(\widetilde{\varphi}^v_t)_{t\in\mathbb{R}}$ on the $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ has a special representation over the skew product $T_{\psi_{\gamma,I}}:I\times\mathbb{Z}^d\to I\times\mathbb{Z}^d$ of the form $T_{\psi_{\gamma,I}}(x,m)=(Tx,m+\psi_{\gamma,I}(x))$, where $\psi_{\gamma,I}:I\to\mathbb{Z}^d$ is a piecewise constant function given by*

$$\psi_{\gamma,I}(x)=\langle\gamma,\xi_\alpha\rangle=\big(\langle\gamma_1,\xi_\alpha\rangle,\dots,\langle\gamma_d,\xi_\alpha\rangle\big)$$

if $x\in I_\alpha$ for $\alpha\in\mathcal{A}$. Moreover, the corresponding roof function $\widetilde{\tau}:I\times\mathbb{Z}^d\to\mathbb{R}_{>0}$ is given by $\widetilde{\tau}(x,m)=\tau(x)$ for $(x,m)\in I\times\mathbb{Z}^d$.

Remark 3.2. Since the roof function $\widetilde{\tau}$ is bounded and uniformly separated from zero, the absence of invariant sets of finite and positive measure for the flow $(\widetilde{\varphi}^v_t)_{t\in\mathbb{R}}$ on $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ is equivalent the absence of invariant sets of finite and positive measure for the skew product $T_{\psi_{\gamma,I}}$.

Cocycles for transformations and essential values. Given an ergodic automorphism T of a standard probability space $(X,\mathcal{B},\mu)$, a locally compact abelian second countable group G and a measurable map $\psi:X\to G$, called a *cocycle* for T, consider the skew-product extension T_ψ acting on $(X\times G,\mathcal{B}\times\mathcal{B}_G,\mu\times m_G)$ ($\mathcal{B}_G$ is the Borel σ-algebra on G) by

$$T_\psi(x,y)=(Tx,y+\psi(x)).$$

Clearly T_ψ preserves the product of μ and the Haar measure m_G on G. Moreover, for any $n\in\mathbb{Z}$ we have

$$T^n_\psi(x,y)=(T^nx,y+\psi^{(n)}(x)),$$

where

$$\psi^{(n)}(x)=\left\{\begin{array}{ll}\sum_{0\le j<n}\psi(T^jx) & \text{if} \quad n\ge 0\\ -\sum_{n\le j<0}\psi(T^jx) & \text{if} \quad n<0.\end{array}\right.$$

The cocycle $\psi:X\to G$ is called a *coboundary* for T if there exists a measurable map $h:X\to G$ such that $\psi=h-h\circ T$. Then $\psi^{(n)}=h-h\circ T^n$ for every $n\in\mathbb{Z}$.

An element $g\in G$ is said to be an *essential value* of $\psi:X\to G$, if for each open neighborhood V_g of g in G and each $B\in\mathcal{B}$ with $\mu(B)>0$, there exists $n\in\mathbb{Z}$ such that

$$\mu\big(B\cap T^{-n}B\cap\{x\in X:\psi^{(n)}(x)\in V_g\}\big)>0.$$

Proposition 3.3 (see Theorem 3.9 in [25])**.** *The set of essential values $E_G(\psi)$ is a closed subgroup of G. If ψ is a coboundary then $E_G(\psi)=\{0\}$.*

Proposition 3.4 (see Proposition 3.30 in [3])**.** *If T is an ergodic automorphism of $(X,\mathcal{B},\mu)$ then the cocycle $\psi:X\to G$ for T is a coboundary if and only if the skew product $T_\psi:X\times G\to X\times G$ has an invariant set of positive and finite measure.*

Proposition 3.5 (see Corollary 2.8 in [5])**.** *Let $\mathcal{B}$ be the σ–algebra of Borel sets of a compact metric space (X,d) and let μ be a probability measure on $\mathcal{B}$. Suppose that T is an ergodic measure–preserving automorphism of $(X,\mathcal{B},\mu)$ for which there exist a sequence of Borel sets $(C_n)_{n\geq 1}$ and an increasing sequence of natural numbers $(h_n)_{n\geq 1}$ such that*

$$\mu(C_n)\to\alpha>0,\ \mu(C_n\triangle T^{-1}C_n)\to 0\quad\text{and}\quad \sup_{x\in C_n} d(x,T^{h_n}x)\to 0.$$

If $\psi:X\to G$ is a measurable cocycle such that $\psi^{(h_n)}(x)=g_n$ for all $x\in C_n$ and $g_n\to g$, then $g\in E(\psi)$.

4. Teichmüller flow and Kontsevich-Zorich cocycle

Given a compact connected oriented surface M, denote by $\mathrm{Diff}^+(M)$ the group of orientation-preserving homeomorphisms of M. Denote by $\mathrm{Diff}^+_0(M)$ the subgroup of elements $\mathrm{Diff}^+(M)$ which are isotopic to the identity. Let $\Gamma(M):=\mathrm{Diff}^+(M)/\mathrm{Diff}^+_0(M)$ be the *mapping-class* group. We will denote by $\mathcal{T}(M)$ the *Teichmüller space of Abelian differentials*, that is the space of orbits of the natural action of $\mathrm{Diff}^+_0(M)$ on the space of all Abelian differentials on M. We will denote by $\mathcal{M}(M)$ the *moduli space of Abelian differentials*, that is the space of orbits of the natural action of $\mathrm{Diff}^+(M)$ on the space of Abelian differentials on M. Thus $\mathcal{M}(M)=\mathcal{T}(M)/\Gamma(M)$.

The group $SL(2,\mathbb{R})$ acts naturally on $\mathcal{T}(M)$ and $\mathcal{M}(M)$ as follows. Given a translation structure ω, consider charts for M given by local primitives of the holomorphic 1-form. New charts defined by the post-composition of these charts with an element of $SL(2,\mathbb{R})$ and their derivative yield a new complex structure and a new differential which is holomorphic with respect to this new complex structure, thus a new translation structure. We denote by $g\cdot\omega$ the translation structure on M obtained acting by $g\in SL(2,\mathbb{R})$ on a translation structure ω on M. The *Teichmüller flow* $(g_t)_{t\in\mathbb{R}}$ is the restriction of this action to the diagonal subgroup $(\mathrm{diag}(e^t,e^{-t}))_{t\in\mathbb{R}}$ of $SL(2,\mathbb{R})$ on $\mathcal{T}(M)$ and $\mathcal{M}(M)$. We will deal also with the rotations $(r_\theta)_{\theta\in S^1}$ that acts on $\mathcal{T}(M)$ and $\mathcal{M}(M)$ by $r_\theta\omega=e^{i\theta}\omega$. Then the flow $(\varphi^\theta_t)_{t\in\mathbb{R}}$ on (M,ω) coincides with the vertical flow on $(M,r_{\pi/2-\theta}\omega)$. Moreover, for any $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ the directional flow $(\widetilde{\varphi}^\theta_t)_{t\in\mathbb{R}}$ on $\underbrace{(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)}$ in the direction $\theta\in S^1$ coincides with the vertical flow $(\widetilde{\varphi}^v_t)_{t\in\mathbb{R}}$ on $(\widetilde{M}_\gamma,\widetilde{(r_{\pi/2-\theta}\omega)}_\gamma)$.

Kontsevich-Zorich cocycle. The *Kontsevich-Zorich (KZ) cocycle* $(A_g)_{g\in SL(2,\mathbb{R})}$ is the quotient of the product action $(g\times\mathrm{Id})_{g\in SL(2,\mathbb{R})}$ on $\mathcal{T}(M)\times H_1(M,\mathbb{R})$ by the action of the mapping-class group $\Gamma(M)$. The mapping class group acts on the fiber $H_1(M,\mathbb{R})$ by induced maps. The cocycle $(A_g)_{g\in SL(2,\mathbb{R})}$ acts on the homology vector bundle

$$\mathcal{H}_1(M,\mathbb{R})=(\mathcal{T}(M)\times H_1(M,\mathbb{R}))/\Gamma(M)$$

over the $SL(2,\mathbb{R})$-action on the moduli space $\mathcal{M}(M)$.

Clearly the fibers of the bundle $\mathcal{H}_1(M,\mathbb{R})$ can be identified with $H_1(M,\mathbb{R})$. The space $H_1(M,\mathbb{R})$ is endowed with the symplectic form given by the algebraic intersection number. This symplectic structure is preserved by the action of the mapping-class group and hence it is invariant under the action of $(A_g)_{g\in SL(2,\mathbb{R})}$.

The standard definition of KZ-cocycle bases on cohomological bundle. A correspondence between the homological and cohomological settings is established by the Poincaré duality $\mathcal{P}:H_1(M,\mathbb{R})\to H^1(M,\mathbb{R})$. This correspondence allow us to define so called Hodge norm (see [13] for the cohomological bundle) on each fiber of the bundle $\mathcal{H}_1(M,\mathbb{R})$. The Hodge norm on the fiber $H_1(M,\mathbb{R})$ over $\omega\in\mathcal{M}(M)$ will be denoted by $\|\cdot\|_\omega$.

Generic directions. Let $\omega \in \mathcal{M}(M)$ and denote by $\mathcal{M} = \overline{SL(2,\mathbb{R})\omega}$ the closure of the $SL(2,\mathbb{R})$-orbit of ω in $\mathcal{M}(M)$. The celebrated result of Eskin, Mirzakhani and Mohammadi, proved in [12] and [11], says that $\mathcal{M} \subset \mathcal{M}(M)$ is an affine $SL(2,\mathbb{R})$-invariant submanifold. Denote by $\nu_{\mathcal{M}}$ the corresponding affine $SL(2,\mathbb{R})$-invariant probability measure supported on $\mathcal{M}$. The measure $\nu_{\mathcal{M}}$ is ergodic under the action of the Teichmüller flow.

Theorem 4.1 (see Theorem 1.1 in [4])**.** *For every $\phi \in C_c(\mathcal{M})$ and a.e. $\theta \in S^1$ we have*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \phi(g_t r_\theta \omega)\, dt = \int_{\mathcal{M}} \phi\, d\nu_{\mathcal{M}}. \tag{4.1}$$

Theorem 4.2 (see Theorem 2 in [19])**.** *For a.e. direction $\theta \in S^1$ the directional flows $(\varphi_t^v)_{t\in\mathbb{R}}$ and $(\varphi_t^h)_{t\in\mathbb{R}}$ on $(M, r_\theta\omega)$ are uniquely ergodic.*

All directions $\theta \in S^1$ for which the assertion of Theorems 4.1 and 4.2 hold are called *Birkhoff-Masur generic* for the translation surface (M,ω).

5. Directional flows on $\mathbb{Z}^d$-covers and weak mixing of their Poisson suspensions

Suppose that the direction $0 \in S^1$ is Birkhoff-Masur generic for (M,ω). Then the vertical and horizontal flows on (M,ω) are uniquely ergodic. Let $I \subset M \setminus \Sigma$ (Σ is the set of zeros of ω) be a horizontal interval. Then the interval I has no self-intersections and the Poincaré return map $T : I \to I$ for the flow $(\varphi_t^v)_{t\in\mathbb{R}}$ is a uniquely ergodic IET. Denote by I_α, $\alpha \in \mathcal{A}$ the intervals exchanged by T. Let $\lambda_\alpha(\omega, I)$ stands for the length of the interval I_α.

Denote by $\tau : I \to \mathbb{R}_{>0}$ the map of the first return time to I for the flow $(\varphi_t^v)_{t\in\mathbb{R}}$. Then τ is constant on each I_α and denote by $\tau_\alpha = \tau_\alpha(\omega, I) > 0$ its value on I_α, $\alpha \in \mathcal{A}$. Let us denote by $\delta(\omega, I) > 0$ the maximal number $\Delta > 0$ for which the set $\mathcal{R}^\omega(I,\Delta) := \{\varphi_t^v x : t \in [0,\Delta), x \in I\}$ is a rectangle in (M,ω) without any singular point (from Σ).

Suppose that $J \subset I$ is a subinterval. Denote by $S : J \to J$ the Poincaré return map to J for the flow $(\varphi_t^v)_{t\in\mathbb{R}}$. Then S is also an IET and suppose it exchanges intervals $(J_\alpha)_{\alpha\in\mathcal{A}}$. The IET S is the induced transformation of T on J. Moreover, all elements of J_α have the same time of the first return to J for the transformation T and let us denote this return time by $h_\alpha \geq 0$ for $\alpha \in \mathcal{A}$. Then I is the union of disjoint towers $\{T^j J_\alpha : 0 \leq j < h_\alpha\}$, $\alpha \in \mathcal{A}$, i.e. the sets $T^j J_\alpha$, for $\alpha \in \mathcal{A}$ and $0 \leq j < h_\alpha$, are pairwise disjoint intervals.

The following result follows directly from Lemmas 4.12 and 4.13 in [15].

Lemma 5.1. *Assume that for some $\Delta > 0$ the set $\mathcal{R}^\omega(J,\Delta)$ is a rectangle in (M,ω) without any singular point. Let $h = \left[\Delta/\max_{\alpha\in\mathcal{A}}\tau_\alpha\right]$. Then for every $\gamma \in H_1(M,\mathbb{Z})$ and $\alpha \in \mathcal{A}$ we have*

$$\psi_{\gamma,I}^{(h_\alpha)}(x) = \langle \gamma, \xi_\alpha(\omega, J)\rangle \text{ and } |T^{h_\alpha}x - x| \leq |J| \quad \text{for} \quad x \in C_\alpha := \bigcup_{0\leq j\leq h} T^j J_\alpha. \tag{5.1}$$

The following result follows directly from Lemmas A.3 and A.4 in [14].

Lemma 5.2. *If $0 \in S^1$ is Birkhoff-Masur generic for (M,ω) then there exist positive constants $A, C, c > 0$, a sequence of nested horizontal intervals $(I_k)_{k\geq 0}$ in (M,ω) and an increasing to infinity sequence of real numbers $(t_k)_{k\geq 0}$ with $t_0 = 0$ such that for every $k \geq 0$ we have*

$$\frac{1}{c}\|\xi\|_{g_{t_k}\omega} \leq \max_\alpha |\langle \xi_\alpha(g_{t_k}\omega, I_k), \xi\rangle| \leq c\|\xi\|_{g_{t_k}\omega} \quad \text{for every} \quad \xi \in H_1(M,\mathbb{R}), \tag{5.2}$$

$$\lambda_\alpha(g_{t_k}\omega, I_k)\,\delta(g_{t_k}\omega, I_k) \geq A \ \textit{and}\ \frac{1}{C} \leq \tau_\alpha(g_{t_k}\omega, I_k) \leq C \ \textit{for any}\ \alpha \in \mathcal{A}. \tag{5.3}$$

Lemma 5.3. *If $0 \in S^1$ is Birkhoff-Masur generic for (M,ω) then for every non-zero $\gamma \in H_1(M,\mathbb{Z})$ the cocycle $\psi_{\gamma,I} : I \to \mathbb{Z}$ (the interval $I := I_0$ comes from Lemma 5.2) is not a coboundary.*

Proof. By Lemma 5.2, there exist a sequence of nested horizontal intervals $(I_k)_{k\geq 0}$ in (M,ω) and an increasing to infinity sequence of real numbers $(t_k)_{k\geq 0}$ such that (5.2) and (5.3) hold for $k \geq 0$ and $t_0 = 0$. Let $I := I_0$ and denote by $T : I \to I$ the Poincaré return map to I for the vertical flow $(\varphi^v_t)_{t\in\mathbb{R}}$. Suppose, contrary to our claim, that $\psi_{\gamma,I} : I \to \mathbb{Z}$ is a coboundary with a measurable transfer function $u : I \to \mathbb{R}$, i.e. $\psi_{\gamma,I} = u - u \circ T$.

For every $k \geq 1$ the Poincaré return map $T_k : I_k \to I_k$ to I_k for the vertical flow $(\varphi^v_t)_{t\in\mathbb{R}}$ on (M,ω) is an IET exchanging intervals $(I_k)_\alpha$, $\alpha \in \mathcal{A}$. The length of $(I_k)_\alpha$ in (M,ω) is equal to $\lambda_\alpha(\omega, I_k) = e^{-t_k}\lambda_\alpha(g_{t_k}\omega, I_k)$ for $\alpha \in \mathcal{A}$. In view of (5.3), the length of I_k in (M,ω) is

$$|I_k| = \sum_{\alpha\in\mathcal{A}} e^{-t_k}\lambda_\alpha(g_{t_k}\omega, I_k) \leq Ce^{-t_k}\sum_{\alpha\in\mathcal{A}}\lambda_\alpha(g_{t_k}\omega, I_k)\tau_\alpha(g_{t_k}\omega, I_k) = Ce^{-t_k}\mu_\omega(M).$$

By the definition of δ, the set $\mathcal{R}^\omega(I_k, e^{t_k}\delta(g_{t_k}\omega, I_k)) = \mathcal{R}^{g_{t_k}\omega}(I_k, \delta(g_{t_k}\omega, I_k))$ is a vertical rectangle in $(M, g_{t_k}\omega)$ without any singular point. It follows that the set $\mathcal{R}^\omega(I_k, e^{t_k}\delta(g_{t_k}\omega, I_k))$ is a rectangle in (M,ω) without any singular point.

Denote by $h^k_\alpha \geq 0$ the first return time of the interval $(I_k)_\alpha$ to I_k for the IET T. Let

$$h_k := \left[e^{t_k}\delta(g_{t_k}\omega, I_k)/\max_{\alpha\in\mathcal{A}}\tau_\alpha(\omega, I)\right] \text{ and } C^k_\alpha := \bigcup_{0\leq j\leq h_k} T^j(I_k)_\alpha.$$

Now Lemma 5.1 applied to $J = I_k$ and $\Delta = e^{t_k}\delta(g_{t_k}\omega, I_k)$ gives

$$\psi^{(h^k_\alpha)}_{\gamma,I}(x) = \langle\gamma, \xi_\alpha(\omega, I_k)\rangle \text{ and } |T^{h^k_\alpha}x - x| \leq |I_k| \leq Ce^{-t_k}\mu_\omega(M) \ \text{ for }\ x \in C^k_\alpha \tag{5.4}$$

for every $k \geq 1$ and $\alpha \in \mathcal{A}$. Moreover, by (5.3),

$$Leb(C^k_\alpha) = (h_k{+}1)|(I_k)_\alpha| \geq \frac{e^{t_k}\delta(g_{t_k}\omega, I_k)}{\max_{\alpha\in\mathcal{A}}\tau_\alpha}e^{-t_k}\lambda_\alpha(g_{t_k}\omega, I_k) \geq \frac{A}{\max_{\alpha\in\mathcal{A}}\tau_\alpha} =: a > 0.$$

By assumption, in view of (5.2), we have

$$\|\gamma\|_{g_{t_k}\omega} \leq c\max_{\alpha\in\mathcal{A}}|\langle\gamma, \xi_\alpha(g_{t_k}\omega, I_k)\rangle|.$$

Choose $B > 0$ such that $Leb(U_B) < a/2$ for $U_B = \{x \in I : |u(x)| > B\}$. For every $m \geq 1$ let $J_m := I \setminus (U_B \cup T^{-m}U_B)$. Then $Leb(I \setminus J_m) < a$ and for every $x \in J_m$ we have both $|u(x)| \leq B$, $|u(T^m x)| \leq B$. As $Leb(I \setminus J_{h^k_\alpha}) < a$ and $Leb(C^k_\alpha) \geq a$, there exists $x^k_\alpha \in C^k_\alpha \cap J_{h^k_\alpha}$. Therefore, by (5.4), for all $k \geq 1$ and $\alpha \in \mathcal{A}$ we have

$$|\langle\gamma, \xi_\alpha(\omega, I_k)\rangle| = |\psi^{(h^k_\alpha)}_{\gamma,I}(x^k_\alpha)| = |u(x^k_\alpha) - u(T^{h^k_\alpha}x^k_\alpha)| \leq |u(x^k_\alpha)| + |u(T^{h^k_\alpha}x^k_\alpha)| \leq 2B.$$

Since $\langle\gamma, \xi_\alpha(\omega, I_k)\rangle\rangle \in \mathbb{Z}$, passing to a subsequence, if necessary, we can assume that for every $\alpha \in \mathcal{A}$ the sequence $(\langle\gamma, \xi_\alpha(\omega, I_k)\rangle)_{k\geq 1}$ is constant. Since (5.4) holds and $Leb(C^k_\alpha) \geq a > 0$ for $k \geq 1$ and $\alpha \in \mathcal{A}$, we can apply Proposition 3.5 to $\psi = \psi_{\gamma,I}$, $C_k = C^k_\alpha$ and $h_k = h^k_\alpha$. This gives $\langle\gamma, \xi_\alpha(\omega, I_k)\rangle \in E(\psi_{\gamma,I})$ for all $k \geq 1$ and $\alpha \in \mathcal{A}$. In view of Proposition 3.3, as $\psi_{\gamma,I}$ is a coboundary, we have $E(\psi_{\gamma,I}) = \{0\}$, so $\langle\gamma, \xi_\alpha(\omega, I_k)\rangle = 0$ for all $k \geq 1$ and $\alpha \in \mathcal{A}$. Since $\langle\gamma, \xi_\alpha(g_{t_k}\omega, I_k)\rangle = \langle\gamma, \xi_\alpha(\omega, I_k)\rangle$, (5.2) gives

$$\|\gamma\|_{g_{t_k}\omega} \leq c\max_{\alpha\in\mathcal{A}}|\langle\gamma, \xi_\alpha(g_{t_k}\omega, I_k)\rangle| = 0.$$

It follows that $\gamma = 0$, contrary to $\gamma \neq 0$. Consequently, the cocycle $\psi_{\gamma,I}$ is not a coboundary for the IET $T : I \to I$. □

Theorem 5.4. *Let (M,ω) be a compact connected translation surface and let $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ be its non-trivial $\mathbb{Z}^d$-cover (i.e. $\gamma\in H_1(M,\mathbb{Z})^d$ is non-zero). Then for a.e. $\theta\in S^1$ the Poisson suspension of the directional flow $(\widetilde{\varphi}^\theta_t)_{t\in\mathbb{R}}$ flow on $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ is weakly mixing.*

Proof. By Theorems 4.1 and 4.2, the set $\Theta\subset S^1$ of all $\theta\in S^1$ for which $\pi/2-\theta$ is Birkhoff-Masur generic for (M,ω) has full Lebesgue measure in S^1. We will show that for every $\theta\in\Theta$ the directional flow $(\widetilde{\varphi}^\theta_t)_{t\in\mathbb{R}}$ flow on $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ has no invariant set of positive and finite measure. In view of Proposition 2.1, this gives weak mixing of the corresponding Poisson suspension.

Suppose that $\theta\in\Theta$. Then $0\in S^1$ is a Birkhoff-Masur generic direction for $(M,r_{\pi/2-\theta}\omega)$ and the flow $(\widetilde{\varphi}^\theta_t)_{t\in\mathbb{R}}$ on $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ coincides with the vertical flow $(\widetilde{\varphi}^v_t)_{t\in\mathbb{R}}$ on $(\widetilde{M}_\gamma,\widetilde{(r_{\pi/2-\theta}\omega)}_\gamma)$.

Assume that $\gamma=(\gamma_1,\ldots,\gamma_d)$ and $\gamma_j\in H_1(M,\mathbb{Z})$ is non-zero for some $1\leq j\leq d$. By Lemmas 5.2 and 5.3, there exists a horizontal interval in $(M,r_{\pi/2-\theta}\omega)$ such that $\psi_{\gamma_j,I}:I\to\mathbb{Z}$ is not a coboundary for the Poincaré return map $T:I\to I$ for the vertical flow on $(M,r_{\pi/2-\theta}\omega)$. Since $\psi_{\gamma_j,I}$ is the j-th coordinate function of $\psi_{\gamma,I}:I\to\mathbb{Z}^d$, the latter is also not a coboundary for T. In view of Proposition 3.4, the skew product $T_{\psi_{\gamma,I}}$ on $I\times\mathbb{Z}^d$ has no invariant set of positive and finite measure. By Proposition 3.1 and Remark 3.2, the vertical flow on $(\widetilde{M}_\gamma,\widetilde{(r_{\pi/2-\theta}\omega)}_\gamma)$ has no invariant set of positive and finite measure as well. As the vertical flow $(\widetilde{\varphi}^v_t)_{t\in\mathbb{R}}$ on $(\widetilde{M}_\gamma,\widetilde{(r_{\pi/2-\theta}\omega)}_\gamma)$ coincides with the directional flow $(\widetilde{\varphi}^\theta_t)_{t\in\mathbb{R}}$ on $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$, this completes the proof. □

Proof of Theorem 1.1. The first part follows directly from Theorem 5.4 applied to the $\mathbb{Z}^2$-cover $(M_\mathcal{T},\omega_\mathcal{T})$. Non-triviality of the $\mathbb{Z}^2$-cover follows from the connectivity of $M_\mathcal{T}$.

The second part is based on the fact that the billiard flow $(b_t)_{t\in\mathbb{R}}$ of $\mathcal{T}^1$ is metrically isomorphic to the flow $(\varphi^\mathcal{T}_t)_{t\in\mathbb{R}}$ on $M_\mathcal{T}\times S^1/\Gamma$ given by $\varphi^\mathcal{T}_t(x,\theta)\mapsto(\varphi^{\mathcal{T},\theta}_t x,\theta)$. By Theorem 5.4, for a.e. $\theta\in S^1/\Gamma$ the flow $(\varphi^{\mathcal{T},\theta}_t)_{t\in\mathbb{R}}$ has no invariant subset of positive and finite measure. In view Lemma 2.2, the flow $(\varphi^\mathcal{T}_t)_{t\in\mathbb{R}}$ enjoys the same property. The proof is completed by applying Proposition 2.1. □

6. Absence of mixing

Let (M,ω) be a compact connected translation surface and let $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ be its $\mathbb{Z}^d$-cover determined by $\gamma\in H_1(M,\mathbb{Z})^d$. Denote by $p_\gamma:\widetilde{M}_\gamma\to M$ the covering map. Let d^ω_γ be the geodesic distance on $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$. Of course, $d^\omega_\gamma=d^{r_\theta\omega}_\gamma$ for every $\theta\in S^1$. Denote by $(\widetilde{\varphi}^v_t)_{t\in\mathbb{R}}$ the vertical flow on $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$.

Definition (cf. Definition 1 in [2]). Given real numbers $c,L,\delta>0$, the $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ is called *(c,L,δ)-recurrent* if there exists a horizontal interval $I\subset M\setminus\Sigma$ such that

- the set $\mathcal{R}^\omega(I,L)=\{\varphi^v_t x:x\in I,t\in[0,L)\}$ is a vertical rectangle (without singularities and overlaps) in (M,ω);
- $\mu_\omega(\mathcal{R}^\omega(I,L))\geq c$;
- for every $\widetilde{x}\in p_\gamma^{-1}(\mathcal{R}^\omega(I,L))$ the points $\widetilde{x}$ and $\widetilde{\varphi}^v_L\widetilde{x}$ belong to the same horizontal leaf on $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ and the distance between them along this leaf is smaller than δ.

Let $\mathcal{M}=\overline{SL(2,\mathbb{R})\omega}$ and let us consider the bundle $\mathcal{H}_1^{\mathcal{M}}(M,\mathbb{R})\to\mathcal{M}$ which is the restriction of the homological bundle to $\mathcal{M}$. Assume that

$$\mathcal{H}_1^{\mathcal{M}}(M,\mathbb{R})=\mathcal{K}\oplus\mathcal{K}^{\perp} \tag{6.1}$$

is a continuous symplectic orthogonal splitting of the bundle which is $(A_g)_{g\in SL(2,\mathbb{R})}$-invariant. Denote by $H_1(M,\mathbb{R})=K_{\omega'}\oplus K_{\omega'}^{\perp}$ the corresponding splitting of the fiber over any $\omega'\in\mathcal{M}$.

A cylinder C on (M,ω) is a maximal open annulus filled by homotopic simple closed geodesics. The direction of C is the direction of these geodesics and the homology class of them is denoted by $\sigma(C)\in H_1(M,\mathbb{Z})$. A cylinder C on $(M,\omega')\in\mathcal{M}$ is called $\mathcal{K}$-*good* if $\sigma(C)\in K_{\omega'}^{\perp}\cap H_1(M,\mathbb{Z})$. If a cylinder C on (M,ω) is $\mathcal{K}$-good and $\gamma\in(K_\omega\cap H_1(M,\mathbb{Z}))^d$ then C lifts to a cylinder on the $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$.

Proposition 6.1 (see the proof of Proposition 2 in [2])**.** *Suppose that $(M,\omega_*)\in\mathcal{M}$ has a vertical $\mathcal{K}$-good cylinder. If the positive $(g_t)_{t\in\mathbb{R}}$ orbit of (M,ω) accumulates on (M,ω_*) then for any $\gamma\in(K_\omega\cap H_1(M,\mathbb{Z}))^d$ there exists $c>0$ and two sequences of positive numbers $(L_n)_{n\geq1}$, $(\delta_n)_{n\geq1}$ such that $L_n\to+\infty$, $\delta_n\to0$ and the $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ is (c,L_n,δ_n)-recurrent for $n\geq1$.*

For every $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ let $D_\gamma^\omega\subset\widetilde{M}_\gamma$ be a fundamental domain for the deck group action so that the boundary of D_γ^ω is a finite union of intervals. Then, $\mu_{\widetilde{\omega}_\gamma}(D_\gamma^\omega)=\mu_\omega(M)\in(0,+\infty)$.

Theorem 6.2. *Suppose that (M,ω) has a $\mathcal{K}$-good cylinder C. If $\pi/2-\theta\in S^1$ is a Birkhoff generic direction then for every $\gamma\in(K_\omega\cap H_1(M,\mathbb{Z}))^d$ we have*

$$\liminf_{t\to+\infty}\mu_{\widetilde{\omega}_\gamma}(D_\gamma^\omega\cap\widetilde{\varphi}_t^\theta D_\gamma^\omega)>0.$$

Proof. Denote by $\theta_0\in S^1$ the direction of the cylinder C on (M,ω). Since the splitting (6.1) is $(A_g)_{g\in SL(2,\mathbb{R})}$-invariant, C is a vertical $\mathcal{K}$-good cylinder on the translation surface $(M,r_{\pi/2-\theta_0}\omega)\in\mathcal{M}$. Since $\pi/2-\theta\in S^1$ is Birkhoff generic, applying (4.1) to a sequence $(\phi_k)_{k\geq1}$ in $C_c(\mathcal{M})$ such that $(\mathrm{supp}(\phi_k))_{k\geq1}$ is a decreasing nested sequence of non-empty compact subsets with the intersection $\{r_{\pi/2-\theta_0}\omega\}$, there exists $t_n\to+\infty$ such that $g_{t_n}(r_{\pi/2-\theta}\omega)\to r_{\pi/2-\theta_0}\omega$. By Proposition 6.1, there exists $c>0$ and two sequences of positive numbers $(L_n)_{n\geq1}$, $(\delta_n)_{n\geq1}$ such that $L_n\to+\infty$, $\delta_n\to0$ and the $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma,\widetilde{r_{\pi/2-\theta}\omega_\gamma})$ is (c,L_n,δ_n)-recurrent for $n\geq1$. Let us denote by $(\widetilde{\varphi}_t^v)_{t\in\mathbb{R}}$ the vertical flow on $(\widetilde{M}_\gamma,\widetilde{r_{\pi/2-\theta}\omega_\gamma})$ which coincides with the flow $(\widetilde{\varphi}_t^\theta)_{t\in\mathbb{R}}$ in direction $\theta\in S^1$ on $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$. Then there exists a sequence $(I_n)_{n\geq1}$ of horizontal intervals in $(M,r_{\pi/2-\theta}\omega)$ such that $\mathcal{R}^{r_{\pi/2-\theta}\omega}(I_n,L_n)$ is a rectangle in $(M,r_{\pi/2-\theta}\omega)$ such that $\mu_\omega(\mathcal{R}^{r_{\pi/2-\theta}\omega}(I_n,L_n))=\mu_{r_{\pi/2-\theta}\omega}(\mathcal{R}^{r_{\pi/2-\theta}}(I_n,L_n))>c$ and

$$\text{for every } \widetilde{x}\in p_\gamma^{-1}(\mathcal{R}^{r_{\pi/2-\theta}\omega}(I_n,L_n)) \text{ we have } d_\gamma^\omega(\widetilde{x},\widetilde{\varphi}_{L_n}^v\widetilde{x})=d_\gamma^{r_{\pi/2-\theta}\omega}(\widetilde{x},\widetilde{\varphi}_{L_n}^v\widetilde{x})<\delta_n. \tag{6.2}$$

As $D_\gamma^\omega\subset\widetilde{M}_\gamma$ is a fundamental domain for the $\mathbb{Z}^d$-action of the deck group, we have

$$\mu_{\widetilde{\omega}_\gamma}(D_\gamma^\omega\cap p_\gamma^{-1}(\mathcal{R}^{r_{\pi/2-\theta}\omega}(I_n,L_n)))=\mu_\omega(\mathcal{R}^{r_{\pi/2-\theta}}(I_n,L_n))>c. \tag{6.3}$$

For every $\delta>0$ denote by $\partial_\delta D_\gamma^\omega$ the δ-neighborhood in $(\widetilde{M}_\gamma,d_\gamma^\omega)$ of the boundary ∂D_γ^ω. Since $\mu_{\widetilde{\omega}_\gamma}(\partial D_\gamma^\omega)=0$, we have

$$\mu_{\widetilde{\omega}_\gamma}(\partial_\delta D_\gamma^\omega)\to0 \quad\text{as}\quad \delta\to0. \tag{6.4}$$

In view of (6.2), we obtain

$$\widetilde{\varphi}_{L_n}^v\big((D_\gamma^\omega\cap p_\gamma^{-1}(\mathcal{R}^{r_{\pi/2-\theta}\omega}(I_n,L_n)))\setminus\partial_{\delta_n}D_\gamma^\omega\big)\subset D_\gamma^\omega.$$

It follows that

$$\begin{aligned}\mu_{\widetilde{\omega}_\gamma}(D^\omega_\gamma \cap \widetilde{\varphi}^\theta_{L_n} D^\omega_\gamma) &= \mu_{\widetilde{\omega}_\gamma}(D^\omega_\gamma \cap \widetilde{\varphi}^v_{L_n} D^\omega_\gamma)\\ &\geq \mu_{\widetilde{\omega}_\gamma}\big(\widetilde{\varphi}^v_{L_n}\big(\big(D^\omega_\gamma \cap p_\gamma^{-1}(\mathcal{R}^{r_{\pi/2-\theta}\omega}(I_n, L_n))\big) \setminus \partial_{\delta_n} D^\omega_\gamma\big)\big)\\ &\geq \mu_{\widetilde{\omega}_\gamma}\big(D^\omega_\gamma \cap p_\gamma^{-1}(\mathcal{R}^{r_{\pi/2-\theta}\omega}(I_n, L_n))\big) - \mu_{\widetilde{\omega}_\gamma}(\partial_{\delta_n} D^\omega_\gamma).\end{aligned}$$

By (6.3) and (6.4), this gives $\liminf_{n\to+\infty} \mu_{\widetilde{\omega}_\gamma}(D^\omega_\gamma \cap \widetilde{\varphi}^\theta_{L_n} D^\omega_\gamma) \geq c > 0$, which completes the proof. □

In view of Proposition 2.1 and Theorem 4.1, this leads to the following result:

Theorem 6.3. *Suppose that (M,ω) is a compact connected translation surface with a $\mathcal{K}$-good cylinder. Then for every $\gamma \in (K_\omega \cap H_1(M,\mathbb{Z}))^d$ and for a.e. $\theta \in S^1$ the Poisson suspension of the directional flow $(\widetilde{\varphi}^\theta_t)_{t\in\mathbb{R}}$ on the $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ is not mixing.*

Remark 6.4. The notion of $\mathcal{K}$-good cylinder was introduced in [2] and applied to prove recurrence for a.e. directional billiard flow in the standard periodic wind tree model. The existence of $\mathcal{K}$-good cylinders was also shown in more complicated billiards on periodic tables in [14] and [26]. The paper [26] deals with $\mathbb{Z}^2$-periodic patterns of polygonal scatterers with horizontal and vertical sides, moreover the obstacles are horizontally and vertically symmetric. Some Λ-periodic patterns of scatterers with horizontal and vertical sides are considered in [14] for any lattice $\Lambda \subset \mathbb{R}^2$; here obstacles are centrally symmetric. Among others, the existence of $\mathcal{K}$-good cylinders was shown for Λ_λ-periodic wind tree model (obstacles are rectangles), where Λ_λ is any lattice of the form $(1,\lambda)\mathbb{Z} + (0,1)\mathbb{Z}$. In view of Theorem 6.3, we have the absence of mixing for the Poisson suspension of the directional billiard flows $(b^\theta_t)_{t\in\mathbb{R}}$ for a.e. $\theta \in S^1$ on all billiards tables considered in [2, 14, 26].

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Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland
Email address: `fraczek@mat.umk.pl`