On a function that realizes the maximal spectral type

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Abstract

We show that for a unitary operator U on $L^2(X,\mu)$, where X is a compact manifold of class C^r, $r \in \mathbb{N} \cup \{\infty, \omega\}$ and μ is a finite Borel measure on X, there exists a C^r function that realizes the maximal spectral type of U.

Introduction

Let U be a unitary operator on a separable Hilbert space H. For any $f \in H$ we define the *cyclic space* generated by f as $\mathbb{Z}(f) = span\{U^n f : n \in \mathbb{Z}\}$. By the *spectral measure* σ_f of f we mean a Borel measure on the circle $\mathbf{S}^1$ determined by the equalities

$$\hat{\sigma}_f(n) = \int_{S^1} z^n d\sigma_f(z) = (U^n f, f)$$

for every $n \in \mathbb{Z}$.

Theorem 0.1 (spectral theorem). (see [5]) *There exists in H a sequence $f_1, f_2, ...$ such that*

$$H = \bigoplus_{n=1}^{\infty} \mathbb{Z}(f_n) \quad \textit{and} \quad \sigma_{f_1} \gg \sigma_{f_2} \gg \dots \; . \tag{1}$$

Moreover, for any sequence $g_1, g_2, ...$ in H satisfying (1) we have $\sigma_{f_1} \equiv \sigma_{g_1}, \sigma_{f_2} \equiv \sigma_{g_2}, \dots$.

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The spectral type of σ_{f_1} (the equivalence class of measures) is called the *maximal spectral type* of U.

In this paper we will try to answer, what is the "best" function realizing the maximal spectral type of U in the case, where $H = L^2(X, \mu)$. The meaning of to be the best is not rigorously defined; if no other than measurable structure is imposed on X we will seek a function from $L^\infty(X, \mu)$. However, if X admits a structure of a compact manifold we will look for a function that is sufficiently smooth (and this is not connected to possible smooth properties of U or an invariant measure μ).

In case of U acting on $L^2(X, \mu)$, Alexeyev in [1] proved the existence of a function $f \in L^\infty(X, \mu)$ that realizes the maximal spectral type. In this note we will show that Alexeyev's arguments are in fact of pure spectral nature and this will let us generalize his result in various directions.
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1 What is the "best" function realize the maximal spectral type

The theorem below is a general version the classical Alexeyev theorem ([1]).

Theorem 1.1. *Let U be a unitary operator on a separable Hilbert space H. Let F be a linear subspace of H and let $E \subset F$ be a dense linear subspace of H such that for any sequence $\{g_k\}_{k=1}^\infty$ in E there exist a strictly increasing sequence $\{r_k\}_{k=1}^\infty$ of natural numbers and a real number $0 < a \leq 1$ such that for every complex u, $|u| < a$ the series*

$$\sum_{k=1}^\infty u^{r_k} g_k$$

(converges in H) belongs to F. Then for every $f \in H$ and every $\varepsilon > 0$ there exists $g \in F$ such that $\| f - g \|_H < \varepsilon$ and $\sigma_f \ll \sigma_g$. In particular, there exists $g \in F$ that realizes the maximal spectral type of U.

Proof. Let $\sigma_1 \gg \sigma_2 \gg ...$ be a spectral sequence of U. Hence the unitary operator $U' : \bigoplus_{n=1}^\infty L^2(\mathbf{S}^1, \sigma_n) \to \bigoplus_{n=1}^\infty L^2(\mathbf{S}^1, \sigma_n)$ given by

$$U'(\sum_{n=1}^\infty \xi_n(z_n)) = \sum_{n=1}^\infty z_n \xi_n(z_n) \quad \text{for} \quad \sum_{n=1}^\infty \xi_n(z_n) \in \bigoplus_{n=1}^\infty L^2(\mathbf{S}^1, \sigma_n)$$

and U are unitarily equivalent. Let $V : H \to \bigoplus_{n=1}^{\infty} L^2(\mathbf{S}^1, \sigma_n)$ be a unitary isomorphism of U and U'. For $f \in H$ let $\{\tilde{f}^n\}_{n=1}^{\infty}$ denote a sequence given by

$$\tilde{f}^n = P_{L^2(S^1,\sigma_n)} \circ Vf$$

where $P_{L^2(S^1,\sigma_n)} : \bigoplus_{n=1}^{\infty} L^2(\mathbf{S}^1, \sigma_n) \to L^2(\mathbf{S}^1, \sigma_n)$ is the natural projection.

Let $f \in H$ and $\varepsilon > 0$. Then there exists a sequence $\{g_m\}_{m=0}^{\infty}$ in E such that

$$\| f - g_m \|_H < \frac{\varepsilon}{2^{m+2}} \ \text{ for } \ m \geq 0.$$

Since $\lim_{m\to\infty} \| f - g_m \|_H = 0$ and the operator $P_{L^2(S^1,\sigma_n)}$ is bounded,

$$\lim_{m\to\infty} \int_{S^1} |\tilde{f}^n - \tilde{g}^n_m|^2 d\sigma_n = 0 \ \text{ for } \ n \geq 1.$$

Since convergence in the norm L^2 implies the almost everywhere convergence for a subsequence we can construct by a diagonal process a sequence $\{m_k\}_{k=0}^{\infty}$, $m_0 = 1$ and $m_k \to \infty$ and sets $N_n \subset \mathbf{S}^1$ such that

$$\lim_{k\to\infty} \tilde{g}^n_{m_k}(z) = \tilde{f}^n(z) \ \text{ for } \ z \in N_n, \ \sigma_n(\mathbf{S}^1 \setminus N_n) = 0 \tag{2}$$

Let $\{r_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of natural numbers and let a be a real positive number such that for any complex u, $|u| < a$ we have

$$\sum_{k=1}^{\infty} u^{r_k}(g_{m_k} - g_{m_{k-1}}) \in F \tag{3}$$

where the series converges in H.
Denote $\mathcal{D} = \{\sqcap \in \mathbb{C} : |\sqcap| < \infty\}$. Given $u \in \overline{\mathcal{D}}$ put

$$g(u) = g_0 + \sum_{k=1}^{\infty} u^{r_k}(g_{m_k} - g_{m_{k-1}}). \tag{4}$$

Since

$$\sum_{k=1}^{\infty} \| g_{m_k} - g_{m_{k-1}} \|_H < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{m_{k-1}+1}} \leq \frac{\varepsilon}{2},$$

for $u \in \overline{\mathcal{D}}$ the series (4) converges absolutely in H and $g(u) \in H$. From

$$\| g(u) - g_0 \|_H \leq \sum_{k=1}^{\infty} \| g_{m_k} - g_{m_{k-1}} \|_H < \frac{\varepsilon}{2}$$

for $u \in \overline{\mathcal{D}}$ we obtain $\| g(u) - f \|_H < \varepsilon$.
It follows from (4) that for $n \geq 1$

$$\tilde{g}^n(u,z) = \tilde{g}^n_0(z) + \sum_{k=1}^{\infty} u^{r_k}(\tilde{g}^n_{m_k}(z) - \tilde{g}^n_{m_{k-1}}(z)). \tag{5}$$

We deduce from (2) that for $z \in N_n$ and $u = 1$ the series (5) converges in the usual sense. Hence $\tilde{g}^n(\cdot, z)$ is an analytic function in $\mathcal{D}$ and

$$\tilde{g}^n(1,z) = \tilde{f}^n(z).$$

Hence for $z \in N_n$ one of the two possibilities holds: either

- $\tilde{g}^n(\cdot, z) \equiv 0$, then $\tilde{g}^n_{m_k}(z) = 0$ for $k \geq 1$ and consequently $\tilde{f}^n(z) = \tilde{g}^n(1,z) = 0$ or
- the function $\tilde{g}^n(\cdot, z)$ has at most a countable number of zeroes in $\mathcal{D}$.

Let $A_{n,u} = \{z \in N_n : \tilde{g}^n(u,z) \neq 0\}$. Then for $z \in A_{n,1}$, the function $\tilde{g}^n(\cdot, z)$ has at most a countable number of zeroes in $\mathcal{D}$.
Let us consider the Cartesian product $\mathcal{D} \times \mathcal{A}_{\backslash,\infty}$ of $\mathcal{D}$ with Lebesgue measure λ and the set $A_{n,1}$ with measure σ_n. In that product the set $\{(u,z) : \tilde{g}^n(u,z) = 0\}$ has $\lambda \times \sigma_n$–measure zero because every z–section of that set has λ–measure zero (consists of at most countable many points). Therefore, for λ–a.e. $u \in \mathcal{D}$ we have $\tilde{g}^n(u,z) \neq 0$ for σ_n–a.e. $z \in A_{n,1}$. That implies that for almost all $u \in \mathcal{D}$

$$\sigma_n(A_{n,1} \setminus A_{n,u}) = 0. \tag{6}$$

Choosing $u_0 \in \mathcal{D}$, $|u_0| < a$ such that (6) holds for all $n \geq 1$ we obtain

$$\sigma_{\tilde{f}^n} \ll \sigma_{\tilde{g}^n(u_0,\cdot)} \text{ for } n \geq 1$$

and consequently

$$\sigma_f \ll \sigma_{g(u_0)}.$$

It follows from (3) and (4) that $g(u_0) \in F$. This completes the proof. ∎

Lemma 1.2. *Let $\langle F, \| \ \| \rangle$ be a Fréchet space. Then for any sequence $\{f_k\}_{k=1}^{\infty}$ in F there exists strictly increasing sequence $\{r_k\}_{k=1}^{\infty}$ of natural numbers such that for every complex number u, $|u| < \frac{1}{2}$ the series*

$$\sum_{k=1}^{\infty} u^{r_k} f_k \tag{7}$$

converges in F.

Proof. Choose a sequence $\{s_k\}_{k=1}^{\infty}$ of natural numbers such that for any complex number α and for any $k \geq 1$

$$\text{if } |\alpha| < \frac{1}{2^{s_k}} \text{ then } \| \alpha f_k \| < \frac{1}{2^k}.$$

Fix $r_k = s_1 + ... + s_k$. Then the sequence $\{r_k\}_{k=1}^{\infty}$ is strictly increasing and for any complex u, $|u| < \frac{1}{2}$ we have $|u^{r_k}| < \frac{1}{2^{r_k}}$ and consequently $\| u^{r_k} f_k \| < \frac{1}{2^k}$. Since the series $\sum_{k=1}^{\infty} \| u^{r_k} f_k \|$ converges, the series (7) converges in the Fréchet space F. ■

Corollary 1.1. *Let $\langle F, \| \ \| \rangle$ be a Fréchet space where F is a dense subspace of H and $\| \ \| \geq \| \ \|_H$. Then there exists $f \in F$ that realizes the maximal spectral type of U.*

Now, we will consider a special case, where $H = L^2(X, \mu)$ (μ is a positive finite Borel measure on X and U is an arbitrary unitary operator) putting better and better assumptions of the regularity of X. In the most general case, where (X, μ) is only assumed to be a Lebesgue space we obtain the following.

Corollary 1.2. *There exists $f \in L^{\infty}(X, \mu)$ that realizes the maximal spectral type of U.*

Above result have been proved by Alexeyev in [1].

Corollary 1.3. *Let X be a compact metric space. Then there exists a continuous function that realizes the maximal spectral type of U.*

Proof. Since μ is a regular measure, $C(X)$ is a dense subspace of the space $L^2(X, \mu)$. Apply Corollary 1.1 to $F = C(X)$. ■

Corollary 1.4. *Let X be a compact differentiable manifold of class C^r where $r \in \mathbb{N} \cup \{\infty\}$. Then there exists $f \in C^r(X)$ that realizes the maximal spectral type of U.*

Proof. For $r < \infty$ there exists a norm $\| \ \|_{C^r}$ on $C^r(X)$ such that $\| \ \|_{C^r} \geq \| \ \|_{\infty}$ and $\langle C^r(X), \| \ \|_{C^r} \rangle$ is a Banach space. For $r = \infty$ there exists an F-norm $\| \ \|_{C^{\infty}}$ on $C^{\infty}(X)$ such that $\| \ \|_{C^{\infty}} \geq \| \ \|_{\infty}$ and $\langle C^{\infty}(X), \| \ \|_{C^{\infty}} \rangle$ is a Fréchet space. Since $C^r(X)$ is a dense subspace of $\langle C(X), \| \ \|_{\infty} \rangle$, $C^r(X)$ is a dense subspace of $L^2(X, \mu)$. Apply Corollary 1.1, and the proof is complete. ■

We will now pass to the case of real analytic manifolds. We need some auxiliary results.

Lemma 1.3. *Let A be a real number such that $0 \leq A < 1$. Then for any natural k we have*

$$\sum_{n=1}^{\infty} n^k A^n \leq \frac{A}{1-A}\frac{k!}{(1-A)^k}. \blacksquare$$

Let W be an open subset of $\mathbb{R}^d$. We call a function $f : W \to \mathbb{C}$ *real analytic (analytic)* on W if in a neighborhood of every $x^0 \in W$, $f(x)$ can be represented by a power series of the form

$$\sum_{l_1,...,l_d=0}^{\infty} a_{l_1...l_d}(x_1 - x_1^0)^{l_1}...(x_d - x_d^0)^{l_d} \tag{8}$$

where $a_{l_1...l_d} \in \mathbb{C}$.

Lemma 1.4. *Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of analytic functions on W such that there exist some strictly increasing sequences $\{M_k\}_{k=1}^{\infty}$, $\{R_k\}_{k=1}^{\infty}$ of natural numbers satisfying*

$$\sup_{x\in W} \mid \frac{\partial^{l_1+...+l_d} f_k(x)}{\partial x_1^{l_1}...\partial x_d^{l_d}} \mid \leq R_k M_k^{l_1+...+l_d}$$

for every $l_1, ..., l_d \geq 0$, $k \geq 1$. Then for any $u \in \mathcal{D} = \{\sqcap \in \mathbb{C} : |\sqcap| < \infty\}$ the series

$$\sum_{k=1}^{\infty} u^{R_k M_k} f_k$$

converges uniformly on any compact subset of W and $f = \sum_{k=1}^{\infty} u^{R_k M_k} f_k$ is an analytic function on W.

Proof. By Lemma 1.3 for any $l_1, ..., l_d \geq 0$ such that $l_1 + ... + l_d > 0$ and

$x \in W$ we have

$$\begin{aligned}
\sum_{k=1}^{\infty} |u|^{R_k M_k} \mid \frac{\partial^{l_1+...+l_d} f_k(x)}{\partial x_1^{l_1} ... \partial x_d^{l_d}} \mid &\leq \sum_{k=1}^{\infty} |u|^{R_k M_k} R_k M_k^{l_1+...+l_d} \\
&\leq \sum_{k=1}^{\infty} |u|^{R_k M_k} (R_k M_k)^{l_1+...+l_d} \\
&\leq \sum_{n=1}^{\infty} |u|^n n^{l_1+...+l_d} \\
&\leq (l_1 + ... + l_d)! \frac{|u|}{1-|u|} (\frac{1}{1-|u|})^{l_1+...+l_d}
\end{aligned}$$

and for any $x \in W$

$$\sum_{k=1}^{\infty} |u|^{R_k M_k} |f_k(x)| \leq \sum_{k=1}^{\infty} |u|^{R_k M_k} R_k \leq \sum_{n=1}^{\infty} |u|^n n \leq \frac{|u|}{(1-|u|)^2}.$$

It follows that for any $l_1, ..., l_d \geq 0$ and $x \in W$

$$\begin{aligned}
\mid \frac{\partial^{l_1+...+l_d} f(x)}{\partial x_1^{l_1} ... \partial x_d^{l_d}} \mid &\leq (l_1 + ... + l_d)! \frac{|u|}{(1-|u|)^2} (\frac{1}{1-|u|})^{l_1+...+l_d} \\
&\leq l_1! ... l_d! \frac{|u|}{(1-|u|)^2} (\frac{d}{1-|u|})^{l_1+...+l_d}
\end{aligned}$$

and finally f is an analytic function on W. ■

Lemma 1.5. *Let X be a compact real analytic manifold. Then there exists a subspace E of the space of all analytic functions on X such that E is a dense subspace of $C(X)$ and for any sequence $\{g_k\}_{k=1}^{\infty}$ in E there exists a strictly increasing sequence $\{r_k\}_{k=1}^{\infty}$ of natural numbers such that for any complex $u \in \mathcal{D}$ the series*

$$\sum_{k=1}^{\infty} u^{r_k} g_k$$

converges in $C(X)$ and $\sum_{k=1}^{\infty} u^{r_k} g_k$ is an analytic function on X.

Proof. Let $\varphi : X \to \mathbb{R}^d$ be an analytic injection in an Euclidean space (see [4]). Set $E' = \{P : P \text{ is a polynomial on } \mathbb{R}^d\}$ and set $E = \{f : X \to \mathbb{C} : f \circ \varphi^{-1} \in E'\}$. By Stone–Weierstrass Theorem, E' is a dense subset of

$C(\varphi(X))$. Hence E is a dense subspace of $C(X)$ and every function $f \in E$ is analytic. Given a sequence $\{g_k\}_{k=1}^{\infty}$ in E choose P_k so that $g_k = P_k \circ \varphi$ for $k \geq 1$. Let W be an open bounded subset of $\mathbb{R}^d$ such that $\varphi(X) \subset W$. Then there exist some strictly increasing sequences $\{M_k\}_{k=1}^{\infty}$, $\{R_k\}_{k=1}^{\infty}$ of natural numbers such that

$$\sup_{x \in W} \left| \frac{\partial^{l_1+\ldots+l_d} P_k(x)}{\partial x_1^{l_1} \ldots \partial x_d^{l_d}} \right| \leq R_k M_k^{l_1+\ldots+l_d}$$

for every $l_1, ..., l_d \geq 0$, $k \geq 1$. Fix $r_k = R_k M_k$. By Lemma 1.4 the series $\sum_{k=1}^{\infty} u^{r_k} P_k$ converges in $C(\varphi(X))$ and $\sum_{k=1}^{\infty} u^{r_k} P_k$ is an analytic function on W. It follows that the series $\sum_{k=1}^{\infty} u^{r_k} g_k$ converges in $C(X)$ and

$$\sum_{k=1}^{\infty} u^{r_k} g_k = \sum_{k=1}^{\infty} u^{r_k} P_k \circ \varphi$$

is an analytic function on X. ■

Corollary 1.5. *Let X be a compact real analytic manifold and μ is a finite Borel measure on X. Then there exists an analytic function on X that realizes the maximal spectral type of U.*

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